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RESEARCH ARTICLE

The moduli of sections has a canonical obstruction theory

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Abstract

We give a detailed proof that locally Noetherian moduli stacks of sections carry canonical obstruction theories. As part of the argument, we construct a dualising sheaf and trace map, in the lisse-étale topology, for families of tame twisted curves when the base stack is locally Noetherian.

Contents

1. Introduction

1.1. Overview

Let $\mathcal{M}$ be a locally Noetherian algebraic stack, and let $\mathcal{C} \to \mathcal{M}$ be a family of twisted curves as in [AOV11, Def 2.1]. Let $\mathcal{Z} \to \mathcal{C}$ be a morphism of algebraic stacks such that $\mathcal{Z} \to \mathcal{M}$ is locally of finite presentation, is quasi-separated and has affine stabilisers. By [HR19, Thm 1.3], there is an algebraic stack $\mathrm{Sec}_{\mathcal{M}}(\mathcal{Z}/\mathcal{C})$ over $\mathcal{M}$ whose fibre over a scheme $T \to \mathcal{M}$ is

$$\mathrm{Sec}_{\mathcal{M}}(\mathcal{Z}/\mathcal{C})(T) := \mathrm{Hom}_{\mathcal{C}}(\mathcal{C} \times_{\mathcal{M}} T, \mathcal{Z}),$$

where the right-hand side is the groupoid of morphisms of stacks over $\mathcal{C}$. Recall that an obstruction theory for $\mathrm{Sec}_{\mathcal{M}}(\mathcal{Z}/\mathcal{C})$ is a morphism of complexes $\phi : E \to \mathbb{L}_{\mathrm{Sec}/\mathcal{M}}$ in the derived category of $\mathrm{Sec} := \mathrm{Sec}_{\mathcal{M}}(\mathcal{Z}/\mathcal{C})$ whose mapping cone has vanishing cohomology sheaves in degrees $[-1, \infty)$ (see Section 4.3). An implication of our main theorem is the following.

Theorem 1.1. *The stack* $\mathrm{Sec}_{\mathcal{M}}(\mathcal{Z}/\mathcal{C})$ *carries a canonical obstruction theory.*

We define the canonical obstruction theory in Section 4.4. Theorem 1.1 is generalised and stated more precisely as Theorem 4.13 below. An important feature of the canonical obstruction theory is its functoriality, as explained in [CJW21, Appendix A].

When the obstruction theory in Theorem 1.1 is perfect and $\mathrm{Sec} := \mathrm{Sec}_{\mathcal{M}}(\mathcal{Z}/\mathcal{C})$ is Deligne-Mumford, quasi-separated and locally finite type over a field, the machinery in [BF97] and [Kre99, Sec 5.2] defines a virtual fundamental class on Sec. This is a key part of the construction of Gromov-Witten theory and related enumerative theories: see, for example, [Beh97; AGV08; CCK15; CL12]. On the other hand, Theorem 1.1 is used with a non-Deligne-Mumford instance of Sec to functorially compare different obstruction theories on quasimap moduli spaces in [CJW21, Lem A.2.5]. This comparison is crucial for the application of [CJW21] to quasimap theory and also for the computations of quasimap I-functions in [Web18; Web21].

1.2. Discussion of Theorem 1.1

The usual argument supporting Theorem 1.1 when Sec is Deligne-Mumford is as follows (this is used, for example, in [BF97, Prop 6.2]). First reduce to showing that for each affine $f : T \to \mathrm{Sec}$ and square-zero quasi-coherent ideal sheaf I on T, the induced map

$$\mathrm{Ext}^i(\mathsf{L}f^*\mathbb{L}_{\mathrm{Sec}/\mathcal{M}}, I) \to \mathrm{Ext}^i(\mathsf{L}f^*E, I) \tag{1.1}$$

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has the following properties (see [BF97, Thm 4.5]):

- When $i = 1$, equation (1.1) is injective on obstructions. (1.2)
- When $i = 0$ and there exists a deformation of f by I, equation (1.1) is an isomorphism.

Second, use standard deformation theory to relate the groups $\mathrm{Ext}^i(\mathsf{L}f^*\mathbb{L}_{\mathrm{Sec}/\mathcal{M}}, I)$ (respectively, $\mathrm{Ext}^i(\mathsf{L}f^*E, I)$) to deformations of the morphism $f : T \to \mathrm{Sec}$ (respectively, $\mathcal{C} \times_{\mathcal{M}} T \to \mathcal{Z}$). Since morphisms $T \to \mathrm{Sec}$ are equivalent to morphisms $\mathcal{C} \times_{\mathcal{M}} T \to \mathcal{Z}$ by definition of Sec, one concludes that equation (1.1) is an injection (on obstructions) when $i = 1$ and that *the groups in equation (1.1) are isomorphic* (if the obstruction vanishes) when $i = 0$. We note that this falls just shy of the second requirement in list (1.2) since it is not clear that the morphism in equation (1.1) is itself an isomorphism.

In this paper, we copy the first step above in Lemma 4.11. However, in the second step, we analyse the functoriality of the isomorphism of Picard categories

$$\underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{X}, I) \simeq \mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}, I[1])) \tag{1.3}$$

due to Illusie and Olsson ([Ill71; Ols06]), for $\mathcal{X} \to \mathcal{Y}$ a representable morphism of algebraic stacks. (See Section 4.1 for the notation and Theorem 4.4 for the precise statement.) Our proof shows that when $i = 0$, not only are the groups in equation (1.1) isomorphic (in the case of vanishing obstruction), but in fact *the morphism in equation (1.1) is an isomorphism*, completing the proof of the second requirement in list (1.2). Our proof also covers the case when $\mathrm{Sec} \to \mathcal{M}$ is not representable or even relatively Deligne-Mumford.

The correct approach to Theorem 1.1 is likely through derived algebraic geometry, as in [STV15, Sec 2.2]. The functoriality properties of the obstruction theory proved in [CJW21, Appendix A] would be natural consequences of such a construction. Unfortunately, this author is not equipped to produce the argument. Although the statement of Theorem 1.1 is certainly familiar, we note that there does not seem to be a reference in the literature for the generality in which we have stated it here.

1.3. Duality for twisted curves

A key ingredient for the construction of the obstruction theory in Theorem 1.1 is the following (stated more precisely as Proposition 3.14 below).

Theorem 1.2. *For every family $p : \mathcal{C} \to \mathcal{M}$ of tame twisted curves on a locally Noetherian algebraic stack $\mathcal{M}$, there is a functorial pair $(\omega_{\mathcal{M}}, tr_{\mathcal{M}})$ with $\omega_{\mathcal{M}}$ a quasi-coherent sheaf on $\mathcal{C}$ and $tr_{\mathcal{M}} : \mathsf{R}p_*\omega_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}[-1]$. When $\mathcal{M}$ is a quasi-separated Noetherian algebraic space, the pair $(\omega_{\mathcal{M}}, tr_{\mathcal{M}})$ agrees with the right adjoint to $\mathsf{R}p_*$.*

We restate the last sentence of the theorem more precisely: if $p : \mathcal{C} \to \mathcal{M}$ is a family of twisted curves, a right adjoint $p^!$ to $\mathsf{R}p_*$ exists by [HR17, Thm 4.14(1)] (see also Lemma 3.6 below). The last sentence of Theorem 1.2 says that if $\mathcal{M}$ is a quasi-separated Noetherian algebraic space, we have that $\omega_{\mathcal{M}}[1] = p^!\mathcal{O}_{\mathcal{M}}$ and $tr_{\mathcal{M}}$ is the counit of the $(\mathsf{R}p_*, p^!)$ adjunction.

The reason we do not have this agreement for arbitrary locally Noetherian $\mathcal{M}$ is that it seems difficult to show that $p^!$ is compatible with arbitrary basechange. Following the exposition of [Lip09] for schemes, we prove basechange for the right adjoint to pushforward for certain morphisms of algebraic stacks in Lemma 3.7 below (see also [Nee17] for a complementary result). However, in applications, one would like to have basechange for $p^!$ for families of curves over arbitrary morphisms of algebraic stacks.

The basechange problem arises for nontwisted prestable curves as well, and [Stacks, Tag 0E5W] addresses the issue by 'glueing' the pairs $(\omega_{\mathcal{M}}, tr_{\mathcal{M}})$ to get a functorial construction of a dualising

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complex and trace map. We adopt the same strategy to prove Theorem 1.2. Again, while the statement of Theorem 1.2 is well-known, we do not know a reference for twisted curves, even over the complex numbers.

1.4. Contents of the paper

The main goal of Section 2 is to derive a certain commuting diagram (Lemma 2.14), which will be used in our proof of Theorem 1.1. Because the notation is simpler and because we can reuse various parts of the argument in other parts of the paper, we work in the setting of abstract closed symmetric monoidal categories.

In Section 3, we prove Theorem 1.2. The proof requires us to construct a special kind of hypercover of an algebraic stack and an associated lisse-étale topos. This is an application of the general results proved in [Stacks], and we explain the details in Appendix A.

We explain equation (1.3) and prove Theorem 1.1 in Section 4. The proof itself is fairly short, granting the existence of the dualising complex and the functoriality of equation (1.3). We reserve our proof of the functoriality of equation (1.3) for Appendix B. Since the functoriality is critical to our argument, we include the details, but said details are unsurprising.

1.5. The locally Noetherian hypothesis

We expect that the locally Noetherian assumption on $\mathcal{M}$ can be relaxed. It is used only in the proof of Lemma 3.4 to show that pushing forward to the coarse moduli space of a twisted curve preserves pseudo-coherent objects. See Remark 3.5.

1.6. Conventions and notation

We collect some conventions and recurring notation. Our list of notation here is not exhaustive.

Algebraic stacks. We follow the conventions in [Stacks, Tag 0260]; in particular, an algebraic stack need not be quasicompact or quasi-separated.

Twisted curves. A morphism $p : \mathcal{C} \to \mathcal{M}$ of algebraic stacks is a *family of twisted curves* if smooth-locally on $\mathcal{M}$ it is a twisted curve in the sense of [AOV11, Def 2.1].

Notation for closed categories and internal hom

Notation	Category	Internal hom
$\mathrm{Mod}(A)$	Category of A-modules for a sheaf of rings A on a site $\mathcal{S}$	$\mathcal{H}om_A$
$\mathrm{D}(A)$	Unbounded derived category of $\mathrm{Mod}(A)$	$\mathrm{R}\mathcal{H}om_A$
	Derived global hom functor (valued in the derived category of $\Gamma(\mathcal{S}, A)$*-modules).*	*Notated* RHom_A
$\mathcal{X}_{\text{lis-et}}$ (respectively, $\mathcal{X}_{\text{et}}$)	Category of sheaves on the lisse-étale (respectively, étale) site of an algebraic (respectively, Deligne-Mumford) stack $\mathcal{X}$	Not needed
$\mathrm{QCoh}(\mathcal{X}_{\text{lis-et}})$ (respectively, $\mathrm{QCoh}(\mathcal{X}_{\text{et}})$)	Category of quasi-coherent sheaves on the lisse-étale (respectively, étale) site of an algebraic (respectively, Deligne-Mumford) stack $\mathcal{X}$	Not needed
$\mathrm{D}(\mathcal{X}_{\text{lis-et}})$ (respectively, $\mathrm{D}(\mathcal{X}_{\text{et}})$)	Unbounded derived category of $\mathcal{O}_{\mathcal{X}}$-modules in $\mathcal{X}_{\text{lis-et}}$ (respectively, $\mathcal{X}_{\text{et}}$)	$\mathrm{R}\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}$
	Derived global hom functor (valued in the derived category of $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$*-modules)*	*Notated* $\mathrm{RHom}_{\mathcal{O}_{\mathcal{X}}}$
$\mathrm{D}_{\text{qc}}(\mathcal{X}_{\text{lis-et}})$ (respectively, $\mathrm{D}_{\text{qc}}(\mathcal{X}_{\text{et}})$)	Full subcategory of $\mathrm{D}(\mathcal{X}_{\text{lis-et}})$ (respectively, $\mathrm{D}(\mathcal{X}_{\text{et}})$) on objects with quasi-coherent cohomology sheaves	$\mathrm{R}\mathcal{H}om^{\text{qc}}_{\mathcal{O}_{\mathcal{X}}}$

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Operations on sheaves on topoi and algebraic stacks

Notation	Meaning
$H^i(\mathcal{F})$	The i^{th} cohomology sheaf of a complex $\mathcal{F}$.
$\mathsf{R}\Gamma(\mathcal{F})$	The derived global sections functor applied to a complex $\mathcal{F}$, commonly notated $\mathsf{R}\Gamma(X, \mathcal{F})$, where X is a topos.
(f^{-1}, f_*)	Adjoint functors defined by a morphism of topoi $f : \mathscr{C} \to \mathscr{D}$.
f^*	For $f : (\mathscr{C}, \mathscr{O}_{\mathscr{C}}) \to (\mathscr{D}, \mathscr{O}_{\mathscr{D}})$ a morphism of ringed topoi, set $f^*(-) := f^{-1}(-) \otimes_{f^{-1}\mathscr{O}_{\mathscr{D}}} \mathscr{O}_{\mathscr{C}}$.
$\mathsf{L}f^*$	For $f : \mathcal{X} \to \mathcal{Y}$ a morphism of algebraic stacks, we denote by $\mathsf{L}f^* : \mathrm{D}_{\mathrm{qc}}(\mathcal{Y}_{\text{lis-et}}) \to \mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}})$ the functor $\mathsf{L}f_{\mathrm{qc}}^*$ in [HR17, Sec 1.3].
$\mathsf{R}f_*$	For $f : (\mathscr{C}, \mathscr{O}_{\mathscr{C}}) \to (\mathscr{D}, \mathscr{O}_{\mathscr{D}})$ a morphism of ringed topoi, this is the usual direct image functor $\mathrm{D}(\mathscr{O}_{\mathscr{C}}) \to \mathrm{D}(\mathscr{O}_{\mathscr{D}})$.
$\mathsf{R}f_*$	For $f : \mathcal{X} \to \mathcal{Y}$ a *concentrated* morphism of algebraic stacks, this is the functor $\mathsf{R}(f_{\mathrm{qc}})_* : \mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}}) \to \mathrm{D}_{\mathrm{qc}}(\mathcal{Y}_{\text{lis-et}})$ of [HR17, Sec 1.3] that is right adjoint to $\mathsf{L}f^*$. By [HR17, Thm 2.6(2)], it agrees with the restriction of the usual direct image functor $\mathsf{R}(f_{\text{lis-et}})_* : \mathrm{D}(\mathcal{X}_{\text{lis-et}}) \to \mathrm{D}(\mathcal{Y}_{\text{lis-et}})$.

2. A formal framework for dualising objects and trace maps

2.1. *Closed symmetric monoidal categories*

Because notation is simpler in an abstract setting, we work for a moment with closed symmetric monoidal categories. If $\mathscr{C}$ is such a category, we will write $\mathscr{O}_{\mathscr{C}}$ for the unit, $\otimes$ for the product and Hom for internal hom, using $\mathscr{C}(X, Y)$ to denote the set of morphisms between two objects $X, Y \in \mathscr{C}$ and 1_X to denote the identity morphism on X. We will suppress mention of the associativity, commutativity and identity isomorphisms that are part of the definition of $\mathscr{C}$. If $\mathscr{C}$ and $\mathscr{D}$ are any two categories and $R : \mathscr{C} \to \mathscr{D}$ is a functor with a left adjoint L, then for $X \in \mathscr{D}$ and $Y \in \mathscr{C}$, we will denote the unit and counit of the adjunction by

$$\eta_X^L : X \to RL(X) \qquad \epsilon_Y^L : LR(Y) \to Y,$$

omitting the decorations on η and ϵ when there is no risk of confusion.

We will use many specific instances of the following abstract situation.

Situation 2.1. *We are given $\mathscr{C}$, $\mathscr{D}$ be closed symmetric monoidal categories with $f^* : \mathscr{D} \to \mathscr{C}$ strong monoidal and f_* a right adjoint.*[1] *This means we have natural isomorphisms*

$$f^*(X) \otimes f^*(Y) \to f^*(X \otimes Y) \tag{2.1}$$

$$\mathscr{O}_{\mathscr{C}} \to f^*\mathscr{O}_{\mathscr{D}}. \tag{2.2}$$

When we are in Situation 2.1, we have the following three morphisms at our disposal. The first we recall from [FHM03, (3.4)]: given $Y \in \mathscr{C}$ and $X \in \mathscr{D}$, there is a functorial isomorphism

$$\mathrm{Hom}(X, f_*Y) \xrightarrow{\sim} f_*(\mathrm{Hom}(f^*(X), Y)). \tag{2.3}$$

The second is the composition

$$f_* \mathrm{Hom}(X, Y) \to f_* \mathrm{Hom}(f^* f_* X, Y) \xleftarrow[\sim]{(2.3)} \mathrm{Hom}(f_* X, f_* Y), \tag{2.4}$$

where the first morphism is induced by the counit of the adjunction. The third is

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, f_* f^* Y) \xrightarrow[\sim]{(2.3)} f_* \mathrm{Hom}(f^* X, f^* Y), \tag{2.5}$$

[1] Moreover, f_* is lax monoidal by [FHM03, (3.2)], so our setup is consistent with that used in [Lip09, Sec 3.5].

