INNER BOGOLIUBOV AUTOMORPHISMS OF THE MINIMAL C^* WEYL ALGEBRA

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(Received 12 March, 1991; revised 25 July, 1991)

Introduction. Within the context of orthogonal geometry, isometries of a real inner product space induce Bogoliubov automorphisms of its associated Clifford algebras. The question whether or not such automorphisms are inner is of considerable interest and importance. Inner Bogoliubov automorphisms were fully characterized for the C^* Clifford algebra by Shale and Stinespring [**14**] and for the W^* Clifford algebra by Blattner [**2**]: each case engenders a corresponding notion of spin group, constructed as a group of units inside the Clifford algebra [**4**].

Symplectic automorphisms of a real symplectic vector space induce Bogoliubov automorphisms of its various associated Weyl algebras. Segal [**11**] introduced what might be called the tame C^* Weyl algebra; its inner Bogoliubov automorphisms were analyzed by Shale [**13**]. The minimal C^* Weyl algebra was introduced by Manuceau [**5**] and Slawny [**15**]; see also [**6**]. Here we determine the inner Bogoliubov automorphisms of this minimal C^* Weyl algebra. The result is perhaps a little disappointing: it may be phrased as follows.

THEOREM. *Among the Bogoliubov automorphisms of the minimal C^* Weyl algebra, only the identity is inner.*

Actually, we establish rather more: the result remains true when the symplectic automorphisms themselves are merely additive and when the minimal C^* Weyl algebra is replaced by the von Neumann algebra generated in its trace representation.

Thus, the minimal C^* Weyl algebra really is quite small. In particular, it does not contain an Mp^c group or metaplectic group of units; as noted above, this is in contrast with the situation for Clifford algebras. We should remark that Mp^c groups and metaplectic groups have been constructed externally as groups of unitary operators implementing Bogoliubov automorphisms in Fock representations [**9**], [**13**].

Building upon our main theorem, we refine an earlier result of Plymen [**7**] concerning outer automorphic group representations. In addition, we place the tame C^* Weyl algebra in context and compare it with the minimal C^* Weyl algebra.

The author is grateful to the referee for encouraging suggestions and to the National Science Foundation for partial financial support.

The minimal C^* Weyl algebra. Let (V, Ω) be a real symplectic vector space: the real vector space V may be infinite-dimensional; the skew bilinear form Ω is nonsingular in having trivial kernel.

The (*exponential*) *Weyl algebra* $A(V, \Omega)$ is the complex associative algebra of all finitely-supported maps $V \to \mathbb{C}$, with pointwise linear operations and with product given by

$$(\phi\psi)(v) = \sum_{x+y=v} \varepsilon(x, y)\phi(x)\psi(y)$$

for $\phi, \psi \in A(V, \Omega)$ and $v \in V$; here

$$x, y \in V \Rightarrow \varepsilon(x, y) = \exp \frac{1}{2i\hbar} \Omega(x, y)$$

where $h = 2\pi\hbar$ is a positive scalar. It is known [8] that $A(V, \Omega)$ is central simple and lacks zero-divisors. Further, $A(V, \Omega)$ carries a canonical involution given by

$$\phi \in A(V, \Omega), \qquad v \in V \Rightarrow \phi^*(v) = \overline{\phi(-v)}$$

and a canonical trace given by evaluation at zero:

$$\phi \in A(V, \Omega) \Rightarrow \tau(\phi) = \phi(0).$$

It is convenient to denote by δ_v the element of $A(V, \Omega)$ taking value 1 at $v \in V$ and 0 elsewhere. Note that

$$x, y \in V \Rightarrow \delta_x \delta_y = \varepsilon(x, y)\delta_{x+y}:$$

in particular, $\delta_0 = 1$ is the multiplicative identity of $A(V, \Omega)$ and each δ_v is invertible; indeed, each δ_v is unitary. Moreover, it is known [8] that the invertibles in $A(V, \Omega)$ are precisely the elements having singleton support: those of the form $\lambda\delta_v$ for $0 \neq \lambda \in \mathbb{C}$ and $v \in V$.