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where the first morphism is induced by the unit of the adjunction; it is an isomorphism if f^* is fully faithful. One can check that equation (2.5) is functorial in X, Y and the adjoint pair (f^*, f_*) (see [Lip09, Exercise 3.7.1.1]).

We present Example 2.2 as the first instance of Situation 2.1; more instances can be found in Examples 3.1, B.3, B.4, and B.8.

Example 2.2. The following is an example of Situation 2.1. Let $B' \to B$ be a homomorphism of rings, and set $\mathscr{C} = \mathrm{Mod}(B')$ and $\mathscr{D} = \mathrm{Mod}(B)$. Define f^* to be the extension of scalars functor $- \otimes_{B'} B$, and define f_* to be restriction of scalars $(-)_{B'}$. The functor $- \otimes_{B'} B$ is strong symmetric monoidal. One can check from the definition in [FHM03, (3.4)] that equation (2.3) sends $X \to (Y)_{B'}$ to its adjoint arrow $X \otimes_{B'} B \to Y$, that equation (2.4) sends a B-module homomorphism $g : X \to Y$ to $(g)_{B'} : (X)_{B'} \to (Y)_{B'}$ and that equation (2.5) sends a B'-module homomorphism $h : X \to Y$ to $h \otimes_{B'} B : X \otimes_{B'} B \to Y \otimes_{B'} B$.

We recall a formal framework for basechange. We do not need monoidal structures here.

Situation 2.3. *We have a diagram of categories and functors*

$$\begin{tikzcd} \mathscr{S} \arrow[r, "m'_*"] \arrow[d, "g_*"] & \mathscr{C} \arrow[d, "f_*"] \\ \mathscr{T} \arrow[r, "m_*"] & \mathscr{D} \end{tikzcd} \tag{2.6}$$

where the functors f_*, g_*, m_*, m'_* have left adjoints f^*, g^*, m^*, m'^*, and we are given a natural transformation $m'^* f^* \simeq g^* m^*$.

In this situation, we get a unique natural transformation $m_* g_* \simeq f_* m'_*$ such that the adjunctions for $(m'^* f^*, f_* m'_*)$ and $(m_* g_*, g^* m^*)$ are compatible (see [Lip09, Sec 3.6]). We define the basechange map

$$m^* f_* X \to g_* m'^* X \qquad \text{for } X \in \mathscr{S} \tag{2.7}$$

as in [Lip09, Prop 3.7.2(i)]. It may not be an isomorphism in general.

Lemma 2.4. *For $Y \in \mathscr{D}$ and $X \in \mathscr{C}$, there are commuting diagrams*

$$\begin{tikzcd} m'^* f^* f_* X \arrow[r, "m'^* \epsilon"] \arrow[d, equal] & m'^* X \\ g^* m^* f_* X \arrow[r, "(2.7)"] & g^* g_* m'^* X \arrow[u, "\epsilon"'] \end{tikzcd} \qquad \begin{tikzcd} m^* Y \arrow[r, "m^* \eta"] \arrow[d, "\eta"] & m^* f_* f^* Y \arrow[d, "(2.7)"] \\ g_* g^* m^* Y \arrow[r, equal] & g_* m'^* f^* Y \end{tikzcd} \tag{2.8}$$

Proof. We show that the first diagram commutes; the second one may be checked similarly. The commutativity of the first follows from the following commuting diagram:

$$\begin{tikzcd} g^* m^* f_* X \arrow[r, "g^* m^* f_* \eta"] \arrow[d, equal] & g^* m^* f_* m'_* m'^* X \arrow[r, equal] \arrow[d, equal] & g^* m^* m_* g_* m'^* X \arrow[r, "g^* \epsilon^{m^*}"] & g^* g_* m'^* X \arrow[dd, "\epsilon^{g^*}"] \\ m'^* f^* f_* X \arrow[r, "m'^* f^* f_* \eta"] \arrow[d, "m'^* \epsilon"] & m'^* f^* f_* m'_* m'^* X \arrow[d, "m'^* \epsilon"] & & \\ m'^* X \arrow[r, "m'^* \eta"] & m'^* m'_* m'^* X \arrow[rr, "\epsilon^{m'^*}"] & & m'^* X \end{tikzcd} \tag{2.9}$$

The perimeter of the diagram from $m'^* f^* f_* X$ to $m'^* X$ along the bottom is equal to $m'^* \epsilon$ using a triangle identity, while the composition along the top is equal to the composition of the other three arrows in the desired square. The commutativity of the big cell in diagram (2.9) is compatibility of the $(m'^* f^*, f_* m'_*)$ and $(m_* g_*, g^* m^*)$ adjunctions—see [Lip09, (3.6.2)]. □

2.2. An ideal setup

We recall the formal framework of [FHM03, Rmk 5.10].

Situation 2.5. *We are given closed symmetric monoidal categories $\mathscr{C}, \mathscr{D}$ and functors $f_*, f_! : \mathscr{C} \to \mathscr{D}$ and $f^*, f^! : \mathscr{D} \to \mathscr{C}$ such that (f^*, f_*) and $(f_!, f^!)$ are adjoint pairs. Moreover, these functors satisfy*

- *f^* is strong symmetric monoidal,*
- *$f_* = f_!$,*
- *The canonical projection formula morphism*

$$\pi : Y \otimes f_*(X) \to f_*(f^*Y \otimes X) \tag{2.10}$$

defined in [Hal, Appendix A] is an isomorphism,

- *The object $C := f^!\mathscr{O}_{\mathscr{D}}$ is invertible, and*
- *The canonical morphism*

$$\varphi : f^*Y \otimes f^!\mathscr{O}_{\mathscr{D}} \to f^!Y \tag{2.11}$$

defined in [FHM03, (5.5)] is an isomorphism.

Example 2.6. Let $p : \mathcal{C} \to T$ be a family of prestable curves on a quasi-separated Noetherian scheme T, in the sense of [Stacks, Tag 0E6T]. Let $\mathscr{C} = \mathrm{D}_{\mathrm{qc}}(\mathcal{C}_{\mathrm{et}})$, $\mathscr{D} = \mathrm{D}_{\mathrm{qc}}(T_{\mathrm{et}})$, $f_* = \mathsf{R}p_*$, and $f^* = \mathsf{L}p^*$. Then f_* has a right adjoint $f^!$ and $f^!\mathscr{O}_T$ is equal to the relative dualising sheaf $\omega_{\mathcal{C}/T}[1]$. These data are an example of Situation 2.5. We will extend this example to families $\mathcal{C} \to T$ of twisted curves in Example 3.11.

Lemma 2.7. *For every $X \in \mathscr{C}$, the $(\otimes, \mathrm{Hom})$-unit*

$$\eta_X : X \to \mathrm{Hom}(C, X \otimes C) \tag{2.12}$$

is an isomorphism.

Proof. Since C is invertible, it follows from [May01, Lem 2.9] that C is dualisable and that the coevaluation map defined there is an isomorphism. It follows from the definition of the coevaluation map that the unit in equation (2.12) is an isomorphism when $X = \mathscr{O}_{\mathscr{C}}$. For general X, there is a commuting square

$$\begin{tikzcd}
\mathscr{O}_{\mathscr{C}} \otimes X \arrow[r, "\eta_{\mathscr{O}_{\mathscr{C}}} \otimes 1_X"] \arrow[d, equal] & \mathrm{Hom}(C, \mathscr{O}_{\mathscr{C}} \otimes C) \otimes X \arrow[r, equal] & \mathrm{Hom}(C, C) \otimes X \arrow[d, "\nu", "\sim"'] \\
X \arrow[r, "\eta_X"] & \mathrm{Hom}(C, X \otimes C) \arrow[r, equal] & \mathrm{Hom}(C, C \otimes X)
\end{tikzcd}$$

where the map labelled ν (defined in [Lew86, p. 120]) is an isomorphism since C is dualisable (see [Lew86, Prop III.1.3(ii)]). This implies that η_X is an isomorphism. The commutativity of the square follows immediately from the definition of ν and the functoriality of η. $\square$

Following [FHM03, Def 5.6], we define twisted functors $f^!_C(X) := \mathrm{Hom}(C, f^!(X))$ and $f^C_!(X) := f_!(X \otimes C)$. We have an isomorphism $f^*Y \to f^!_C(Y)$ for $Y \in \mathscr{C}$ equal to the composition

$$f^*(Y) \xrightarrow{(2.12)} \mathrm{Hom}(C, f^*Y \otimes C) \xrightarrow{\varphi} \mathrm{Hom}(C, f^!Y), \tag{2.13}$$

where equation (2.12) is an isomorphism by Lemma 2.7 and φ is an isomorphism by assumption.

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Since $(f_!^C, f_C^!)$ is an adjoint pair, we've realised $f_!^C$ as a left adjoint to pullback. Moreover, there is a projection isomorphism

$$\pi_C : Y \otimes f_!^C(X) \xrightarrow{\sim} f_!^C(f^*(Y) \otimes X)$$

defined by replacing X with $X \otimes C$ in π.

In this setting, we prove commutativity of some diagrams that will be useful to us.

Lemma 2.8. *There is a commuting diagram*

$$\begin{tikzcd}
X \otimes f_!^C f^*(Y) \arrow[r, "(2.13)", "\sim"'] \arrow[d, "\pi_C", "\sim"'] & X \otimes f_!^C f_C^!(Y) \arrow[r, "\epsilon"] & X \otimes Y \\
f_!^C(f^*(X) \otimes f^*(Y)) & f_!^C f^*(X \otimes Y) \arrow[l, "\sim"] \arrow[r, "(2.13)", "\sim"'] & f_!^C f_C^!(X \otimes Y) \arrow[u, "\epsilon"']
\end{tikzcd} \tag{2.14}$$

where the arrows labelled ϵ are counits for the $(f_!^C, f_C^!)$ adjunction.

Proof. For any $Z \in \mathscr{D}$, the composition

$$f_!^C f^*(Z) \xrightarrow{(2.13)} f_!^C f_C^!(Z) \xrightarrow{\epsilon^{f_!^C}} Z$$

is equal to

$$f_!^C f^*(Z) = f_*(f^*(Z) \otimes C) \underset{\sim}{\xleftarrow{\pi}} Z \otimes f_*(C) \xrightarrow{\epsilon^{f_!}_{\mathscr{O}_{\mathscr{D}}}} Z.$$

To see this, expand the $(f_!^C, f_!^C)$-counit in terms of the $(\otimes, \mathrm{Hom})$-counit and the $(f_!, f^!)$-counit; commute the morphism φ in the definition of equation (2.13) with the $(\otimes, \mathrm{Hom})$-counit; and finally, use the triangle identity $1_{f^*Z\otimes C} = \epsilon^{\otimes}_{f^*Z\otimes C} \circ (\eta^{\otimes}_{f^*Z} \otimes 1_C)$, where η and ϵ here denote the unit and counit of the $(\otimes, \mathrm{Hom})$ adjunction. Now equation (2.14) is equivalent to the diagram

$$\begin{tikzcd}
X \otimes f_*(f^*(Y) \otimes C) \arrow[d, "\pi"] & X \otimes Y \otimes f_*(C) \arrow[l, "\pi"'] \arrow[r, "\epsilon"] \arrow[d, "\pi"] & X \otimes Y \\
f_*(f^*(X) \otimes f^*(Y) \otimes C) \arrow[r, equal] & f_*(f^*(X \otimes Y) \otimes C) &
\end{tikzcd}$$

whose commutativity is proved in [Lip09, Lem 3.4.7(iv)]. □

Lemma 2.9. *Suppose we are in Situation 2.5. Then there is an isomorphism $\gamma : f_* \mathrm{Hom}(f^*X, f_C^!(Y)) \to \mathrm{Hom}(f_!^C(f^*X), Y)$ making the following diagram commute:*

$$\begin{tikzcd}
\mathrm{Hom}(f_!^C f_C^!(X), Y) & \mathrm{Hom}(X, Y) \arrow[l, "\mathrm{Hom}(\epsilon, 1_Y)"'] \arrow[d, "(2.5)"] \\
& f_* \mathrm{Hom}(f^*X, f^*Y) \arrow[ul, "\gamma"]
\end{tikzcd} \tag{2.15}$$

Here, ϵ is the counit for the $(f_!^C, f_C^!)$-adjunction.

Proof. The definition of γ will come out in the course of the proof: it will be 'conjugate' to π via various adjoints (see also [FHM03, (4.1)] and [Stacks, Tag 0A9Q]). For future reference, we summarise it in the final paragraph of the proof. To simplify notation, when there is no risk of confusion, if F is a functor

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between categories and α is a morphism of the source category, we will notate $F(\alpha)$ by α. For example, we may use ϵ as the label for the horizontal arrow in diagram (2.15).

To show the commutativity of diagram (2.15), we use the Yoneda embedding: for an arbitrary $T \in \mathscr{D}$, it suffices to show the commutativity of

$$\begin{tikzcd}
{}^{3}\mathscr{D}(T, \mathrm{Hom}(f_!^C f_C^!(X), Y) & {}^{1}\mathscr{D}(T, \mathrm{Hom}(X, Y)) \arrow[l, "\epsilon^{f_!^C}"'] \arrow[d, "(2.5)"] \\
& {}^{2}\mathscr{D}(T, f_* \mathrm{Hom}(f^*X, f^*Y)) \arrow[ul, "\gamma"]
\end{tikzcd}$$

We do this by demonstrating that it is equivalent to the commutativity of

$$\begin{tikzcd}
{}^{6}\mathscr{D}(T \otimes f_!^C f_C^!(X), Y) & {}^{4}\mathscr{D}(T \otimes X, Y) \arrow[l, "\epsilon^{f_!^C}"'] \arrow[d, "\epsilon^{f_!^C}"] \\
& {}^{5}\mathscr{D}(f_!^C f_C^!(T \otimes X), Y) \arrow[ul, "\tilde{\gamma}"]
\end{tikzcd}$$

where $\tilde{\gamma}$ is defined to equal the isomorphisms in diagram (2.14). This second diagram commutes by Lemma 2.8.

There are isomorphisms $1 = 4$ and $3 = 6$ given by $(\otimes, \mathrm{Hom})$ adjunction and an isomorphism $2 = 5$ given by

$$\begin{aligned}
{}^{2}\mathscr{D}(T, f_* \mathrm{Hom}(f^*X, f^*Y)) &= \mathscr{C}(f^*T, \mathrm{Hom}(f^*X, f^*Y)) = \mathscr{C}(f^*T \otimes f^*X, f^*Y) \\
&= {}^{7}\mathscr{C}(f^*(T \otimes X), f^*Y) = \mathscr{C}(f_C^!(T \otimes X), f_C^!Y) = {}^{5}\mathscr{D}(f_!^C(f_C^!(T \otimes X)), Y),
\end{aligned} \tag{2.16}$$

where the equalities are (f^*, f_*)-adjunction, $(\otimes, \mathrm{Hom})$-adjunction, the isomorphism in equation (2.1), the isomorphism in equation (2.13) and $(f_!^C, f_C^!)$-adjunction. The square with corners $1, 3, 4$, and 6 commutes by functoriality of the $(\otimes, \mathrm{Hom})$ adjunction. We take the commutativity of the square with corners $2, 3, 5$, and 6 as the definition of γ. The final square commutes as follows. By the definition of the vertical map in diagram (2.15) and adjunction, the composition $1 \to 2 \to 7$ is equal to

$$\begin{aligned}
\mathscr{D}(T, \mathrm{Hom}(X, Y)) &\xrightarrow{f^*} \mathscr{C}(f^*T, f^* \mathrm{Hom}(X, Y)) \xrightarrow{\otimes f^*X} \mathscr{C}(f^*T \otimes f^*X, f^* \mathrm{Hom}(X, Y) \otimes f^*X) \\
&\overset{(2.1)}{=\!=} \mathscr{C}(f^*(T \otimes X), f^*(\mathrm{Hom}(X, Y) \otimes X)) \xrightarrow{f^*\epsilon^{\otimes}} \mathscr{C}(f^*(T \otimes X), f^*Y).
\end{aligned}$$

By functoriality of equation (2.1), this composition is equivalent to

$${}^{1}\mathscr{D}(T, \mathrm{Hom}(X, Y)) \xrightarrow{\otimes X} \mathscr{D}(T \otimes X, \mathrm{Hom}(X, Y) \otimes X) \xrightarrow{\epsilon^{\otimes}} {}^{4}\mathscr{D}(T \otimes X, Y) \xrightarrow{f^*} {}^{7}\mathscr{C}(f^*(T \otimes X), f^*Y).$$

The first two arrows are precisely the $(\otimes, \mathrm{Hom})$ adjunction $1 = 4$. Finally, the composition

$${}^{4}\mathscr{D}(T \otimes X, Y) \xrightarrow{f^*} {}^{7}\mathscr{C}(f^*(T \otimes X), f^*Y) \overset{(2.13)}{=\!=} \mathscr{C}(f_C^!(T \otimes X), f_C^!Y) = {}^{5}\mathscr{D}(f_!(f_C^!(T \otimes X)), Y),$$

where the arrow comes from the previous formula and the two equalities come from equation (2.16), is induced by the counit $\epsilon^{f_!^C}$ as desired.

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To conclude, we summarise the definition of γ as promised. After cancelling assorted isomorphisms with their inverses, we see that it is given under the Yoneda embedding by

$$\begin{aligned}\mathscr{D}(T, f_* \operatorname{Hom}(f^*X, f^*Y)) &= \mathscr{C}(f^*T, \operatorname{Hom}(f^*X, f^*Y)) = \mathscr{C}(f^*T \otimes f^*X, f^*Y)\\ &= \mathscr{C}(f^*T \otimes f^*X, f^!_CY) = \mathscr{D}(f^C_!(f^*T \otimes f^*X), Y) \xrightarrow{\sim} \mathscr{D}(T \otimes f^C_! f^*X, Y)\\ &= \mathscr{D}(T, \operatorname{Hom}(f^C_! f^*X, Y)) = \mathscr{D}(T, \operatorname{Hom}(f^C_! f^!_C X, Y)),\end{aligned}$$

where the equalities are (f^*, f_*) adjunction, $(\otimes, \operatorname{Hom})$ adjunction, the isomorphism in equation (2.13), $(f^C_!, f^!_C)$ adjunction, the projection formula, $(\otimes, \operatorname{Hom})$ adjunction and finally the isomorphism in equation (2.13) again. In particular, if we follow γ with the inverse of the last equality, we have defined a natural isomorphism

$$f_* \operatorname{Hom}(Z, f^*Y) \xrightarrow{\sim} \operatorname{Hom}(f^C_! Z, Y) \tag{2.17}$$

that is functorial in both arguments. (The content of this statement is that to define equation (2.17), it is not necessary for Z to be of the form f^*X.) □

2.3. A modification of the ideal situation

When $\mathcal{C} \to \mathcal{X}$ is a family of twisted curves on an algebraic stack $\mathcal{X}$, we would like to apply Situation 2.5 by setting $\mathscr{D} = \mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}})$ and $\mathscr{C} = \mathrm{D}_{\mathrm{qc}}(\mathcal{C}_{\text{lis-et}})$. Unfortunately, we do not know a proof that the right adjoint $f^!$ in Situation 2.5 exists in the generality we would like (see Lemma 3.6 and the discussion following it). Instead, we work in the following weaker situation, replacing $f^!$ with a dualising complex and trace map.

Situation 2.10. *We are given closed symmetric monoidal categories $\mathscr{C}$, $\mathscr{D}$, a functor $f_* : \mathscr{C} \to \mathscr{D}$ with a right adjoint f^*, and an invertible object $C \in \mathscr{C}$ with a trace map $tr : f_*C \to \mathscr{O}_{\mathscr{D}}$. These data satisfy*

- *f^* is strong symmetric monoidal.*
- *The canonical morphism $\pi : Y \otimes f_*(X) \to f_*(f^*Y \otimes X)$ is an isomorphism.*

In this situation, we define an adjunction-like map $a : \mathscr{C}(X, f^*Y) \to \mathscr{D}(f_*(X \otimes C), Y)$ as follows (see also [CJW21, Sec A.2.1]).[2] Given $\phi' \in \mathscr{C}(X, f^*Y)$, define $a(\phi')$ to be the composition

$$f_*(X \otimes 1_C) \xrightarrow{f_*(\phi' \otimes C)} f_*(f^*Y \otimes C) \underset{\sim}{\xleftarrow{\pi}} Y \otimes f_*C \xrightarrow{id \otimes tr_C} Y. \tag{2.18}$$

Observe that a is functorial in both arguments, by which we mean the following:

1. Given $X' \in \mathscr{C}$ and $\psi \in \mathscr{C}(X', X)$, we have $a(\phi' \circ \psi) = a(\phi') \circ f_*(\psi \otimes 1_C)$.
2. Given $Y' \in \mathscr{D}$ and $\psi \in \mathscr{D}(Y, Y')$, we have $a(f^*\psi \circ \phi') = \psi \circ a(\phi')$.

The next example explains why we call a 'adjunction-like'.

Example 2.11. Suppose we are in Situation 2.5, with an adjoint pair (f^*, f_*) and object $C = f^!\mathscr{O}_{\mathscr{D}} \in \mathscr{C}$. Then we have the data of Situation 2.10: we can define $tr_C : f_*C \to \mathscr{O}_{\mathscr{D}}$ to be the counit of the $(f_!, f^!)$ adjunction. Under the isomorphism in equation (2.13), the adjunction-like map $a : \mathscr{C}(X, f^*Y) \to \mathscr{D}(f_*(X \otimes C), Y)$ is identified with the adjunction $\mathscr{C}(X, f^!_C Y) \simeq \mathscr{D}(f^C_! X, Y)$. To see this, let $\phi' \in \mathscr{C}(X, f^*Y)$. The $(f^C_!, f^!_C)$-adjoint of (2.13) $\circ\phi'$ is equal to the $(f_!, f^!)$-adjoint of

$$X \otimes C \xrightarrow{\phi' \otimes 1_C} f^*Y \otimes C \xrightarrow{\varphi} f^!Y.$$

[2]Informally, we think of a as realising $f_!^C = f_*(\cdot \otimes C)$ as a left adjoint to $f^* = f^!_C$.

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Since $f_* = f_!$, said adjoint is equal to the composition

$$f_*(X \otimes C) \xrightarrow{f_*(\phi' \otimes 1_C)} f_*(f^*Y \otimes C) \xrightarrow{\hat{\varphi}} Y,$$

where $\hat{\varphi}$ is the $(f_!, f^!)$-adjoint of φ. By the definition of φ in [FHM03, (5.5)], this last composition is equal to $a(\phi')$.

2.4. Basechange

We introduce a setting where the adjunction-like morphism a is compatible with pullback.

Situation 2.12. *We have a diagram of closed symmetric monoidal categories as in diagram (2.6) such that the left adjoints f^*, g^*, m^*, m'^* are strong symmetric monoidal. We are given objects $S \in \mathcal{S}, C \in \mathcal{C}$ and morphisms $tr_S : g_*S \to \mathcal{O}_{\mathcal{T}}, tr_C : f_*C \to \mathcal{O}_{\mathcal{D}}$ such that the data for each column of diagram (2.6) are in Situation 2.10, and these data are compatible as follows:*

- *The basechange map in equation (2.7) is an isomorphism.*
- *We are given an isomorphism $\alpha : m'^*C \to S$, making this diagram commute:*

$$\begin{tikzcd} m^* f_* C \arrow[r, "m^* tr_C"] \arrow[d, "(2.7)"] & \mathcal{O}_{\mathcal{T}} \\ g_* m'^* C \arrow[r, "g_* \alpha"] & g_* S \arrow[u, "tr_S"'] \end{tikzcd} \tag{2.19}$$

In this situation, Lemma 2.13 explains a precise sense in which $m^*a(\phi') = a(m'^*\phi')$.

Lemma 2.13. *Suppose we are in Situation 2.12. Let $\phi' : X \to f^*(Y)$ be an arrow in $\mathcal{C}$. Then we have $m'^*\phi' : m'^*X \to m'^* f^*Y = g^*m^*Y$, and the following diagram commutes:*

$$\begin{tikzcd} m^* f_*(X \otimes C) \arrow[r, "m^* a(\phi')"] \arrow[d, "\mathfrak{a}_{m^*}"', "\sim"] & m^* Y \\ g_*(m'^* X \otimes S) \arrow[ur, "a(m'^*\phi')"'] & \end{tikzcd}$$

The isomorphism $\mathfrak{a}$ is equal to equation (2.7) followed by equation (2.1) and finally α, and in particular it is functorial in X. Moreover, suppose we have a diagram

$$\begin{tikzcd} \mathcal{C}_1 \arrow[d, "f_{1*}"] & \mathcal{C}_2 \arrow[l, "m'^*_{12}"] \arrow[d, "f_{2*}"] & \mathcal{C}_3 \arrow[l, "m'^*_{23}"] \arrow[d, "f_{3*}"] \\ \mathcal{D}_1 & \mathcal{D}_2 \arrow[l, "m^*_{12}"] & \mathcal{D}_3 \arrow[l, "m^*_{23}"] \end{tikzcd} \tag{2.20}$$

together with distinguished objects $C_i \in \mathcal{C}_i$ and trace maps $tr_i : f_{i*}C_i \to \mathcal{O}_{\mathcal{D}_i}$ for each i, and isomorphisms $\alpha_{ij} : m'^*_{ij}C_j \to C_i$ for $i < j$. Suppose that with these data, both squares and the outer rectangle of diagram (2.20) are in Situation 2.12 and that $\alpha_{13} = \alpha_{12} \circ m'^*_{12}(\alpha_{23})$. Then $\mathfrak{a}_{m^*_{12} \circ m^*_{23}} = \mathfrak{a}_{m^*_{12}} \circ m'^*_{12}(\mathfrak{a}_{m^*_{23}})$.

Proof. The proof of [CJW21, Lem A.2.1] works in this more general situation. □

Lemma 2.14. *Suppose we are in Situation 2.12, except the left column of diagram (2.6) is actually in Situation 2.5: this means we are given a right adjoint $g^!$ for $g_* =: g_!$ with $g^!\mathcal{O}_{\mathcal{T}} = S$ and $tr_S : g_*S \to \mathcal{O}_{\mathcal{T}}$*

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equal to the counit for the $(g_!, g^!)$ *adjunction. Let* $\phi' : X \to f^*(Y)$ *be an arrow in* $\mathscr{C}$*, and set* $\phi := a(\phi')$*. Let* I *be an object of* $\mathscr{T}$*. Then the following diagram commutes:*

$$\begin{tikzcd}
\operatorname{Hom}(m^* f_*(X \otimes C), I) \arrow[d, equal] & \operatorname{Hom}(m^* Y, I) \arrow[l, "m^*\phi"'] \arrow[d] \\
g_* \operatorname{Hom}(m'^* X, g^* I) & g_* \operatorname{Hom}(m'^* f^* Y, g^* I) \arrow[l, "m'^*\phi'"]
\end{tikzcd} \tag{2.21}$$

In this diagram, the equality is comprised of equation (2.7), α *and equation (2.17). The right vertical arrow is equation (2.5) followed by equation (2.7).*

Proof. The desired commuting diagram is derived from the composition of two. On the left, we have

$$\begin{tikzcd}
\operatorname{Hom}(g_!^S(m'^* X), I) & \operatorname{Hom}(g_!^S(m'^* f^* Y), I) \arrow[l] \arrow[r, equal, "(2.7)"] & \operatorname{Hom}(g_!^S g^*(m^* Y), I) \\
g_* \operatorname{Hom}(m'^* X, g^* I) \arrow[u, "(2.17)"] & g_* \operatorname{Hom}(m'^* f^* Y, g^* I) \arrow[l] \arrow[u, "(2.17)"] \arrow[r, equal, "(2.7)"] & g_* \operatorname{Hom}(g^* m^* Y, g^* I) \arrow[u, "(2.17)"]
\end{tikzcd} \tag{2.22}$$

Here, the arrows pointing left are induced by $m'^*\phi'$. On the right, we have

$$\begin{tikzcd}
\operatorname{Hom}(g_!^S g^*(m^* Y), I) \arrow[r, "\pi_S", "\sim"'] \arrow[d, equal, "(2.13)"] & \operatorname{Hom}(m^* Y \otimes g_!^S(\mathscr{O}_{\mathscr{T}}), I) & \operatorname{Hom}(m^* Y, I) \arrow[l, "tr_S"'] \arrow[dll, "\epsilon"'] \arrow[ddll, "(2.5)"] \\
\operatorname{Hom}(g_!^S g_S^!(m^* Y), I) & & \\
g_* \operatorname{Hom}(g^* m^* Y, g^* I) \arrow[u, "\gamma"] & &
\end{tikzcd} \tag{2.23}$$

The commutativity of the top triangle is Lemma 2.8 with $X = m^*Y$ and $Y = \mathscr{O}_{\mathscr{T}}$, and the commutativity of the bottom triangle is Lemma 2.9. We note that the definition in equation (2.17) is equal to γ followed by the equality induced by equation (2.13). Finally, by Lemma 2.13, the top row of diagram (2.22) followed by the top row of diagram (2.23) agrees with the top row of diagram (2.21) (after inserting a copy of equation (2.7)). $\square$

3. Duality for twisted curves

We explain how the formal discussion of Section 2 will generally be used in the remainder of this article. In this section and the remainder of the paper, we define pseudo-coherent and perfect objects of lisse-étale sites as in [Stacks, Tag 08FT] and [Stacks, Tag 08G5], respectively. Note that if X is an algebraic stack, pseudo-coherent and perfect objects of $\mathrm{D}(\mathcal{X}_{\text{lis-et}})$ are always in $\mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}})$.

Example 3.1. If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks, we have closed symmetric monoidal categories $\mathscr{C} = \mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}})$ and $\mathscr{D} = \mathrm{D}_{\mathrm{qc}}(\mathcal{Y}_{\text{lis-et}})$ and a strong monoidal functor $\mathsf{L}f^* : \mathscr{D} \to \mathscr{C}$. If f is concentrated, we also have $\mathsf{R}f_* : \mathscr{C} \to \mathscr{D}$ that is a right adjoint to $\mathsf{L}f^*$. In this context, the functor notated Hom in Section 2 translates to internal hom for $\mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}})$. However, we note that by [HR17, Lem 4.3(2)], for any algebraic stack $\mathcal{X}$, we have equality

$$\mathsf{R}\mathcal{H}om^{\mathrm{qc}}_{\mathscr{O}_{\mathcal{X}}}(\mathcal{P}, \mathcal{F}) \simeq \mathsf{R}\mathcal{H}om_{\mathscr{O}_{\mathcal{X}}}(\mathcal{P}, \mathcal{F})$$

for any $\mathcal{F} \in \mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}})$ and any perfect complex $\mathcal{P} \in \mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}})$.

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3.1. Background on twisted curves

Recall from Section 1.6 that a morphism $p : \mathcal{C} \to \mathcal{M}$ of algebraic stacks is a *family of twisted curves* if smooth-locally on $\mathcal{M}$ it is a twisted curve in the sense of [AOV11, Def 2.1]: in particular, p is flat and proper and the diagonal $\mathcal{C} \to \mathcal{C} \times_{\mathcal{M}} \mathcal{C}$ is quasi-finite. If $r : \mathcal{C} \to C$ is the coarse moduli map and $\bar{c} \to C$ is a geometric point, the fibre product $\mathrm{Spec}(\mathcal{O}_{C,\bar{c}}) \times_C \mathcal{C}$ is moreover required to have a certain description (see the full definition in [AOV11, Def 2.1]). We recall some properties of families of twisted curves.

Our first lemma 'spreads out' the local quotient description of a twisted curve at a geometric point to an étale neighbourhood of that point.