We remark that $\{\delta_v : v \in V\}$ is a basis for $A(V, \Omega)$ and that

$$\phi \in A(V, \Omega) \Rightarrow \tau(\phi^*\phi) = \sum_{v \in V} |\phi(v)|^2.$$

It follows that the prescription

$$\phi, \psi \in A(V, \Omega) \Rightarrow \langle \phi \mid \psi \rangle_\tau = \tau(\psi^*\phi)$$

defines a Hermitian form on $A(V, \Omega)$. The Hilbert space completion of $A(V, \Omega)$ in the induced norm $\|\cdot\|_\tau$ will be denoted $\mathbb{H}_\tau$; it has $\{\delta_v : v \in V\}$ as a complete orthonormal set.

Any nonzero representation of the simple algebra $A(V, \Omega)$ is of course automatically faithful. All such Hilbert space representations of $A(V, \Omega)$ as an involutive algebra induce the same pre C^* norm on $A(V, \Omega)$: if $a \in A(V, \Omega)$ then the operator norm $\|\pi(a)\|$ is independent of $\pi : A(V, \Omega) \to B(\mathbb{H})$ and will be denoted $\|a\|$. The completion of $A(V, \Omega)$ in the norm $\|\cdot\|$ is called the *minimal* C^* *Weyl algebra of* (V, Ω) and will be denoted $\mathfrak{A}(V, \Omega)$. Minimality of $\mathfrak{A}(V, \Omega)$ may be expressed in terms of the following universal property: by extension, any nonzero star-homomorphism π from $A(V, \Omega)$ to a C^* algebra $\mathfrak{B}$ induces an isomorphism from $\mathfrak{A}(V, \Omega)$ to the norm closure of the image of π in $\mathfrak{B}$. We remark that $\mathfrak{A}(V, \Omega)$ is a central simple C^* algebra with a unique central state; this faithful state agrees with τ on $A(V, \Omega)$ and will be denoted by the same symbol. See [6] for details on these matters.

A specific (left regular) representation is determined by allowing $A(V, \Omega)$ to act on itself by left multiplication. This extends to define the trace representation

$$\pi_\tau : \mathfrak{A}(V, \Omega) \to B(\mathbb{H}_\tau)$$

whose state corresponding to the cyclic vector $\delta_0 \in A(V, \Omega) \subset \mathbb{H}_\tau$ is precisely τ. Of course, π_τ is essentially the GNS representation of $\mathfrak{A}(V, \Omega)$ constructed from the state τ. Note that

$$a \in A(V, \Omega) \Rightarrow \|a\|_\tau \leq \|\pi_\tau(a)\| = \|a\|.$$

As a consequence, the identity map on $A(V, \Omega)$ extends to a contractive linear map

$$T : \mathfrak{A}(V, \Omega) \to \mathbb{H}_\tau;$$

of course, this is the map given by

$$a \in \mathfrak{A}(V, \Omega) \Rightarrow T(a) = \pi_\tau(a)\delta_0.$$

Our analysis will be facilitated by the decomposition of elements from $\mathfrak{A}(V, \Omega)$. Rather than develop decompositions in $\mathfrak{A}(V, \Omega)$ itself, we prefer to make use of the Fourier decomposition in the Hilbert space $\mathbb{H}_\tau$ relative to its complete orthonormal system $\{\delta_v : v \in V\}$; the details are as follows. If $a \in \mathfrak{A}(V, \Omega)$ then $T(a) \in \mathbb{H}_\tau$ has Fourier decomposition

$$T(a) = \sum_{v \in V} a_v \delta_v$$

in which

$$v \in V \Rightarrow a_v = \langle T(a) \mid \delta_v \rangle_\tau = \langle \pi_\tau(a)\delta_0 \mid \delta_v \rangle_\tau = \tau(\delta_{-v} a).$$

From this it follows that T is injective: indeed, if $T(a) = 0$ then $\tau(\delta_{-v} a) = 0$ for all $v \in V$ so that $\tau(a'a) = 0$ for all $a' \in \mathfrak{A}(V, \Omega)$ by linearity and continuity of τ; thus $a = 0$ by faithfulness of τ.