Lemma 3.2. *Let $\mathcal{C} \to T$ be a family of twisted curves over an affine scheme T, and let $q : C \to T$ be the coarse moduli space. Let $\bar{c} \to C$ be a geometric point. Then there is an integer $n \geq 1$ and an affine scheme $V = \mathrm{Spec}(A)$ with an action of μ_n such that if $R = A^{\mu_n}$ is the ring of invariants and $U := \mathrm{Spec}(R)$, there is a commuting diagram*

$$\begin{tikzcd} V \arrow[r, "\sigma"] & {[V/\mu_n]} \arrow[r, "\tau"] \arrow[d] & U \arrow[d] \\ & \mathcal{C} \arrow[r, "r"] & C \end{tikzcd} \tag{3.1}$$

where the square is fibred and the vertical maps are étale. Moreover, one of the following holds:

1. $A = R[x]/(x^n - t)$ *for some $t \in R$, and μ_n acts by $\zeta \cdot p(x) = p(\zeta x)$.*
2. $A = R[x, y]/(xy - t, x^r - u, y^r - v)$ *for some $t, u, v \in R$, and μ_n acts by $\zeta \cdot p(x, y) = p(\zeta x, \zeta^{-1} y)$.*

Proof. We prove the lemma when $\bar{c}$ maps to a node of C; the case when $\bar{c}$ maps to a smooth point is similar. By definition [AOV11, Def 2.1(v)], there is a fibre square

$$\begin{tikzcd} {[(\mathrm{Spec}(\mathcal{O}_{T,q(\bar{c})}[x,y]/(xy-t))/\mu_n]} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_{C,\bar{c}}) \arrow[r] & C \end{tikzcd} \tag{3.2}$$

for some $t \in \mathcal{O}_{T,q(\bar{c})}$, where $\zeta \in \mu_n$ acts by $x \mapsto \zeta \cdot x$ and $y \mapsto \zeta^{-1} \cdot y$. Since $\mathcal{C}$ is tame, formation of the coarse space commutes with arbitrary basechange [AOV08, Cor 3.3], and we have

$$\mathcal{O}_{C,\bar{c}} \simeq (\mathcal{O}_{T,q(\bar{c})}[x,y]/(xy-t))^{\mu_n} \simeq \mathcal{O}_{T,q(\bar{c})}[x^n, y^n]/(x^n y^n - t^n).$$

If we set $u := x^n$ and $v := y^n$ in $\mathcal{O}_{C,\bar{c}}$, then we may write the top-left corner of diagram (3.2) as the stack

$$[(\mathrm{Spec}(\mathcal{O}_{C,\bar{c}}[x,y]/(x^n - u, y^n - v, xy - t))/\mu_n]. \tag{3.3}$$

Now write $\mathcal{O}_{C,\bar{c}}$ as the inverse limit of affine schemes $\mathrm{Spec}(R_i)$ with étale maps to C. Since $\mathrm{Spec}(\mathcal{O}_{C,\bar{c}}[x,y]/(x^n - u, y^n - v, xy - t)) \to \mathrm{Spec}(\mathcal{O}_{C,\bar{c}})$ is finitely presented, there is an index i_0 and elements $u, v, t \in R_{i_0}$ such that $\mathrm{Spec}(R_{i_0}[x,y]/(x^n - u, y^n - v, xy - t))$ pulls back to the affine scheme in equation (3.3). Define μ_n to act on $\mathrm{Spec}(R_{i_0}[x,y]/(x^n - u, y^n - v, xy - t))$ by the same rule $x \mapsto \zeta \cdot x$ and $y \mapsto \zeta^{-1} \cdot y$.

Let $\mathcal{C}_{R_i}$ denote the pullback of $\mathcal{C}$ to $\mathrm{Spec}(R_i)$. Observe that for $i \geq i_0$, we have two stacks $[(\mathrm{Spec}(R_i[x,y]/(x^n - u, y^n - v, xy - t))/\mu_n]$ and $\mathcal{C}_{R_i}$ defined over $\mathrm{Spec}(R_i)$ and an isomorphism between their pullbacks to $\mathrm{Spec}(\mathcal{O}_{C,\bar{c}})$. By [LM00, Prop 4.18(i)], there is an index $j \geq i_0$ and an isomorphism $[(\mathrm{Spec}(R_j[x,y]/(x^n - u, y^n - v, xy - t))/\mu_n] \simeq \mathcal{C}_{R_j}$. We may set $R := R_j$. □

We refer the reader to Section 1.6 for definitions of the direct and inverse image functors in the next lemma.

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Lemma 3.3. *Let $p : \mathcal{C} \to \mathcal{M}$ be a family of twisted curves on an algebraic stack $\mathcal{M}$.*

1. *The morphism p has cohomological dimension ≤ 1 (in the sense of [HR17, Def 2.1]).*
2. *The morphism p is concentrated (in the sense of [HR17, Def 2.4]).*
3. *For $\mathcal{F} \in \mathrm{D}_{\mathrm{qc}}(\mathcal{C}_{\text{lis-et}})$ and $\mathcal{G} \in \mathrm{D}_{\mathrm{qc}}(\mathcal{M}_{\text{lis-et}})$, the projection morphism $\mathcal{G} \otimes \mathsf{R}p_*(\mathcal{F}) \xrightarrow{(2.10)} \mathsf{R}p_*(p^*\mathcal{G} \otimes \mathcal{F})$ is an isomorphism.*
4. *Given a fibre square of algebraic stacks*

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "m'"] \arrow[d, "p'"] & \mathcal{C} \arrow[d, "p"] \\ \mathcal{M}' \arrow[r, "m"] & \mathcal{M} \end{tikzcd}$$

   and $\mathcal{F} \in \mathrm{D}_{\mathrm{qc}}(\mathcal{C}_{\text{lis-et}})$, the basechange map $\mathsf{L}m^*\mathsf{R}p_*\mathcal{F} \xrightarrow{(2.7)} \mathsf{R}p'_*\mathsf{L}m'^*\mathcal{F}$ is an isomorphism.
5. *If $\mathcal{M}$ is locally Noetherian, then the functor $\mathsf{R}p_*$ sends perfect complexes to perfect complexes.*

Proof. For part (1), by flat basechange [HR17, Lem 1.2(4)], we may assume $\mathcal{M}$ is an affine scheme, but this is [AOV11, Prop 2.6]. Now (1) implies part (2) by definition, part (3) by [HR17, Cor 4.12] and part (4) by [HR17, Cor 4.13]. For part (5), we recall that perfection is a flat-local property of complexes in the sense of [HR17, Lem 4.1], so we may use basechange [HR17, Cor 4.13] to reduce to the case when $\mathcal{M}$ is a Noetherian affine scheme. Now the result follows from Lemma 3.4 below. $\square$

Lemma 3.4. *Let $p : \mathcal{C} \to T$ be a family of twisted curves over a Noetherian affine scheme T, and let $r : \mathcal{C} \to C$ be the map to the coarse moduli space.*

1. *The exact functor r_* sends pseudo-coherent objects in $\mathrm{D}_{\mathrm{qc}}(\mathcal{C}_{\text{lis-et}})$ to pseudo-coherent objects in $\mathrm{D}_{\mathrm{qc}}(C_{\text{lis-et}})$.*
2. *The functor $\mathsf{R}p_*$ sends perfect objects in $\mathrm{D}_{\mathrm{qc}}(\mathcal{C}_{\text{lis-et}})$ to perfect objects in $\mathrm{D}_{\mathrm{qc}}(T_{\text{lis-et}})$.*

Remark 3.5. We expect that the locally Noetherian hypothesis can be removed using absolute Noetherian approximation for algebraic stacks as in [Stacks, Tag 0CN4] (see the proof of [Stacks, Tag 01AH]). We do not, however, know a reference that allows us to assume the approximating morphism has properties (1) and (2) of Lemma 3.4. Since we are not aware of an application of the non-Noetherian setting, we omit this investigation.

Proof of Lemma 3.4. We will repeatedly use the fact that if X is a scheme, there are equivalences of categories $\mathrm{QCoh}(X_{\text{lis-et}}) \simeq \mathrm{QCoh}(X_{\mathrm{zar}})$ and $\mathrm{D}_{\mathrm{qc}}(X_{\text{lis-et}}) \simeq \mathrm{D}_{\mathrm{qc}}(X_{\mathrm{zar}})$, where X_{zar} is the category of sheaves on the small Zariski site of X, and that these equivalences preserve coherence, pseudo-coherence and perfection.

To prove (1), let $\mathcal{F} \in \mathrm{D}_{\mathrm{qc}}(\mathcal{C}_{\text{lis-et}})$ be pseudo-coherent. Let $f : U \to \mathcal{C}$ be a smooth cover by a scheme. Since f defines a morphism of lisse-étale sites and f^* is exact, $f^*\mathcal{F}$ is pseudo-coherent by [Stacks, Tag 08H4]. By [Stacks, Tag 08E8], the sheaves $H^i(f^*\mathcal{F})$ are coherent and vanish for $i \gg 0$. It follows from [Ols07, Rmk 6.10, Prop 6.12] that the sheaves $H^i(\mathcal{F})$ are coherent and vanish for $i \gg 0$. By [Alp13, Thm 4.16(x)], the sheaves $r_*H^i(\mathcal{F})$ have these same properties, but since r_* is exact, we know $r_*H^i(\mathcal{F}) = H^i(r_*\mathcal{F})$. Hence by [Stacks, Tag 08E8] again, the object $r_*\mathcal{F}$ is pseudo-coherent.

To prove (2), let $\mathcal{F} \in \mathrm{D}_{\mathrm{qc}}(\mathcal{C}_{\text{lis-et}})$ be perfect—by [Stacks, Tag 08G8] this is equivalent to pseudo-coherent and locally of finite tor dimension. By part (1) of this lemma and [Stacks, Tag 0CTL], the pushforward $\mathsf{R}p_*\mathcal{F}$ is pseudo-coherent. To see that $\mathsf{R}p_*\mathcal{F}$ locally has finite tor dimension, by [Stacks, Tag 08EA], it suffices to show that for $\mathcal{G} \in \mathrm{QCoh}(T)$, the sheaves $H^i(\mathsf{R}p_*\mathcal{F} \overset{\mathsf{L}}{\otimes} \mathcal{G})$ vanish for i outside a finite range. By the projection formula [HR17, p. 4.12] and flatness of p, these are equal to

$H^i(\mathsf{R}p_*(\mathcal{F} \overset{\mathsf{L}}{\otimes} p^*\mathcal{G}))$. Since $\mathcal{F}$ is a perfect complex on a quasi-compact space, it has finite tor amplitude, so the spectral sequence

$$\mathsf{R}^m p_* H^n(\mathcal{F}) \implies \mathsf{R}^{m+n} p_* \mathcal{F}$$

of [Stacks, Tag 015J] and the fact that p is concentrated finish the proof. □

3.2. Background on right adjoint to pushforward

We recall some statements about right adjoint to pushforward that hold for purely formal reasons.

Lemma 3.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a concentrated morphism of algebraic stacks. Then a right adjoint $f^!$ to $\mathsf{R}f_* : \mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}}) \to \mathrm{D}_{\mathrm{qc}}(\mathcal{Y}_{\text{lis-et}})$ exists, and for dualisable $\mathcal{G} \in \mathrm{D}_{\mathrm{qc}}(\mathcal{Y}_{\text{lis-et}})$ the canonical morphism $f^*\mathcal{G} \otimes f^!\mathcal{O}_{\mathcal{Y}} \to f^!\mathcal{G}$ defined in equation (2.11) is an isomorphism. Moreover, for $\mathcal{F} \in \mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}})$ and $\mathcal{G} \in \mathrm{D}_{\mathrm{qc}}(\mathcal{Y}_{\text{lis-et}})$, there is a functorial isomorphism*

$$\mathsf{R}f_*\mathsf{R}\mathcal{H}om^{\mathrm{qc}}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, f^!\mathcal{G}) \to \mathsf{R}\mathcal{H}om^{\mathrm{qc}}_{\mathcal{O}_{\mathcal{Y}}}(\mathsf{R}f_*\mathcal{F}, \mathcal{G}). \tag{3.4}$$

Proof. Existence of $f^!$ is [HR17, Thm 4.14(1)], and that equation (2.11) is an isomorphism follows from [FHM03, Prop 5.4]. The isomorphism in equation (3.4) is [FHM03, Prop 4.3] (see also [Stacks, Tag 0A9Q]). □

We now explain what it means for $f^!$ to be compatible with basechange. While lemma 3.6 applies to arbitrary families of twisted curves $\mathcal{C} \to \mathcal{M}$, we will see that we need additional assumptions for the basechange property to hold.

Suppose we have a fibre square of algebraic stacks as below with m and f tor-independent (see [HR17, Sec 4.5]) and f concentrated.

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "m'"] \arrow[d, "g"] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r, "m"] & \mathcal{Y} \end{tikzcd} \tag{3.5}$$

Then $f^!$ and $g^!$ exist as recalled in Lemma 3.6. By [HR17, Cor 4.13], the basechange map in equation (2.7) is an isomorphism (we take the closed symmetric monoidal categories in equation (2.6) to be $\mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}})$, etc.). This lets us define the functorial basechange map $\mathsf{L}m'^* f^! \to g^!\mathsf{L}m^*$ to be the composition

$$\mathsf{L}m'^* f^! \to g^!\mathsf{R}g_*\mathsf{L}m'^* f^! \xrightarrow{(2.7)} g^!\mathsf{L}m^*\mathsf{R}f_* f^! \to g^!\mathsf{L}m^*. \tag{3.6}$$

We are interested in when equation (3.6) is an isomorphism.

Lemma 3.7. *Suppose we have the tor-independent fibre square (3.5) of quasi-compact algebraic stacks with quasi-finite and separated diagonals, and suppose $\mathcal{Y}'$ and $\mathcal{Y}$ are concentrated with quasi-affine diagonals. If f is concentrated and $\mathsf{R}f_*$ sends perfect complexes to perfect complexes, then equation (3.6) is an isomorphism.*

Remark 3.8. The hypotheses of the lemma are satisfied if all the stacks in diagram (3.5) are quasi-compact tame Deligne-Mumford with separated diagonals, with additional conditions on f as above.

Remark 3.9. The preprint [Nee17] proves that equation (3.6) is an isomorphism under very general conditions. Compared with [Nee17], our Lemma 3.7 imposes stricter conditions on the stacks $\mathcal{X}$, $\mathcal{X}'$, $\mathcal{Y}$, $\mathcal{Y}'$ and morphism f, but we allow m to be arbitrary, whereas [Nee17] requires m to be flat.

Remark 3.10. The proof of Lemma 3.7 relies on our ability to find a compact generator for the algebraic stack $\mathcal{X}$. By [HR17, Thm A], our assumptions that $\mathcal{X}$ is quasi-compact with quasi-finite and separated

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diagonal imply that $D_{qc}(\mathcal{X}_{\text{lis-et}})$ is compactly generated by a single perfect complex $\mathcal{P}$. This means for any $\mathcal{F} \in D_{qc}(\mathcal{X}_{\text{lis-et}})$, we have $\mathcal{F} = 0$ if and only if $\mathrm{Hom}_{D_{qc}(\mathcal{X}_{\text{lis-et}})}(\mathcal{P}[n], \mathcal{F}) = 0$ for every $n \in \mathbb{Z}$ (here, $\mathrm{Hom}_{D_{qc}(\mathcal{X}_{\text{lis-et}})}$ denotes the hom-set in the (additive) category $D_{qc}(\mathcal{X}_{\text{lis-et}})$). Since $D_{qc}(\mathcal{X}_{\text{lis-et}})$ is a full subcategory of $D(\mathcal{X}_{\text{lis-et}})$, we may compute the hom set in the larger category. But these hom sets are computed by the cohomology of the derived global hom functor. We conclude that for any morphism $f : \mathcal{F} \to \mathcal{G}$, we have that f is an isomorphism if and only if $\mathsf{R}\mathrm{Hom}_{\mathcal{O}_\mathcal{X}}(\mathcal{P}, f)$ is an isomorphism.

Proof of Lemma 3.7. We explain why the proof of [Lip09, Cor 4.4.3] also works in this setting.

The first step is to reduce to the case where m is quasi-affine. Indeed, by [Lip09, Prop 4.6.8], the morphism in equation (3.6) satisfies a cocycle condition for squares stacked horizontally. This implies that it is enough to prove the Lemma when $\mathcal{Y}'$ is an affine scheme and m is smooth or when $\mathcal{Y}'$ and $\mathcal{Y}$ are both affine (see [Lip09, pp. 182–4] for more details). By assumption, $\mathcal{Y}$ has quasi-affine diagonal, so in either case, the morphism m is quasi-affine.

Now we assume m is quasi-affine. Let $\mathcal{F} \in D_{qc}(\mathcal{Y}_{\text{lis-et}})$, and let $\mathcal{P}$ be a perfect, compact generator for $D_{qc}(\mathcal{X}_{\text{lis-et}})$ (see Remark 3.10). Since $\mathcal{Y}'$ and $\mathcal{Y}$ are concentrated by assumption, the morphisms m and m' are also concentrated [HR17, Lem 2.5], and we have functors $\mathsf{R}m_*$ and $\mathsf{R}m'_*$. To show that equation (3.6) is an isomorphism, we claim that it suffices to show the induced map

$$\mathsf{R}f_*\mathsf{R}m'_*\mathsf{R}\mathcal{H}om^{qc}_{\mathcal{O}_{\mathcal{X}'}}(\mathsf{L}m'^*\mathcal{P}, \mathsf{L}m'^*f^!\mathcal{F}) \to \mathsf{R}f_*\mathsf{R}m'_*\mathsf{R}\mathcal{H}om^{qc}_{\mathcal{O}_{\mathcal{X}'}}(\mathsf{L}m'^*\mathcal{P}, g^!\mathsf{L}m^*\mathcal{F}) \tag{3.7}$$

is an isomorphism. First, $\mathsf{L}m'^*\mathcal{P}$ is perfect,[3] so we may replace the functors $\mathsf{R}\mathcal{H}om^{qc}_{\mathcal{O}_{\mathcal{X}'}}$ with $\mathsf{R}\mathcal{H}om_{\mathcal{O}_{\mathcal{X}'}}$ (see Example 3.1). Next, if equation (3.7) is an isomorphism, we get an isomorphism of global derived homs by applying the global sections functor. But $\mathsf{L}m'^*\mathcal{P}$ is a perfect generator for $D_{qc}(\mathcal{X}'_{\text{lis-et}})$ by [HR17, Cor 2.8]—this is where we use that m (hence m') is quasi-affine. We conclude that equation (3.6) is an isomorphism (see Remark 3.10).

To show that equation (3.7) is an isomorphism, we cite the bottom two cells of the commuting diagram on [Lip09, p. 182] to reduce to proving a certain morphism

$$\mathsf{R}m_*\mathsf{R}g_*\mathsf{R}\mathcal{H}om^{qc}_{\mathcal{O}_{\mathcal{X}'}}(\mathsf{L}m'^*\mathcal{P}, \mathsf{L}m'^*f^!\mathcal{F}) \xrightarrow{\mathsf{R}m_*(4.4.1)^*_{pc}} \mathsf{R}m_*\mathsf{R}\mathcal{H}om^{qc}_{\mathcal{O}_{\mathcal{Y}'}}(\mathsf{R}g_*\mathsf{L}m'^*\mathcal{P}, \mathsf{L}m^*\mathcal{F})$$

is an isomorphism.[4] (In the cited diagram, the map notated $u^*\delta$ is, in our notation, equal to $\mathsf{L}m^*$ applied to the isomorphism in equation (3.4).) We will not bother to write the definition of $\mathsf{R}m_*(4.4.1)^*_{pc}$ because [Lip09, Lem 4.6.4] gives a commuting diagram

$$\begin{tikzcd}
\mathsf{R}f'_*\mathsf{R}\mathcal{H}om^{qc}_{\mathcal{O}_{\mathcal{X}'}}(\mathsf{L}m'^*\mathcal{P}, \mathsf{L}m'^*f^!cF) \arrow[r, "(4.4.1)^*_{pc}"] & \mathsf{R}\mathcal{H}om^{qc}_{\mathcal{O}_{\mathcal{Y}'}}(\mathsf{R}f'_*\mathsf{L}m'^*\mathcal{P}, \mathsf{L}m^*\mathcal{F}) \\
\mathsf{R}f'_*\mathsf{L}m'^*\mathsf{R}\mathcal{H}om^{qc}_{\mathcal{O}_\mathcal{X}}(\mathcal{P}, f^!\mathcal{F}) \arrow[u, "\rho"] & \mathsf{R}\mathcal{H}om^{qc}_{\mathcal{O}_{\mathcal{Y}'}}(\mathsf{L}m^*\mathsf{R}f_*\mathcal{P}, \mathsf{L}m^*\mathcal{F}) \arrow[u, "(2.7)"'] \\
\mathsf{L}m'^*\mathsf{R}f_*\mathsf{R}\mathcal{H}om^{qc}_{\mathcal{O}_\mathcal{X}}(\mathcal{P}, f^!\mathcal{F}) \arrow[u, "(2.7)"] \arrow[r, "\mathsf{L}m^*(3.4)"] & \mathsf{L}m^*\mathsf{R}\mathcal{H}om^{qc}_{\mathcal{O}_\mathcal{Y}}(\mathsf{R}f_*\mathcal{P}, \mathcal{F}) \arrow[u, "\rho"']
\end{tikzcd} \tag{3.8}$$

The arrows labelled (2.7) are isomorphisms by [HR17, Cor 4.13]. The arrows labelled ρ are defined in [FHM03, (3.3)]; and by [FHM03, Prop 3.2], all instances in this diagram are isomorphisms since $\mathcal{P}$ and $\mathsf{R}f_*\mathcal{P}$ are perfect complexes by assumption. We note that our definition of ρ agrees with the definition

[3]To see this, note that the equivalence in Proposition A.4 sends perfect objects to perfect objects—one reason is that this is an equivalence of symmetric monoidal categories, and the perfect objects are the dualisables [Stacks, Tag 0FPP]. Now apply [Stacks, Tag 08H6] to the morphism of strictly simplicial étale topoi.

[4]One may check that Lipman's discussion of the relevant commuting diagram here and in diagram (3.8) uses only formal properties of adjoint symmetric functors as discussed in [Lip09, Sec 3.5]. The setup in [Lip09, Sec 3.5] is compatible with our Situation 2.1 by footnote 1.

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in [Lip09, (3.5.4.5)] by [Lip09, Exercise 3.5.6(a)]. Finally, in diagram (3.8), we know that equation (3.4) is an isomorphism; this concludes the proof. □

3.3. Example of Situation 2.5

We realise Situation 2.5 as duality for families of twisted curves on Noetherian algebraic spaces.

Example 3.11. Let $p : \mathcal{C} \to T$ be a family of twisted curves on a quasi-separated Noetherian algebraic space T. Let $\mathscr{C} = \mathrm{D}_{\mathrm{qc}}(\mathcal{C}_{\mathrm{et}})$, $\mathscr{D} = \mathrm{D}_{\mathrm{qc}}(T_{\mathrm{et}})$, $f_* = \mathsf{R}p_*$, and $f^* = \mathsf{L}p^*$. We will write p^* for $\mathsf{L}p^*$ since p is flat (this is justified by [HR17, (1.9)]). The projection map in equation (2.10) is an isomorphism by Lemma 3.3.

The right adjoint $p^!$ exists by Lemma 3.6. Moreover, by [HR17, Thm A], the category $\mathrm{D}_{\mathrm{qc}}(\mathcal{C}_{\text{lis-et}})$ is compactly detected by a single perfect complex. Since $\mathsf{R}p_*$ preserves perfect complexes (by Lemma 3.3) and perfect objects in $\mathrm{D}_{\mathrm{qc}}(T_{\mathrm{et}})$ are also compact, it follows from [FHM03, Thm 8.4] and [FHM03, Lem 7.4] that equation (2.11) is an isomorphism for all Y.

It remains to show that $p^!\mathscr{O}_T$ is invertible. This follows from Lemma 3.7 and Lemma 3.12 below.

Lemma 3.12. *Let $p : \mathcal{C} \to T$ be a family of twisted curves on a Noetherian affine scheme T. Then $p^!\mathscr{O}_T$ is represented by a rank one locally free sheaf in degree -1.*

Proof. Let $q : C \to T$ be the coarse moduli space of $\mathcal{C}$, and let $r : \mathcal{C} \to C$ be the coarse moduli map. By [Stacks, Tag 0E6P, 0E6R], we know $q^!\mathscr{O}_T$ is invertible and supported in degree -1. In particular, it is dualisable, so we have

$$p^!\mathscr{O}_T = r^!q^!\mathscr{O}_T = r^*q^!\mathscr{O}_T \otimes r^!\mathscr{O}_C,$$

where the second equality uses [FHM03, Thm 8.4] and the fact that $q^!\mathscr{O}_T$ is dualisable. Hence to prove the lemma, it suffices to show that $r^!\mathscr{O}_C$ is invertible and supported in degree 0.

Let $\bar{c} \to C$ be a geometric point. By Lemma 3.2, we have a local description of $\mathcal{C} \to C$ near $\bar{c}$ given by the diagram (3.1). Note that a right adjoint to pushforward exists for every horizontal map in diagram (3.1). It follows from [Nee17, Lem 0.1] that the pullback of $r^!\mathscr{O}_C$ to $[V/\mu_n]$ is equal to $\tau^!\mathscr{O}_U$ (note that [Nee17, Lem 0.1] applies since $U \to C$ is étale and r_* preserves pseudo-coherent objects by Lemma 3.4). Since $[V/\mu_n] \to \mathcal{C}$ is flat, the complex $r^!\mathscr{O}_C$ is represented by a quasi-coherent sheaf if and only if $\tau^!\mathscr{O}_U$ is; and by [Stacks, Tag05B2] (applied on strictly simplicial étale sites as in Proposition A.4), $r^!\mathscr{O}_C$ is invertible if and only if $\tau^!\mathscr{O}_U$ is.

To compute $\tau^!\mathscr{O}_U$, set $\rho = \tau \circ \sigma$; we observe that we have an equality

$$\rho^!\mathscr{O}_U = \sigma^*\tau^!\mathscr{O}_U \otimes \sigma^!\mathscr{O}_{[V/\mu_n]},$$

so it suffices to show that $\rho^!\mathscr{O}_U$ and $\sigma^!\mathscr{O}_{[V/\mu_n]}$ are both invertible and supported in degree 0. In Lemma 3.13 below, we prove the statement about $\rho^!\mathscr{O}_U$ as well as the statement that $pr^!\mathscr{O}_V$ is a line bundle in degree 0, where $pr : \mu_n \times V \to V$ is the projection. The statement about $pr^!\mathscr{O}_V$ is equivalent to the statement about $\sigma^!\mathscr{O}_{[V/\mu_n]}$ by an argument identical to the one used in the previous paragraph. □

Lemma 3.13. *The complexes $\rho^!\mathscr{O}_U$ and $pr^!\mathscr{O}_V$ are represented by line bundles supported in degree 0.*

Proof. We use the statement of finite duality in [Stacks, Tag 0AX2] and translate it to a statement about rings using [Stacks, Tag 06Z0]. These results imply that for a morphism of affine schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, the image of $\mathscr{O}_{\mathrm{Spec}(A)}$ under the right adjoint to pushforward is induced by the complex of B-modules

$$\mathsf{R}\mathcal{H}om_A(B, A). \tag{3.9}$$

For pr, the relevant ring map is the diagonal $A \to \prod_{g\in G} A$, and $B = \prod_{g\in G} A$ is a free A-module so equation (3.9), is supported in degree 0. One checks that there is an isomorphism $B \to \mathsf{R}\mathcal{H}om_A(B, A)$ given by sending 1_B to the projection to the identity factor.

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For ρ, let $R = A^G$, the ring of G-invariants. Lemma 3.2 lists two possibilities for A. In case (1), the computation of equation (3.9) is similar to that for pr since in this case, A is a free R-module with basis $1, x, \dots, x^{r-1}$ and $\mathsf{R}\mathcal{H}om_R(A, R)$ is generated as an A-module by projection to the x^{r-1}-factor.

The computation in case (2) is more involved since we have to take a free resolution of A. One may use the resolution

$$\dots \xrightarrow{d_3} R^{\oplus 2r-2} \xrightarrow{d_2} R^{\oplus 2r-2} \xrightarrow{d_1} R^{\oplus 2r-1} \xrightarrow{d_0} A \to 0,$$

with maps given as follows. If $\{f_i, g_i\}_{i=1}^{r-1}$ denotes a free basis for $R^{\oplus 2r-2}$ and e is the additional basis element of $R^{\oplus 2r-1}$, then d_i is defined by

$$\begin{aligned} d_0 : e &\mapsto 1 \\ f_i &\mapsto x^i \\ g_i &\mapsto y^i \end{aligned} \qquad \begin{aligned} d_i,\ i \text{ odd} : f_i &\mapsto v f_i - t^i g_{r-i} \\ g_{r-i} &\mapsto u g_{r-i} - t^{r-i} f_i \end{aligned} \qquad \begin{aligned} d_i,\ i > 0 \text{ even: } f_i &\mapsto u f_i + t^i g_{r-i} \\ g_{r-i} &\mapsto t^{r-i} f_i + v g_{r-i} \end{aligned}$$

For details, see [Web20, pp. 25–28]. □

3.4. Example of Situation 2.12

We realise Situation 2.12 for families of twisted curves on algebraic stacks. We use the dualising sheaf and trace map (as in Situation 2.10) as a substitute for the full duality in Example 3.11 because we are unable to show that the basechange morphism in equation (3.6) is an isomorphism in general.

Proposition 3.14. *For every family $\mathcal{C} \to \mathcal{M}$ of twisted curves on a locally Noetherian algebraic stack $\mathcal{M}$, there is a pair $(\omega^\bullet_{\mathcal{M}}, tr_{\mathcal{M}})$ with $\omega^\bullet_{\mathcal{M}} = \omega_{\mathcal{M}}[1]$, where $\omega_{\mathcal{M}} \in \mathrm{QCoh}(\mathcal{C}_{\text{lis-et}})$ is locally free and $tr_{\mathcal{M}} : \mathsf{R}p_*\omega^\bullet_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}$, such that the following hold:*

1. *The pair is functorial in the following sense. Given a fibre square*

$$\begin{tikzcd} \mathcal{C}_{\mathcal{N}} \arrow[r, "m'"] \arrow[d] & \mathcal{C}_{\mathcal{M}} \arrow[d] \\ \mathcal{N} \arrow[r, "m"] & \mathcal{M} \end{tikzcd} \tag{3.10}$$

there is a canonical isomorphism

$$m'^*\omega^\bullet_{\mathcal{M}} \xrightarrow{\sim} \omega^\bullet_{\mathcal{N}} \tag{3.11}$$

such that the following square commutes:

$$\begin{tikzcd} \mathsf{L}m^*\mathsf{R}p_*\omega^\bullet_{\mathcal{M}} \arrow[r, "\mathsf{L}m^*tr_{\mathcal{M}}"] \arrow[d, "\sim"', "(2.7)"] & \mathcal{O}_{\mathcal{N}} \\ \mathsf{R}p_*\mathsf{L}m'^*\omega^\bullet_{\mathcal{M}} \arrow[r, "(3.11)", "\sim"'] & \mathsf{R}p_*\omega^\bullet_{\mathcal{N}} \arrow[u, "tr_{\mathcal{N}}"] \end{tikzcd} \tag{3.12}$$

Moreover, if $n : \mathcal{K} \to \mathcal{N}$ is a morphism of algebraic stacks and $\mathcal{C}_{\mathcal{K}} = \mathcal{C}_{\mathcal{N}} \times_{\mathcal{N}} \mathcal{K}$ is the pullback and $n' : \mathcal{C}_{\mathcal{K}} \to \mathcal{C}_{\mathcal{N}}$ the projection, then the isomorphism $(m' \circ n')^*\omega^\bullet_{\mathcal{M}} \to \omega_{\mathcal{N}}$ is equal to the composition $n'^*m'^*\omega^\bullet_{\mathcal{M}} \to n'^*\omega^\bullet_{\mathcal{N}} \to \omega^\bullet_{\mathcal{K}}$.

2. *If $\mathcal{M}$ is a quasi-separated Noetherian algebraic space, then $\omega^\bullet_{\mathcal{M}} = p^!\mathcal{O}_{\mathcal{M}}$ and $tr_{\mathcal{M}}$ is the counit of the $(\mathsf{R}p_*, p^!)$ adjunction.*

For a general base $\mathcal{M}$, we do not know if our construction of $(\omega^\bullet_{\mathcal{M}}, tr_{\mathcal{M}})$ agrees with the right adjoint to pushforward.

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Remark 3.15. To see that Proposition 3.14 gives an example of Situation 2.12 compatible with Example 3.11, we use the fact that $\mathsf{L}m^* : \mathrm{D}_{\mathrm{qc}}(\mathcal{M}_{\text{lis-et}}) \to \mathrm{D}_{\mathrm{qc}}(\mathcal{N}_{\text{lis-et}})$ has a right adjoint even when m is not concentrated; see [HR17, Sec 1.3].

Proof of Proposition 3.14. The idea is as follows. We will define the pair $(\omega^\bullet_{\mathcal{M}}, tr_{\mathcal{M}})$ when $\mathcal{M}$ is an algebraic space as required by part (2) of the proposition. When $\mathcal{M}$ is an algebraic stack, we will take this as the smooth-local definition of $(\omega^\bullet_{\mathcal{M}}, tr_{\mathcal{M}})$; and using the notion of a very smooth hypercover explained in Appendix A, we will show that these local pairs 'glue' to a global one with the correct properties.

We now proceed with the proof. When $\mathcal{M}$ is a quasi-separated Noetherian algebraic space, we define $\omega^\bullet_{\mathcal{M}}$ and $tr_{\mathcal{M}}$ as required in part (2) of the proposition (see Example 3.11). When both $\mathcal{N}$ and $\mathcal{M}$ are quasi-separated Noetherian algebraic spaces, we define equation (3.11) to be the basechange map in equation (3.6) (it is an isomorphism by Lemma 3.7). The commuting diagram (3.12) follows from the definition of equation (3.6); see [Lip09, Rmk 4.4(d)]. The cocycle condition on equation (3.11) is [Lip09, Prop 4.6.8].