THEOREM. *The contractive linear map*

$$T : \mathfrak{A}(V, \Omega) \to \mathbb{H}_\tau : a \mapsto \pi_\tau(a)\delta_0$$

is injective; if $a \in \mathfrak{A}(V, \Omega)$, then

$$T(a) = \sum_{v \in V} \tau(\delta_{-v} a)\delta_v.$$

A basic commutation property will conclude our preliminary material on the minimal C^* Weyl algebra. We claim that if $a \in \mathfrak{A}(V, \Omega)$ and $x, y \in V$ then $a\delta_x = \delta_y a$ forces either $a = 0$ or $x = y$. To see this, pass to $\mathbb{H}_\tau$ by application of T: thus, $T(a\delta_x) = T(\delta_y a)$ so that

$$v \in V \Rightarrow \tau(\delta_{-v} a \delta_x) = \tau(\delta_{-v}\delta_y a) \Rightarrow \varepsilon(v, x)\tau(\delta_{x-v} a) = \varepsilon(y, v)\tau(\delta_{y-v} a)$$

since τ is central. Taking absolute values now yields

$$v \in V \Rightarrow |\tau(\delta_{x-v} a)| = |\tau(\delta_{y-v} a)|$$

since $|\varepsilon| = 1$. Thus

$$|\tau(\delta_{-v+n(y-x)} a)| = |\tau(\delta_{-v} a)|$$

whenever $v \in V$ and $n \in \mathbb{Z}$. If $a \neq 0$ then $T(a) \neq 0$ so that some $\tau(\delta_{-v} a) \neq 0$; if also $x \neq y$ then an infinity of the Fourier coefficients of $T(a)$ have equal nonzero modulus. This is absurd.

THEOREM. *Let $a \in \mathfrak{A}(V, \Omega)$ and $x, y \in V$. If $a\delta_x = \delta_y a$, then either $a = 0$ or $x = y$.*

This result forms the basis for our determination of the inner Bogoliubov automorphisms of the minimal C^* Weyl algebra.

Inner Bogoliubov automorphisms. Denote by $Sp_+(V, \Omega)$ the group of all additive automorphisms g of V such that $\Omega(gx, gy) = \Omega(x, y)$ for all $x, y \in V$ and denote by $Sp(V, \Omega)$ its subgroup of real-linear elements. Each $g \in Sp_+(V, \Omega)$ induces an automorphism θ_g of the involutive algebra $A(V, \Omega)$ by the rule

$$\phi \in A(V, \Omega) \Rightarrow \theta_g(\phi) = \phi \circ g^{-1}$$

so that in particular

$$v \in V \Rightarrow \theta_g(\delta_v) = \delta_{gv}.$$

In turn, θ_g extends to an automorphism Θ_g of the minimal C^* Weyl algebra $\mathfrak{A}(V, \Omega)$. Traditionally, the term Bogoliubov automorphism is applied to an automorphism of $\mathfrak{A}(V, \Omega)$ having the form Θ_g for some g in the real symplectic group $Sp(V, \Omega)$; we shall be more liberal, referring to Θ_g as a Bogoliubov automorphism whenever g lies in the additive symplectic group $Sp_+(V, \Omega)$.

Our main result concerns necessary and sufficient conditions on $g \in Sp_+(V, \Omega)$ in order that the Bogoliubov automorphism Θ_g of $\mathfrak{A}(V, \Omega)$ be inner. These conditions turn out to be rather stringent.

THEOREM. *Let $g \in Sp_+(V, \Omega)$. The Bogoliubov automorphism Θ_g of the minimal C^* Weyl algebra $\mathfrak{A}(V, \Omega)$ is inner if and only if $g = I$.*

Proof. If Θ_g is the inner automorphism determined by the unit $a \in \mathfrak{A}(V, \Omega)$ then

$$v \in V \Rightarrow a\delta_v = \delta_{gv}a;$$

since $a \neq 0$, it follows from our closing result in the preceding section that $gv = v$ whenever $v \in V$ and so $g = I$. The converse is plain. □

In fact, this result can be pressed somewhat further. The von Neumann algebra $\mathscr{A}_\tau(V, \Omega)$ generated by $\mathfrak{A}(V, \Omega)$ or $A(V, \Omega)$ in the trace representation π_τ on $\mathbb{H}_\tau$ will be called the tracial W^* Weyl algebra of (V, Ω). Each $g \in Sp_+(V, \Omega)$ induces a Bogoliubov automorphism of $\mathscr{A}_\tau(V, \Omega)$ which we shall again denote by Θ_g. Being a cyclic trace vector, $\delta_0 \in \mathbb{H}_\tau$ is separating for $\mathscr{A}_\tau(V, \Omega)$ in its action on $\mathbb{H}_\tau$; as a consequence, the contractive linear map $T: \mathscr{A}_\tau(V, \Omega) \to \mathbb{H}_\tau$ defined by evaluation against δ_0 is injective. Arguing as for the minimal C^* Weyl algebra, if $a \in \mathscr{A}_\tau(V, \Omega)$ and $x, y \in V$ then $a\delta_x = \delta_y a$ forces either $a = 0$ or $x = y$. Thus the Bogoliubov automorphism Θ_g of $\mathscr{A}_\tau(V, \Omega)$ is inner if and only if $g = I$.

THEOREM. *Let $g \in Sp_+(V, \Omega)$. The Bogoliubov automorphism Θ_g of the tracial W^* Weyl algebra $\mathscr{A}_\tau(V, \Omega)$ is inner if and only if $g = I$.*

Of course, the result for the minimal C^* Weyl algebra is a direct corollary of this result for the tracial W^* Weyl algebra; however, we wished to offer a separate proof. In the interests of variety, we give an independent proof of the weaker analogous result for the plain Weyl algebra $A(V, \Omega)$. Write $\mathrm{bar}(\phi)$ for the barycentre of the support of a nonzero $\phi \in A(V, \Omega)$; note that if $v \in V$ then multiplication by δ_v on left or right translates barycentres through v. Now, if $0 \neq a \in A(V, \Omega)$ and $x, y \in V$, then

$$a\delta_x = \delta_y a \Rightarrow \mathrm{bar}(a\delta_x) = \mathrm{bar}(\delta_y a) \Rightarrow \mathrm{bar}(a) + x = y + \mathrm{bar}(a) \Rightarrow x = y.$$

As with the minimal C^* Weyl algebra, this commutation property forces $g = I$ if $g \in Sp_+(V, \Omega)$ and θ_g is inner.

Thus, among the Bogoliubov automorphisms of the Weyl algebras $\mathfrak{A}(V, \Omega)$ and $\mathscr{A}_\tau(V, \Omega)$ naturally associated to the real symplectic vector space (V, Ω), only the identity is inner. These results provoke mixed reactions. On the one hand, they are satisfying in their decisive nature. On the other hand, they demonstrate that $\mathfrak{A}(V, \Omega)$ and $\mathscr{A}_\tau(V, \Omega)$ are really rather small. Contrary to what might be expected by analogy with the case for Clifford algebras, these Weyl algebras do not contain metaplectic groups as groups of units.