Let $\mathcal{M}$ be a locally Noetherian algebraic stack. In this paragraph, we define $\omega_{\mathcal{M}}$. Let $M_\bullet \to \mathcal{M}$ be a very smooth hypercover (see Definition A.12), and let $\mathcal{C}_{M,\bullet}$ be its pullback to $\mathcal{C}_{\mathcal{M}}$ (see Remark A.13). We have associated categories of quasi-coherent sheaves $\mathrm{QCoh}(M_{\bullet,\text{lis-et}})$ and $\mathrm{QCoh}(\mathcal{C}_{M,\bullet,\text{lis-et}})$ as in Section A.3.2. By Remark A.14, we may assume that each M_i is a disjoint union of affine schemes (each Noetherian by [Stacks, Tag 06R6]). In particular, each M_i is a disjoint union of qcqs Noetherian schemes. For each $n \in \mathbb{Z}_{\geq 0}$, we have families of twisted curves $\mathcal{C}_{M,n} \to M_n$, and hence the system of locally free sheaves ω_{M_n} (defined by applying the construction in the previous paragraph to the Noetherian components of M_n) together with the isomorphisms in equation (3.11) defines an object $\omega_{M,\bullet}$ of $\mathrm{QCoh}(\mathcal{C}_{M,\bullet,\text{lis-et}})$. By Proposition A.18, the sheaf $\omega_{M,\bullet}$ corresponds to a unique quasi-coherent sheaf $\omega_{\mathcal{M}}$ in $\mathrm{QCoh}(\mathcal{C}_{\mathcal{M},\text{lis-et}})$ whose restriction to $\mathcal{C}_{M_i}$ is ω_{M_i}. Let $\omega^\bullet_{\mathcal{M}} = \omega_{\mathcal{M}}[1]$.

In this paragraph, we define $tr_{\mathcal{M}}$. By Remark A.20, the complex $\mathsf{R}p_*\omega^\bullet_{\mathcal{M}}$ is represented by the element of $\mathrm{D}_{\mathrm{qc}}(M_{\bullet,\text{lis-et}})$ whose n^{th} component is $\mathsf{R}p_*\omega^\bullet_{M_n}$ (see also [Stacks, Tag 0D9P]). We have trace maps $tr_{M_n} : \mathsf{R}p_*\omega^\bullet_{M_n} \to \mathcal{O}_{M_n}$ for each n, and these are compatible with the transition maps of $M_\bullet$ by diagram (3.12). Now, from Proposition A.18 combined with the argument in [Stacks, Tag 0DL9], we obtain $tr_{\mathcal{M}} : \mathsf{R}p_*\omega^\bullet_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}$ (the required Ext groups vanish since $\mathsf{R}p_*\omega^\bullet_{\mathcal{M}}$ is a complex in degrees [-1,0] by Lemma 3.3).

Now we check that the pair $(\omega_{\mathcal{M}}, tr_{\mathcal{M}})$ has the properties required in part (1) of the proposition. Suppose we have a fibre square (3.10) where $\mathcal{N}$ and $\mathcal{M}$ are algebraic stacks. Let $N_\bullet \to \mathcal{N}$ and $M_\bullet \to \mathcal{M}$ be very smooth hypercovers with M_i and N_i disjoint unions of affine schemes, with a morphism $N_\bullet \to M_\bullet$ commuting with the augmentations and $m : \mathcal{N} \to \mathcal{M}$ (see Remark A.15). Let $\mathcal{C}_{M,\bullet}$ and $\mathcal{C}_{N,\bullet}$ be the pullbacks of $M_\bullet$ and $N_\bullet$ to $\mathcal{C}_{\mathcal{M}}$ and $\mathcal{C}_{\mathcal{N}}$, respectively. For each $n \in \mathbb{Z}_{\geq 0}$, the twisted curve $\mathcal{C}_{N,n} \to N_n$ is the pullback of $\mathcal{C}_{M,n} \to M_n$, and we have isomorphisms $m'^*_n\omega_{M_n} \xrightarrow{(3.11)} \omega_{N_n}$. Under the identifications $(a^*m'^*\omega_{\mathcal{M}})|_{M_n} \simeq m'^*_n\omega_{M_n}$ of Remark A.20, these isomorphisms are compatible with the transition maps for the sheaves $a^*m'^*\omega_{\mathcal{M}}$ and $a^*\omega_{\mathcal{N}}$ in $\mathrm{QCoh}(\mathcal{C}_{N,\bullet,\text{lis-et}})$ because equation (3.11) satisfies the cocycle condition. By descent, we get an isomorphism $m'^*\omega^\bullet_{\mathcal{M}} \to \omega^\bullet_{\mathcal{N}}$. To check that this definition makes diagram (3.12) commute, apply the equivalences a^* and use Remark A.20 to get a collection of commuting diagrams indexed by $n \in \mathbb{Z}_{\geq 0}$. □

4. Obstruction theories via the Fundamental Theorem

4.1. Some Picard categories

Let $\mathcal{S}$ be a site. We recall the notion of Picard stacks from [73, Sec XVIII.1.4.5] and observe that a Picard category is just a Picard stack on the punctual site (see also [73, Def XVIII.1.4.2]). If $f : \mathcal{P} \to \mathcal{Q}$ is a morphism of Picard stacks on $\mathcal{S}$, we define the *kernel* to be the fibre product $\mathcal{K} = \bullet \times_{e,\mathcal{Q},f} \mathcal{P}$, where $\bullet$ is the trivial Picard stack (a constant sheaf with all its fibres equal to a single point) and $e : \bullet \to \mathcal{Q}$ is the identity.

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Example 4.1. Let $\mathrm{D}(\mathcal{S})$ be the unbounded derived category of abelian sheaves on $\mathcal{S}$. As in [73, Sec XVIII.1.4.11], we have a functor ch from the subcategory $\mathrm{D}^{[-1,0]}(\mathcal{S})$ to the category of Picard stacks on $\mathcal{S}$ (in the latter category, arrows are isomorphism classes of morphisms of stacks). Suppose A is a sheaf of rings on $\mathcal{S}$ and $\mathrm{D}(A)$ is the unbounded derived category of sheaves of A-modules. For two complexes $F \in \mathrm{D}^{[-\infty,a]}(A)$ and $G \in \mathrm{D}^{[a-1,\infty]}(A)$, we define

$$\mathrm{Ext}_A^{0/-1}(F,G) := ch(\tau_{\leq 0}\mathrm{RHom}_A(F,G)) = ch(\tau_{\leq 0}\mathrm{R}\Gamma\mathrm{R}\mathcal{H}om_A(F,G)), \tag{4.1}$$

where RHom_A is derived global hom for $\mathrm{D}(A)$; we have omitted the pushforward from the derived category of $\Gamma(\mathcal{S},A)$-modules to the category of abelian groups. Observe that ch is applied here over the site with one object and one morphism, so $\mathrm{Ext}_A^{0/-1}(F,G)$ is actually a Picard category (and the prestack $pch(\tau_{\leq 0}\mathrm{R}\Gamma\mathrm{R}\mathcal{H}om_A(F,G)$ of [73, Sec XVIII.1.4.11] is actually a stack). If the ring A is clear, we will omit it from the notation. It follows from [73, (XVIII.1.4.11.1)] that isomorphism classes of objects of $\mathrm{Ext}_A^{0/-1}(F,G)$ are equal to $\mathrm{Ext}_A^0(F,G)$ and from [73, (XVIII.1.4.11.2)] that automorphisms of the identity element are $\mathrm{Ext}_A^{-1}(F,G)$.

Example 4.2. Let $\mathcal{X} \to \mathcal{Y}$ be a representable morphism of algebraic stacks, and let I be a quasi-coherent sheaf on $\mathcal{X}$. We recall from [Ols06, Sec 2.2, 2.12] the Picard category $\underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{X},I)$ on $\mathcal{X}_{\mathrm{et}}$: objects are square-zero extensions $\mathcal{X} \hookrightarrow \mathcal{X}'$ of stacks over $\mathcal{Y}$, together with an isomorphism $I \to \ker(\mathcal{O}_{\mathcal{X}'} \to \mathcal{O}_{\mathcal{X}})$ (see [Ols06, Sec 2.2] for details, e.g., arrows).

Now suppose we have the following commuting diagram of algebraic stacks where $q : \mathcal{X} \hookrightarrow \mathcal{X}'$ is a square-zero extension by a quasi-coherent sheaf I, the maps f and g are representable, and we have fixed 2-morphism $\gamma : r \circ f \to g \circ q$:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[d, "q"'] & \mathcal{Y} \arrow[d, "r"] \\ \mathcal{X}' \arrow[r, "g"] & \mathcal{Z} \end{tikzcd} \tag{4.2}$$

The morphism r induces a morphism $\underline{R} : \underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{X},I) \to \underline{\mathrm{Exal}}_{\mathcal{Z}}(\mathcal{X},I)$, and the perimeter of diagram (4.2) defines an element of $\underline{\mathrm{Exal}}_{\mathcal{Z}}(\mathcal{X},I)$ (i.e., a functor $\bullet \to \underline{\mathrm{Exal}}_{\mathcal{Z}}(\mathcal{X},I)$, where $\bullet$ is the groupoid with one object and one arrow). We define the Picard category $\underline{\mathrm{Def}}(f)$ to be the fibre product

$$\begin{tikzcd} \underline{\mathrm{Def}}(f) \arrow[r] \arrow[d] & \underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{X},I) \arrow[d, "\underline{R}"] \\ \bullet \arrow[r] & \underline{\mathrm{Exal}}_{\mathcal{Z}}(\mathcal{X},I) \end{tikzcd} \tag{4.3}$$

where the bottom arrow $\bullet \to \underline{\mathrm{Exal}}_{\mathcal{Z}}(\mathcal{X},I)$ is the section induced by diagram (4.2). We use $\mathrm{Def}(f)$ to denote the set of isomorphism classes of $\underline{\mathrm{Def}}(f)$. Explicitly, objects of $\underline{\mathrm{Def}}(f)$ are triples (k,ϵ,δ) such that $k : \mathcal{X}' \to \mathcal{Y}$ is a 1-morphism, and $\epsilon : f \to k \circ q$ and $\delta : r \circ k \to g$ are 2-morphisms satisfying $q^*(\delta) \circ r(\epsilon) = \gamma$. A morphism from $(k_1,\epsilon_1,\delta_1)$ to $(k_2,\epsilon_2,\delta_2)$ is a natural transformation $\tau : k_1 \to k_2$ such that $q^*(\tau) \circ \epsilon_1 = \epsilon_2$ and $\delta_1 = \delta_2 \circ r(\tau)$ (for details see [Web20, Lem 2.4.3]).

Example 4.3. As an example of diagram (4.2), let $\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{r} \mathcal{Z}$ be morphisms of algebraic stacks with $\mathcal{X}$ an algebraic space, and let $I \in \mathrm{QCoh}(\mathcal{X}_{\mathrm{et}})$. Define $q : \mathcal{X} \to \mathcal{X}'$ to be the trivial extension by I, so we have $q' : \mathcal{X}' \to \mathcal{X}$ such that $q' \circ q = 1_{\mathcal{X}}$. Now $g := r \circ f \circ q'$ is representable, and $k = f \circ q'$ defines an element of $\underline{\mathrm{Def}}(f)$.

4.2. The Fundamental Theorem

The fundamental property of the cotangent complex is that it provides a description of the Picard category in Example 4.2 in terms of the construction in Example 4.1.

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Theorem 4.4 [Ols06]. *Let $\mathcal{X} \to \mathcal{Y}$ be a representable morphism of algebraic stacks. Then there is an isomorphism of Picard categories:*

$$\underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{X}, I) \xrightarrow{\sim} \mathrm{Ext}^{0/-1}_{\mathcal{O}_{\mathcal{X}}}(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}, I[1]) \tag{4.4}$$

The definition of equation (4.4) is technical, and we defer it to Section B.3. For us, the key property of equation (4.4) is that it is functorial under pullback and basechange as stated in the next two lemmas.

Lemma 4.5. *Suppose we have maps $\mathcal{Z} \xrightarrow{f} \mathcal{W} \xrightarrow{g} \mathcal{Y}$, with f and $g \circ f$ representable. Then given $I \in \mathrm{QCoh}(\mathcal{Z}_{\text{lis-et}})$, there is a commuting diagram of Picard categories:*

$$\begin{tikzcd}
\mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{Z}/\mathcal{W}}, I[1]) \arrow[r, "A"] & \mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{Z}/\mathcal{Y}}, I[1]) \\
\underline{\mathrm{Exal}}_{\mathcal{W}}(\mathcal{Z}, I) \arrow[u, "(4.4)"] \arrow[r, "B"] & \underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{Z}, I) \arrow[u, "(4.4)"']
\end{tikzcd} \tag{4.5}$$

Here A is induced by the canonical map $\mathbb{L}_{\mathcal{Z}/\mathcal{Y}} \to \mathbb{L}_{\mathcal{Z}/\mathcal{W}}$, and B is induced by composition with g.

Lemma 4.5 is a special case of [Ols06, (2.33.3)], but that result is stated only for isomorphism classes of objects. We will prove Lemma 4.5 in Appendix B. For the second functoriality lemma, suppose we have a fibre square of algebraic stacks

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, "p"] \arrow[d] & \mathcal{X} \arrow[d] \\
\mathcal{W} \arrow[r] & \mathcal{Y}
\end{tikzcd} \tag{4.6}$$

where the map $\mathcal{W} \to \mathcal{Y}$ is flat and $\mathcal{X} \to \mathcal{Y}$ is representable. Then given a quasi-coherent sheaf $I \in \mathrm{QCoh}(\mathcal{X}_{\text{lis-et}})$, there is a morphism of Picard categories

$$\underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{X}, I) \to \underline{\mathrm{Exal}}_{\mathcal{W}}(\mathcal{Z}, p^*I) \tag{4.7}$$

sending $\mathcal{X}' \to \mathcal{Y}$ to the pullback $\mathcal{Z}' := \mathcal{X}' \times_{\mathcal{Y}} \mathcal{W} \to \mathcal{W}$ (observe that, since diagram (4.6) is fibred, we have an induced map $\mathcal{Z} \hookrightarrow \mathcal{Z}'$ with the desired kernel).

Lemma 4.6. *Given the fibre square (4.6) and $I \in \mathrm{QCoh}(\mathcal{X}_{\text{lis-et}})$, there is a commuting diagram of Picard categories:*

$$\begin{tikzcd}
\mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}, I[1]) \arrow[r, "C"] & \mathrm{Ext}^{0/-1}(p^*\mathbb{L}_{\mathcal{X}/\mathcal{Y}}, p^*I[1]) & \mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{Z}/\mathcal{W}}, p^*I[1]) \arrow[l, "D", "\sim"'] \\
\underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{X}, I) \arrow[u, "(4.4)"] \arrow[rr, "E"] & & \underline{\mathrm{Exal}}_{\mathcal{W}}(\mathcal{Z}, p^*I) \arrow[u, "(4.4)"]
\end{tikzcd} \tag{4.8}$$

Here C is induced by equation (2.5) in the context of Example B.4,[5] *the arrow D is induced by the canonical map of cotangent complexes (an isomorphism in this case), and E is equation (4.7).*

We will prove Lemma 4.6 in Appendix B. We conclude this section with a corollary to Theorem 4.4 that may be read as a relative version of the same theorem.

Corollary 4.7. *Consider a diagram (4.2) of algebraic stacks where $\mathcal{X} \to \mathcal{X}'$ is a square-zero extension with ideal sheaf I, and f and g are representable:*

[5]Example B.4 differs from Example 3.1 because it uses general sheaves of $\mathcal{O}$-modules and hence the $\mathsf{R}\mathcal{H}om$ functor instead of $\mathsf{R}\mathcal{H}om^{\mathrm{qc}}$.

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1. *There is an obstruction* $o(f) \in \mathrm{Ext}^1(\mathsf{L}f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I)$ *whose vanishing is necessary and sufficient for the set* $\mathrm{Def}(f)$ *to be nonempty.*
2. *If* $o(f) = 0$*, then there is an isomorphism* $\underline{\mathrm{Def}}(f) \simeq \mathrm{Ext}^{0/-1}(\mathsf{L}f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I)$*.*

Remark 4.8. It follows from the corollary that if $o(f) = 0$, we get an isomorphism of groups between $\mathrm{Ext}^{-1}(\mathsf{L}f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}})$ and the automorphism group of any element of $\underline{\mathrm{Def}}(f)$. One can extract from the proof of the corollary that $\mathrm{Def}(f)$ is a torsor for $\mathrm{Ext}^0(\mathsf{L}f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I)$.