As an application of our result on inner Bogoliubov automorphisms of the minimal C^* Weyl algebra, we simplify and extend a result of Plymen [7] on automorphic group representations. Thus, let $\rho: K \to Sp(V, \Omega)$ be a faithful symplectic representation of a group K on the real symplectic vector space (V, Ω). The composite $\Theta \circ \rho$ is then a faithful representation of K by Bogoliubov automorphisms of the minimal C^* Weyl algebra $\mathfrak{A}(V, \Omega)$; our main theorem automatically ensures that if $k \in K$ then $\Theta \circ \rho(k)$ is inner if and only if k is the identity. We record this as follows.

THEOREM. *Each faithful symplectic representation ρ of a group K on (V, Ω) induces a faithful outer automorphic representation $\Theta \circ \rho$ of K on the minimal C^* Weyl algebra $\mathfrak{A}(V, \Omega)$.*

To be specific, we might let K be a locally compact Hausdorff group and let ρ be its unitary (left) regular representation, with Ω the imaginary part of the Hilbert space inner product on $V = L^2(K)$. In fact, K might be any group at all if we equip it with the discrete topology and counting measure, letting V be the space of all finitely-supported maps $K \to \mathbb{C}$, letting Ω be the imaginary part of the Hermitian inner product given by $\langle f_1 \mid f_2 \rangle = \sum_{l \in K} f_1(l)\overline{f_2(l)}$ for $f_1, f_2 \in V$, and letting ρ be the representation given by $(\rho(k)f)(l) = f(k^{-1}l)$ for $f \in V$ and $k, l \in K$; here, V can be completed if desired.

Our theorem both extends and simplifies Proposition 2 of [7]: on the one hand, the real symplectic vector space can be arbitrary and we lift the hypothesis that the group be separable; on the other hand, we do not need to take a countably infinite direct sum of copies of the regular representation. In reference to this last point, such a summation was called for in [7] because the inner Bogoliubov automorphisms of the minimal C^* Weyl algebra itself were not then known precisely: use was made instead of the characterization of inner Bogoliubov automorphisms of the (larger) tame C^* Weyl algebra.

Having reached this stage, it is appropriate that we discuss the tame C^* Weyl algebra and place it in context. Our account will be primarily descriptive; for details, we refer to the original papers of Segal [11] and Shale [13].

We shall say that the (nonzero) Hilbert space representation $\pi: A(V, \Omega) \to B(\mathbb{H})$ is tame (or regular) if and only if, for each $v \in V$, the one-parameter unitary group $t \mapsto \pi(\delta_{tv})$ is (weakly or strongly) continuous; note that these one-parameter groups then have (Stone) infinitesimal generators satisfying the Heisenberg commutation relations. In these terms, the celebrated uniqueness theorem due to Stone and von Neumann asserts that if V is finite-dimensional then $A(V, \Omega)$ admits precisely one irreducible tame representation, of which any tame representation is a multiple (up to unitary equivalence, of course).

Denote by $\mathscr{F}(V, \Omega)$ the set of all finite-dimensional symplectic subspaces of (V, Ω) and note that $\mathscr{F}(V, \Omega)$ is directed under inclusion. If $M \in \mathscr{F}(V, \Omega)$ then the von Neumann algebra $\mathscr{A}_\mu(M, \Omega) \supset A(M, \Omega)$ generated in any tame representation $\mu: A(M, \Omega) \to B(\mathbb{H}_\mu)$ is actually a factor of type I, as follows from the uniqueness theorem of Stone and von Neumann: indeed, $\mathscr{A}_\mu(M, \Omega) = B(\mathbb{H}_\mu)$ when μ is irreducible, and von Neumann algebras generated in multiple representations are isomorphic. If also $N \in \mathscr{F}(V, \Omega)$ and $M \subset N$, then the uniqueness theorem canonically associates to each tame representation $\nu: A(N, \Omega) \to B(\mathbb{H}_\nu)$ an injective star-homomorphism $\mathscr{A}_\mu(M, \Omega) \to \mathscr{A}_\nu(N, \Omega)$ such that the diagram of inclusions commutes.

$$\begin{array}{ccc} \mathscr{A}_\mu(M,\Omega) & \longrightarrow & \mathscr{A}_\nu(N,\Omega) \\ \uparrow & & \uparrow \\ A(M,\Omega) & \longrightarrow & A(N,\Omega) \end{array}$$

By definition, the tame C^* Weyl algebra $\mathscr{A}(V,\Omega)$ is the C^* inductive limit of the system of von Neumann algebras $\mathscr{A}_\mu(M,\Omega)$ as M ranges over the directed set $\mathscr{F}(V,\Omega)$; here, the choice of tame representation $\mu: A(M,\Omega)\to B(\mathbb{H}_\mu)$ is ultimately immaterial. Of course, as M ranges over $\mathscr{F}(V,\Omega)$ the inclusions $A(M,\Omega)\to\mathscr{A}_\mu(M,\Omega)$ match together to provide a canonical inclusion $A(V,\Omega)\to\mathscr{A}(V,\Omega)$; the C^* algebra generated by the image of this inclusion is a copy of the minimal C^* Weyl algebra $\mathfrak{A}(V,\Omega)$. Note that if $g\in Sp(V,\Omega)$ then θ_g extends to define a Bogoliubov automorphism Θ_g of the tame C^* Weyl algebra $\mathscr{A}(V,\Omega)$; the real-linearity of g ensures that it carries tame representations to tame representations.

We should point out that if $\pi: A(V,\Omega)\to B(\mathbb{H})$ is a tame representation of $A(V,\Omega)$ itself then $\mathscr{A}(V,\Omega)$ can be realized more directly as follows. For each $M\in\mathscr{F}(V,\Omega)$ let $\mathscr{A}_\pi(M,\Omega)$ denote the von Neumann algebra generated by $\{\pi(\delta_v): v\in M\}$ in $B(\mathbb{H})$; now $\mathscr{A}(V,\Omega)$ is the uniform closure in $B(\mathbb{H})$ of the union $\cup\{\mathscr{A}_\pi(M,\Omega): M\in\mathscr{F}(V,\Omega)\}$. Naturally, this fact can be reformulated as a universal mapping property. As noted above, we refer to Segal **[11]** for details on the tame C^* Weyl algebra $\mathscr{A}(V,\Omega)$ and its significance as a universal C^* algebra of field observables over (V,Ω).

Incidentally, if (V,Ω) is arbitrary then the existence problem for tame representations of $A(V,\Omega)$ is substantial: see Chapter IV of Segal **[12]**. If (V,Ω) admits a unitary structure—that is, a Hermitian inner product (complete or not) of which Ω is the imaginary part—then the corresponding Fock representation of $A(V,\Omega)$ is certainly tame. The question of whether or not (V,Ω) admits unitary structures is itself not fatuous. In general, unitary structures can definitely fail to exist: see **[10]**. However, in applications it is often justifiable to assume the existence of a complete unitary structure: see **[1]**, **[3]**, **[12]**.

As a matter of fact, Shale **[13]** characterized the inner Bogoliubov automorphisms of the tame C^* Weyl algebra $\mathscr{A}(V,\Omega)$ in the case that (V,Ω) is provided with a complete unitary structure: if $g\in Sp(V,\Omega)$, then Θ_g is inner if and only if g is tame in restricting to the identity on $M^\perp$ for some $M\in\mathscr{F}(V,\Omega)$. We remark that the inner nature of $\Theta_g\in \mathrm{Aut}(V,\Omega)$ for tame $g\in Sp(V,\Omega)$ is essentially a consequence of the Stone and von Neumann theorem together with the circumstance that $\mathscr{A}_\mu(M,\Omega)$ is constructed as a von Neumann algebra for each $M\in\mathscr{F}(V,\Omega)$; we remark further that the elements of $\mathscr{A}(V,\Omega)$ implementing Θ_g are determined up to scalar multiples and may be assumed unitary.