Proof. Applying Lemma 4.5 to the maps $\mathcal{X} \to \mathcal{Y} \xrightarrow{r} \mathcal{Z}$, we get a commuting diagram

$$\begin{tikzcd} \mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}, I[1]) \arrow[r] \arrow[d, "\sim"] & \mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{X}/\mathcal{Z}}, I[1]) \arrow[d, "\sim"] \\ \underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{X}, I) \arrow[r, "\underline{R}"] & \underline{\mathrm{Exal}}_{\mathcal{Z}}(\mathcal{X}, I) \end{tikzcd} \tag{4.9}$$

where $\underline{R}$ is the same as the map B in the lemma. When we restrict diagram (4.9) to isomorphism classes of objects, we get the commuting square in the following diagram:

$$\begin{tikzcd} \mathrm{Ext}^1(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}, I) \arrow[r] \arrow[d, "\sim"] & \mathrm{Ext}^1(\mathbb{L}_{\mathcal{X}/\mathcal{Z}}, I) \arrow[r, "ob"] \arrow[d, "\sim", "\alpha"'] & \mathrm{Ext}^1(\mathsf{L}f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I) \\ \mathrm{Exal}_{\mathcal{Y}}(\mathcal{X}, I) \arrow[r, "R"] & \mathrm{Exal}_{\mathcal{Z}}(\mathcal{X}, I) & \end{tikzcd} \tag{4.10}$$

The top row of the diagram comes from applying $\mathrm{Ext}^1(-, I)$ to the distinguished triangle

$$\mathsf{L}f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}} \to \mathbb{L}_{\mathcal{X}/\mathcal{Z}} \to \mathbb{L}_{\mathcal{X}/\mathcal{Y}}. \tag{4.11}$$

The set $\mathrm{Def}(f)$ is nonempty if and only if, in diagram (4.10), the fibre of R over the element $[g] \in \mathrm{Exal}_{\mathcal{Z}}(\mathcal{X}, I)$ defined by diagram (4.2) is nonempty. From the long exact sequence for $\mathrm{Ext}^i(-, I)$ applied to equation (4.11), we see that this happens if and only if the image of $[g]$ in $\mathrm{Ext}^1(\mathsf{L}f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I)$ (under the maps given in diagram (4.10)) is 0. We define

$$o(f) = ob(\alpha^{-1}([g])). \tag{4.12}$$

If $\mathrm{Def}(f)$ is not empty, then by Lemma 4.9 below, $\underline{\mathrm{Def}}(f)$ is isomorphic to the kernel of the morphism of Picard categories

$$\underline{R} : \underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{X}, I) \to \underline{\mathrm{Exal}}_{\mathcal{Z}}(\mathcal{X}, I).$$

It follows from diagram (4.9) and [Ols06, Lem 2.29] applied to the distinguished triangle

$$\mathsf{RHom}(\mathbb{L}_{\mathcal{X}/\mathcal{Z}}, I[1]) \xrightarrow{\beta} \mathsf{RHom}(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}, I[1]) \to \mathsf{RHom}(\mathsf{L}f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I[1]) \to$$

induced from equation (4.11) that this kernel is canonically isomorphic to $ch((\tau_{\leq -1}Cone(\tau_{\leq 0}\beta))[-1])$, where $Cone$ denotes the mapping cone of a morphism. But we compute

$$(\tau_{\leq -1}Cone(\tau_{\leq 0}\beta)))[-1] = (\tau_{\leq -1}Cone(\beta))[-1] = \tau_{\leq 0}Cone(\beta[-1]),$$

so we get that this kernel is isomorphic to $\mathrm{Ext}^{0/-1}(\mathsf{L}f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I)$. ◻

Lemma 4.9. *Let* $f : \mathcal{P} \to \mathcal{Q}$ *be a morphism of Picard stacks on a stack* $\mathcal{X}$*, and let* $\mathcal{K}$ *denote the kernel. Let* $q : \mathcal{X} \to \mathcal{Q}$ *be a section and* $\mathcal{F} = \mathcal{P} \times_{\mathcal{Q}, q} \mathcal{X}$ *the fibre product. If the set of global objects of* $\mathcal{F}$ *is not empty, then* $\mathcal{F}$ *is noncanonically isomorphic to* $\mathcal{K}$*.*

Proof. A global object of $\mathcal{F}$ defines a section $\sigma : \mathcal{X} \to \mathcal{F}$. One can check that the composition $\mathcal{K} \times_{\mathcal{X}} \mathcal{F} \to \mathcal{P} \times_{\mathcal{X}} \mathcal{P} \xrightarrow{\mu} \mathcal{P}$, where μ is the group operation, factors through $\mathcal{F}$. We obtain a morphism

$$\mathcal{K} \xrightarrow{(1_{\mathcal{K}},\sigma)} \mathcal{K} \times_{\mathcal{X}} \mathcal{F} \to \mathcal{F}. \tag{4.13}$$

On the other hand, we have the composition

$$\mathcal{F} \xrightarrow{(pr_1,-\sigma)} \mathcal{P} \times_{\mathcal{X}} \mathcal{P} \xrightarrow{\mu} \mathcal{P} \xrightarrow{f} \mathcal{Q}, \tag{4.14}$$

where $pr_1 : \mathcal{F} \to \mathcal{P}$ is the canonical morphism and $-\sigma$ is σ followed by the inverse morphism. The composition of equation (4.14) factors through the identity $e : \mathcal{X} \to \mathcal{Q}$, so we get an induced map $\mathcal{F} \to \mathcal{K}$. One may check that this is inverse to equation (4.13). □

4.3. Equivalent definitions of an obstruction theory

Let $\mathcal{Y} \to \mathcal{Z}$ be a morphism of algebraic stacks. If $\phi : E \to F$ is a morphism in $\mathrm{D}_{\mathrm{qc}}(\mathcal{Y}_{\text{lis-et}})$, let $H^i(\phi) : H^i(E) \to H^i(F)$ denote the induced morphism on cohomology sheaves. The following definition generalises [BF97, Def 4.4].

Definition 4.10. *A morphism $\phi : E \to \mathbb{L}_{\mathcal{Y}/\mathcal{Z}}$ in $\mathrm{D}_{\mathrm{qc}}(\mathcal{Y}_{\text{lis-et}})$ is an obstruction theory if $H^{-1}(\phi)$ is a surjection and $H^0(\phi)$, $H^1(\phi)$ are isomorphisms.*

Given a morphism $\phi : E \to \mathbb{L}_{\mathcal{Y}/\mathcal{Z}}$ in $\mathrm{D}_{\mathrm{qc}}(\mathcal{Y}_{\text{lis-et}})$, for every diagram (4.2), we have induced homomorphisms of groups (computed a priori in the lisse-étale topology)

$$\Phi_i : \mathrm{Ext}^i(\mathsf{L}f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I) \to \mathrm{Ext}^i(\mathsf{L}f^*E, I). \tag{4.15}$$

We now present a well-known local criterion for a morphism ϕ to be an obstruction theory. Similar criteria have appeared in [BF97, Thm 4.5], [AP19, Cor 8.5] and [Pom15, Thm 3.5]. However, we found the wording in these criteria to be vague in that they do not explicitly require compatibility between various morphisms. Since proving said compatibility is a major part of the paper (it comprises the functoriality computations in Appendix B), we give the precise statement of the local criterion and a fully detailed proof.

Lemma 4.11. *The following conditions are equivalent:*

1. *The morphism ϕ is an obstruction theory.*
2. *For every diagram (4.2) with $\mathcal{X}$ a scheme, the following hold:*
   (a) *the element $\Phi_1(o(f)) \in \mathrm{Ext}^1(\mathsf{L}f^*E, I)$ vanishes if and only if $\underline{\mathrm{Def}}(f)$ is nonempty.*
   (b) *if $\Phi_1(o(f)) = 0$, then Φ_0 and Φ_{-1} are isomorphisms.*
3. *For every affine scheme $\mathcal{X}$ and smooth map $\mathcal{X} \to \mathcal{Z}$, the following hold:*
   (a) *For every ambient diagram (4.2) using $\mathcal{X}$, the element $\Phi_1(o(f)) \in \mathrm{Ext}^1(\mathsf{L}f^*E, I)$ vanishes if and only if $\underline{\mathrm{Def}}(f)$ is nonempty.*
   (b) *For every $I \in \mathrm{QCoh}(\mathcal{X}_{\text{lis-et}})$, the maps Φ_0 and Φ_{-1} are isomorphisms.*

Remark 4.12. In Lemma 4.11, conditions (2) and (3) may be computed in $\mathcal{X}_{\text{et}}$—so in (3b), one checks every $I \in \mathrm{QCoh}(\mathcal{X}_{\text{et}})$ (see, e.g., [Ols16, Prop 9.2.16]).

Proof of Lemma 4.11. The proof of this lemma seems to be well-known; many parts were explained to me by Bhargav Bhatt. Let C be the mapping cone of $\phi : E \to \mathbb{L}_{\mathcal{Y}/\mathcal{Z}}$. Then condition (1) is equivalent to

$$\text{(1')} \quad H^i(C) = 0 \text{ for } i \geq -1.$$

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Assume (1'). Then $H^i(\mathsf{L}f^*C)$ also vanish for $i \geq -1$, so a spectral sequence [Stacks, Tag 07AA] for $\mathrm{Ext}^i(-, I)$ implies $\mathrm{Ext}^i(\mathsf{L}f^*C, I) = 0$ for $i \leq 1$ and any I. Now the long exact sequence of Ext groups arising from the distinguished triangle

$$\mathsf{L}f^*E \to \mathsf{L}f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}} \to \mathsf{L}f^*C \to \tag{4.16}$$

implies that Φ_1 is injective and Φ_0 and Φ_{-1} are isomorphisms. Combined with Corollary 4.7, this proves (2) (with $\mathcal{X}$ an arbitrary scheme). Now (2) implies (3) using Example 4.3.

Assume (3). Condition (1') may be checked smooth-locally on $\mathcal{Y}$, so let $f : \mathcal{X} \to \mathcal{Y}$ be a smooth morphism from an affine scheme, and let $I \in \mathrm{QCoh}(\mathcal{X}_{\text{lis-et}})$ be arbitrary. We will show that if $i \geq -1$, then $\mathrm{Ext}^0(H^i(\mathsf{L}f^*C), I) = 0$, which implies $f^*H^i(C) = H^i(\mathsf{L}f^*C) = 0$ (the first equality is [HR17, (1.9)] and uses flatness of f).

By assumption (3b), the morphisms Φ_0 and Φ_{-1} are isomorphisms. We show that Φ_1 is injective. It follows from Corollary 4.7 and assumption (3a) that if $\Phi_1(o(f)) = 0$, then $o(f) = 0$, so it suffices to show that every element of $\mathrm{Ext}^1(\mathsf{L}f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I)$ is equal to $o(f)$ for some diagram (4.2), or equivalently that the map ob in equation (4.12) is surjective. This follows from the long exact sequence

$$\to \mathrm{Ext}^1(\mathbb{L}_{\mathcal{X}/\mathcal{Z}}, I) \xrightarrow{ob} \mathrm{Ext}^1(\mathsf{L}f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I) \to \mathrm{Ext}^2(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}, I) \to$$

since $\mathbb{L}_{\mathcal{X}/\mathcal{Y}} = \Omega^1_{\mathcal{X}/\mathcal{Y}}[0]$ is a locally free sheaf in degree 0.

Since Φ_1 is injective and Φ_0 and Φ_{-1} are isomorphisms, the long exact sequence of Ext groups for equation (4.16) shows that $\mathrm{Ext}^i(\mathsf{L}f^*C, I) = 0$ for every $i \leq 1$. By [Stacks, Tag 07AA], there is a spectral sequence whose second page is

$$\mathrm{Ext}^i(H^{-j}(\mathsf{L}f^*C), I) \implies \mathrm{Ext}^{i+j}(\mathsf{L}f^*(C), I).$$

A priori we know $H^i(\mathsf{L}f^*C) = 0$ for $i \geq 2$. By the above spectral sequence, the group $\mathrm{Ext}^0(H^1(\mathsf{L}f^*C), I)$ is equal to $\mathrm{Ext}^{-1}(\mathsf{L}f^*C, I)$, which vanishes for every I. This forces $H^1(\mathsf{L}f^*C)$ to vanish. Inductively applying the same argument to $\mathrm{Ext}^0(\mathsf{L}f^*C, I)$ and then $\mathrm{Ext}^1(\mathsf{L}f^*C, I)$ shows that $H^0(\mathsf{L}f^*C)$ and $H^{-1}(\mathsf{L}f^*C)$ vanish as well. □

4.4. *Moduli of sections*

Consider a tower of algebraic stacks

$$\mathcal{Z} \to \mathcal{C} \xrightarrow{p} \mathcal{M}$$

as in Section 1. There we defined the moduli of sections $\mathrm{Sec}_{\mathcal{M}}(\mathcal{Z}/\mathcal{C})$. By [HR19, Thm 1.3] and our assumption that $\mathcal{M}$ is locally Noetherian, the stack $\mathrm{Sec}_{\mathcal{M}}(\mathcal{Z}/\mathcal{C})$ is also locally Noetherian. The stack $\mathrm{Sec}_{\mathcal{M}}(\mathcal{Z}/\mathcal{C})$ has a universal curve $\mathcal{C}_{\mathrm{Sec}_{\mathcal{M}}(\mathcal{Z}/\mathcal{C})}$ and a universal section $f_{\mathrm{Sec}_{\mathcal{M}}(\mathcal{Z}/\mathcal{C})} \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}_{\mathrm{Sec}(\mathcal{Z})}, \mathcal{Z})$ (we will omit the subscript on f when possible).

Now suppose we have a tower of algebraic stacks

$$\mathcal{Z} \to \mathcal{W} \to \mathcal{C} \xrightarrow{p} \mathcal{M},$$

where $\mathcal{Z}, \mathcal{C}$ and $\mathcal{M}$ are as before and $\mathcal{W} \to \mathcal{M}$ is locally finitely presented, is quasi-separated and has affine stabilisers. To simplify the notation, let $\mathfrak{S}(\mathcal{Z}) := \mathrm{Sec}_{\mathcal{M}}(\mathcal{Z}/\mathcal{C})$ and $\mathfrak{S}(\mathcal{W}) := \mathrm{Sec}_{\mathcal{M}}(\mathcal{W}/\mathcal{C})$. We have an induced map $\mathfrak{S}(\mathcal{Z}) \to \mathfrak{S}(\mathcal{W})$, and over this map we have a canonical relative obstruction theory defined as follows. We have a morphism in $\mathrm{D}_{\mathrm{qc}}(\mathcal{C}_{\mathfrak{S}(\mathcal{Z})})$ consisting of canonical morphisms of cotangent complexes:

$$\mathsf{L}f^*\mathbb{L}_{\mathcal{Z}/\mathcal{W}} \to \mathbb{L}_{\mathcal{C}_{\mathfrak{S}(\mathcal{Z})}/\mathcal{C}_{\mathfrak{S}(\mathcal{W})}} \xleftarrow{\sim} p^*\mathbb{L}_{\mathfrak{S}(\mathcal{Z})/\mathfrak{S}(\mathcal{W})}. \tag{4.17}$$

Using the pair $(\omega^{\bullet}_{\mathfrak{S}(\mathcal{Z})}, tr_{\mathfrak{S}(\mathcal{Z})})$ defined in Proposition 3.14, we may apply the adjunction-like morphism a defined in Section 2.3 to equation (4.17), obtaining

$$\phi_{\mathfrak{S}(\mathcal{Z})/\mathfrak{S}(\mathcal{W})} : \mathbb{E}_{\mathfrak{S}(\mathcal{Z})/\mathfrak{S}(\mathcal{W})} \to \mathbb{L}_{\mathfrak{S}(\mathcal{Z})/\mathfrak{S}(\mathcal{W})}, \quad \mathbb{E}_{\mathfrak{S}(\mathcal{Z})/\mathfrak{S}(\mathcal{W})} := \mathsf{R}p_*(\mathsf{L}f^*\mathbb{L}_{\mathcal{Z}/\mathcal{W}} \otimes \omega^{\bullet}_{\mathfrak{S}(\mathcal{Z})}). \tag{4.18}$$

For example, when $\mathcal{W} = \mathcal{C}$, we have $\mathfrak{S}(\mathcal{W}) = \mathcal{M}$, and we obtain an obstruction theory on $\mathfrak{S}(\mathcal{Z})$ relative to $\mathcal{M}$. We refer the reader to [CJW21, Appendix A] for functoriality properties of $\mathrm{Sec}_{\mathcal{M}}(\mathcal{Z}/\mathcal{C})$ and the obstruction theories in equation (4.18).

The main theorem of this article is the following.