Now, let $Sp_0(V,\Omega)$ signify the group of all tame real-linear symplectic automorphisms of (V,Ω). The foregoing discussion has the following outcome: the group $Mp_0^c(V,\Omega)$ of all unitary elements of $\mathscr{A}(V,\Omega)$ implementing its inner Bogoliubov automorphisms is a central extension of $Sp_0(V,\Omega)$ by the circle $\mathbb{T}$ of unitary scalars; there is a short exact sequence

$$1\to\mathbb{T}\to Mp_0^c(V,\Omega)\xrightarrow{\sigma} Sp_0(V,\Omega)\to 1$$

where σ sends $u\in Mp_0^c(V,\Omega)$ to $g\in Sp_0(V,\Omega)$ in case $\Theta_g(a)=uau^*$ whenever $a\in$

https://doi.org/10.1017/S001708950000882X Published online by Cambridge University Press

$\mathscr{A}(V, \Omega)$. As is usual [**9**] for Mp^c groups, $Mp_0^c(V, \Omega)$ has a distinguished unitary character η whose restriction to $\mathbb{T}$ is the squaring map; the kernel $Mp_0(V, \Omega)$ of η is a double cover of the tame symplectic group $Sp_0(V, \Omega)$ and so may be called the tame metaplectic group.

Thus, the tame C^* Weyl algebra $\mathscr{A}(V, \Omega)$ does contain the tame metaplectic group $Mp_0(V, \Omega)$ as a group of unitaries implementing inner Bogoliubov automorphisms; of course, it need hardly be said here that $Sp_0(V, \Omega)$ is a decidedly small subgroup of $Sp(V, \Omega)$ when V is infinite-dimensional. By way of contrast, recall that the minimal C^* Weyl algebra $\mathfrak{A}(V, \Omega)$ has the identity as its only inner Bogoliubov automorphism and so cannot possibly host a metaplectic group.

Still supposing V to be a complex Hilbert space of whose inner product Ω is the imaginary part, let us specialize and take $\pi: A(V, \Omega) \to B(\mathbb{H})$ to be the corresponding Fock representation. The main result of Shale [**13**] now asserts that if $g \in Sp(V, \Omega)$ then the Bogoliubov automorphism θ_g of $A(V, \Omega)$ is implemented in π by a unitary operator U on $\mathbb{H}$ (in the sense that $a \in A(V, \Omega)$ implies $\pi(\theta_g a) = U\pi(a)U^*$) if and only if g lies in the restricted symplectic group $Sp_{\text{res}}(V) = Sp(V)$ defined by requiring the commutator $[g, i]$ to be Hilbert–Schmidt. The resulting group $Mp^c(V)$ of all such unitary implementers is a non-split central circle extension of $Sp(V)$; it splits over the unitary group of V since this fixes the Fock state. For further details on $Mp^c(V)$ and a pertinent notion of metaplectic group, see [**9**].

We close by remarking of an arbitrary real symplectic vector space (V, Ω) that the situation as regards unitary implementation of Bogoliubov automorphisms of $A(V, \Omega)$ in the trace representation π_τ is optimally good. By virtue of its uniqueness, the central state τ is invariant under all Bogoliubov automorphisms; consequently, these are all unitarily implemented in π_τ. We can be more explicit: if $g \in Sp_+(V, \Omega)$ then the automorphism θ_g of $A(V, \Omega)$ extends to a unitary operator U_g on $\mathbb{H}_\tau$ fixing δ_0 and with the property that

$$a \in A(V, \Omega) \Rightarrow \pi_\tau(\Theta_g a) = U_g \pi_\tau(a) U_g^*;$$

needless to say, U_g lies in $\mathscr{A}_\tau(V, \Omega)$ if and only if $g = I$. In the terminology of C^* dynamical systems, the Bogoliubov automorphism group $Sp_+(V, \Omega)$ is naturally covariantly represented in π_τ; note that here, the covariant representation U is a true representation and not merely a projective one.

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