Theorem 4.13. *The morphism in equation (4.18) is an obstruction theory.*

4.4.1. Proof of Theorem 4.13

We prove condition (3) of Lemma 4.11. To begin, fix a solid commuting diagram

$$\begin{tikzcd}
T \arrow[r, "m"] \arrow[d] & \mathfrak{S}(\mathcal{Z}) \arrow[d] \\
T' \arrow[r] & \mathfrak{S}(\mathcal{W})
\end{tikzcd} \tag{4.19}$$

with m a smooth morphism and $T \to T'$ a square-zero extension of affine schemes with ideal sheaf $I \in \mathrm{QCoh}(T_{\text{lis-et}})$. Let $\mathcal{C}_T$ (respectively, $\mathcal{C}_{T'}$) denote the pullback of the universal curve to T (respectively, T'). We first observe that from the definition of the moduli stacks, we have a commuting diagram of algebraic stacks:

$$\begin{tikzcd}
 & \mathcal{C}_T \arrow[r, "f_T"] \arrow[dl] \arrow[d, equal] & \mathcal{Z} \arrow[r] & \mathcal{W} \\
\mathcal{C}_{T'} \arrow[urrr] \arrow[d, equal] & \mathcal{C}_T \arrow[r, "m'"] \arrow[d, "p"] & \mathcal{C}_{\mathfrak{S}(\mathcal{Z})} \arrow[u, "f"'] \arrow[r] \arrow[d, "p"] & \mathcal{C}_{\mathfrak{S}(\mathcal{W})} \arrow[u, "f_{\mathfrak{S}(\mathcal{W})}"'] \arrow[d, "p"] \\
\mathcal{C}_{T'} \arrow[d] & T \arrow[r, "m"] \arrow[dl] & \mathfrak{S}(\mathcal{Z}) \arrow[r] & \mathfrak{S}(\mathcal{W}) \\
T' \arrow[urrr] & & &
\end{tikzcd} \tag{4.20}$$

Claim 4.14 (Step 1). *Diagram (4.20) leads to a commuting diagram of Picard categories*

$$\begin{tikzcd}
\underline{\mathrm{Def}}(f_T) \arrow[r] & \underline{\mathrm{Exal}}_{\mathcal{Z}}(\mathcal{C}_T, p^*I) \arrow[r, "B"] & \underline{\mathrm{Exal}}_{\mathcal{W}}(\mathcal{C}_T, p^*I) \\
\underline{\mathrm{Def}}(m) \arrow[r] \arrow[u, "\Psi"] & \underline{\mathrm{Exal}}_{\mathfrak{S}(\mathcal{Z})}(T, I) \arrow[r, "B"] \arrow[u, "B\circ E"] & \underline{\mathrm{Exal}}_{\mathfrak{S}(\mathcal{W})}(T, I) \arrow[u, "B\circ E"]
\end{tikzcd} \tag{4.21}$$

where the arrows B and E are as in Lemmas 4.5 and 4.6, the terms in the leftmost column are fibres of the top and bottom horizontal maps and Ψ is an isomorphism.

Proof. The right square in the diagram follows from the bottom two (fibred) squares of diagram (4.20) and the definitions of B and E. Moreover, the element of $\underline{\mathrm{Exal}}_{\mathfrak{S}(\mathcal{W})}(T, I)$ defined by m and its horizontal square in diagram (4.20) maps under $B \circ E$ to the element of $\underline{\mathrm{Exal}}_{\mathcal{W}}(\mathcal{C}_T, p^*I)$ defined by f_T and its horizontal square. By definition (4.3) of $\underline{\mathrm{Def}}$, we get the left square of diagram (4.21).

To prove that Ψ is an isomorphism, it suffices to check étale-locally on T; that is, it suffices to show that Ψ induces an equivalence of categories $\underline{\mathrm{Def}}(m) \to \underline{\mathrm{Def}}(f_T)$. For this, we construct an inverse functor. Let (k, ϵ, δ) be an element of $\underline{\mathrm{Def}}(f_T)$. We get an arrow $k_\delta : T' \to \mathfrak{S}(\mathcal{Z})$ determined by k and

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δ, making the resulting triangle over $\mathfrak{S}(\mathcal{W})$ strictly commutative. The 2-morphism ϵ determines a 2-morphism (also denoted ϵ) from m to the composition $T \to T' \xrightarrow{k_\delta} \mathfrak{S}(\mathcal{Z})$. Hence our functor sends the object (k, ϵ, δ) to the object (k_δ, ϵ, id). We leave it to the reader to check that this is inverse to Ψ. □

Claim 4.15 (Step 2). *Diagram (4.20) leads to a morphism of distinguished triangles*

$$\begin{tikzcd}
\mathsf{R}p_*\mathsf{R}\mathcal{H}om(\mathsf{L}f_T^*\mathbb{L}_{\mathcal{Z}/\mathcal{W}}, p^*I) \arrow[r] & \mathsf{R}p_*\mathsf{R}\mathcal{H}om(\mathbb{L}_{\mathcal{C}_T/\mathcal{Z}}, p^*I[1]) \arrow[r] & \mathsf{R}p_*\mathsf{R}\mathcal{H}om(\mathbb{L}_{\mathcal{C}_T/\mathcal{W}}, p^*I[1]) \\
\mathsf{R}\mathcal{H}om(\mathsf{L}m^*\mathbb{L}_{\mathfrak{S}(\mathcal{Z})/\mathfrak{S}(\mathcal{W})}, I) \arrow[r] \arrow[u] & \mathsf{R}\mathcal{H}om(\mathbb{L}_{T/\mathfrak{S}(\mathcal{Z})}, I[1]) \arrow[r] \arrow[u] & \mathsf{R}\mathcal{H}om(\mathbb{L}_{T/\mathfrak{S}(\mathcal{W})}, I[1]) \arrow[u]
\end{tikzcd} \tag{4.22}$$

where the leftmost vertical arrow has the property that there exists a composition

$$\mathsf{R}\mathcal{H}om(\mathsf{L}m^*\mathbb{L}_{\mathfrak{S}(\mathcal{Z})/\mathfrak{S}(\mathcal{W})}, I) \to \mathsf{R}p_*\mathsf{R}\mathcal{H}om(\mathsf{L}f_T^*\mathbb{L}_{\mathcal{Z}/\mathcal{W}}, p^*I) \underset{\sim}{\xrightarrow{\exists}} \mathsf{R}\mathcal{H}om(\mathsf{L}m^*(\mathsf{R}p_*\mathbb{L}_{\mathcal{Z}/\mathcal{W}} \otimes \omega^\bullet_{\mathfrak{S}(\mathcal{Z})}), I) \tag{4.23}$$

equal to the map induced by $\phi_{\mathfrak{S}(\mathcal{Z})/\mathfrak{S}(\mathcal{W})} : \mathsf{R}p_*\mathbb{L}_{\mathcal{Z}/\mathcal{W}} \otimes \omega^\bullet_{\mathfrak{S}(\mathcal{Z})} \to \mathbb{L}_{\mathfrak{S}(\mathcal{Z})/\mathfrak{S}(\mathcal{W})}$. Applying the functor $ch \circ \tau_{\leq 0} \circ \mathsf{R}\Gamma$ to diagram (4.22) yields a commuting diagram of Picard categories

$$\begin{tikzcd}
\mathrm{Ext}^{0/-1}(\mathsf{L}f_T^*\mathbb{L}_{\mathcal{Z}/\mathcal{W}}, p^*I) \arrow[r] & \mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{C}_T/\mathcal{Z}}, p^*I[1]) \arrow[r, "A"] & \mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{C}_T/\mathcal{W}}, p^*I[1]) \\
\mathrm{Ext}^{0/-1}(\mathsf{L}m^*\mathbb{L}_{\mathfrak{S}(\mathcal{Z})/\mathfrak{S}(\mathcal{W})}, I) \arrow[r] \arrow[u, "\Phi"] & \mathrm{Ext}^{0/-1}(\mathbb{L}_{T/\mathfrak{S}(\mathcal{Z})}, I[1]) \arrow[r, "A"] \arrow[u, "A\circ D^{-1}\circ C"] & \mathrm{Ext}^{0/-1}(\mathbb{L}_{T/\mathfrak{S}(\mathcal{W})}, I[1]) \arrow[u, "A\circ D^{-1}\circ C"]
\end{tikzcd} \tag{4.24}$$

where the arrows A, D and C are defined as in Lemmas 4.5 and 4.6, the terms in the leftmost column are the kernels of the top and bottom horizontal maps and, if Φ is an isomorphism, then Φ_0 and Φ_{-1} (defined in equation (4.15)) are isomorphisms.

Proof. There is a morphism of distinguished triangles (see [Web20, Lem 2.2.12])

$$\begin{tikzcd}
\mathbb{L}_{\mathcal{C}_T/\mathcal{W}} \arrow[r] \arrow[d] & \mathbb{L}_{\mathcal{C}_T/\mathcal{Z}} \arrow[r] \arrow[d] & \mathsf{L}f_T^*\mathbb{L}_{\mathcal{Z}/\mathcal{W}}[1] \arrow[r] \arrow[d] & {} \\
p^*\mathbb{L}_{T/\mathfrak{S}(\mathcal{W})} \arrow[r] & p^*\mathbb{L}_{T/\mathfrak{S}(\mathcal{Z})} \arrow[r] & p^*\mathsf{L}m^*\mathbb{L}_{\mathfrak{S}(\mathcal{Z})/\mathfrak{S}(\mathcal{W})}[1] \arrow[r] & {}
\end{tikzcd}$$

(note that the vertical arrows are only defined in the derived category). Applying $\mathsf{R}p_*\mathsf{R}\mathcal{H}om^{\mathrm{qc}}(-, p^*I[1])$ to this diagram and composing with the morphism in equation (2.5) yields diagram (4.22), but with $\mathsf{R}\mathcal{H}om^{\mathrm{qc}}$ in place of $\mathsf{R}\mathcal{H}om$. Now Lemma 2.14 (applied in the context of Example 3.1) produces the composition in equation (4.23) that is isomorphic to the map induced by $\phi_{\mathfrak{S}(\mathcal{Z})/\mathfrak{S}(\mathcal{W})}$, but still with $\mathsf{R}\mathcal{H}om^{\mathrm{qc}}$ in place of $\mathsf{R}\mathcal{H}om$. To replace $\mathsf{R}\mathcal{H}om^{\mathrm{qc}}$ with $\mathsf{R}\mathcal{H}om$, we observe that all stacks in in diagram (4.20) are locally Noetherian and all morphisms are locally of finite type; so by [Stacks, Tag 08PZ], all cotangent complexes are pseudo-coherent (in fact, in the derived category D^-_{Coh} of the appropriate topos), and we may make the replacement by [Stacks, Tag 0A6H] (recall that we are working on an affine scheme T). Now diagram (4.24) is produced by applying $ch \circ \tau_{\leq 0} \circ \mathsf{R}\Gamma$ and using [Stacks, Tag 08J6], and arguing as at the end of the proof of Corollary 4.7. The map Φ being an isomorphism implies Φ_0 (respectively, Φ_{-1}) is an isomorphism by restricting Φ to isomorphism classes of objects (respectively, automorphisms of the identity). □

Claim 4.16 (Step 3). *Condition (3) in Lemma 4.11 holds.*

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Proof. We study the commuting cube formed by mapping the right square of diagram (4.24) (on the top floor) to the right square of diagram (4.21) (on the ground) via equation (4.4) (vertical maps):

$$\begin{tikzcd}
& \mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{C}_T/\mathcal{Z}}, p^*I[1]) \arrow[rr] \arrow[dd] & & \mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{C}_T/\mathcal{W}}, p^*I[1]) \arrow[dd] \\
\mathrm{Ext}^{0/-1}(\mathbb{L}_{T/\mathfrak{S}(\mathcal{Z})}, I[1]) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{Ext}^{0/-1}(\mathbb{L}_{T/\mathfrak{S}(\mathcal{W})}, I[1]) \arrow[ur] \arrow[dd] & \\
& \underline{\mathrm{Exal}}_{\mathcal{Z}}(\mathcal{C}_T, p^*I) \arrow[rr] & & \underline{\mathrm{Exal}}_{\mathcal{W}}(\mathcal{C}_T, p^*I) \\
\underline{\mathrm{Exal}}_{\mathfrak{S}(\mathcal{Z})}(T, I) \arrow[ur] \arrow[rr] & & \underline{\mathrm{Exal}}_{\mathfrak{S}(\mathcal{W})}(T, I) \arrow[ur] &
\end{tikzcd} \tag{4.25}$$

This cube commutes by Lemmas 4.5 and 4.6. We note that Theorem 4.4 applies because the maps $\mathcal{C}_T \to \mathcal{Z}$ and $\mathcal{C}_T \to \mathcal{W}$ are representable: for example, representability of $\mathcal{C}_T \to \mathcal{Z}$ follows from the fact that m' is representable and [Stacks, Tag 04Y5].

To prove (3a), restrict diagram (4.25) to isomorphism classes of objects. As in diagram (4.10), we extend this diagram by the obstruction maps, obtaining a commutative diagram

$$\begin{tikzcd}
\mathrm{Exal}_{\mathcal{W}}(\mathcal{C}_T, p^*I) & \mathrm{Ext}^1(\mathbb{L}_{\mathcal{C}_T/\mathcal{W}}, p^*I) \arrow[l, "(4.4)"', "\sim"] \arrow[r, "ob"] & \mathrm{Ext}^1(\mathsf{L}f_T^*\mathbb{L}_{\mathcal{Z}/\mathcal{W}}, p^*I) \\
\mathrm{Exal}_{\mathfrak{S}(\mathcal{W})}(T, I) \arrow[u, "B\circ E"] & \mathrm{Ext}^1(\mathbb{L}_{T/\mathfrak{S}(\mathcal{W})}, I) \arrow[l, "(4.4)"', "\sim"] \arrow[r, "ob"] \arrow[u] & \mathrm{Ext}^1(\mathsf{L}m^*\mathbb{L}_{\mathfrak{S}(\mathcal{Z})/\mathfrak{S}(\mathcal{W})}, I) \arrow[u, "\Phi_1'"]
\end{tikzcd}$$

where the left square is a side of our cube and the right square is obtained by applying the derived global sections functor $\mathsf{R}\Gamma$ to diagram (4.22) and then taking cohomology. By the definition of $B \circ E$ and commutativity of the diagram, the map labelled Φ_1' sends $o(m)$ to $o(f_T)$. By equation (4.23), the map Φ_1' is quasi-isomorphic to Φ_1, where Φ_1 is defined as in Lemma 4.11. By Corollary 4.7, the element $o(f_T)$ (respectively, $o(m)$) vanishes if and only if $\mathrm{Def}(f_T)$ (respectively, $\mathrm{Def}(m)$) is nonempty. Since the map $\Psi : \mathrm{Def}(m) \to \mathrm{Def}(f_T)$ from diagram (4.21) is an isomorphism, we see that (3a) holds.

To prove (3b), we may assume that diagram (4.25) was formed from the trivial example of diagram (4.19) (see Example 4.3). In this case, the terms in the left column of diagram (4.21) are kernels (not just fibres) of the horizontal maps, so diagram (4.25) induces the following commuting square of kernels:

$$\begin{tikzcd}
\mathrm{Ext}^{0/-1}(\mathsf{L}f_T^*\mathbb{L}_{\mathcal{Z}/\mathcal{W}}, p^*I) \arrow[r, "\sim"'] & \underline{\mathrm{Def}}(f_T) \\
\mathrm{Ext}^{0/-1}(\mathsf{L}m^*\mathbb{L}_{\mathfrak{S}(\mathcal{Z})/\mathfrak{S}(\mathcal{W})}, I) \arrow[r, "\sim"'] \arrow[u, "\Phi"] & \underline{\mathrm{Def}}(m) \arrow[u, "\Psi"']
\end{tikzcd}$$

The horizontal maps are induced by the instances of equation (4.4) in diagram (4.25), and they are isomorphisms because equation (4.4) is an isomorphism. Since Ψ is an isomorphism, Φ is an isomorphism as well. ◻

A. Descent theorems for lisse-étale sheaves on algebraic stacks

In this section, we recall the unbounded cohomological descent theorem in [LO08, Ex 2.2.5] for quasi-coherent sheaves in the lisse-étale site of an algebraic stack (Proposition A.4), and then we use it to prove a new descent theorem (Propositions A.18) that is needed in this paper. In this section, if $\mathcal{X}$ is an algebraic stack, we use $\mathrm{Le}(\mathcal{X})$ to denote the lisse-étale site, and if U is an algebraic space, we use $\mathrm{Et}(U)$ to denote its small étale site ([Stacks, Tag 03ED]).

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A.1. Morphisms from étale to lisse-étale sites

If U is an algebraic space and $m : U \to \mathcal{X}$ is a smooth morphism, there is an induced functor of sites $\mathrm{Et}(U) \to \mathrm{Le}(\mathcal{X})$ (also denoted m) that sends a scheme V with an étale map $V \to U$ to the composition $V \to U \to \mathcal{X}$.

Remark A.1. We make the following observations about the functor m:

1. The functor $m : \mathrm{Et}(U) \to \mathrm{Le}(\mathcal{X})$ is cocontinuous and hence induces a morphism of topoi $m : U_{\mathrm{et}} \to \mathcal{X}_{\text{lis-et}}$ by [Stacks, Tag 00XI]. The functor $m^{-1} : \mathcal{X}_{\text{lis-et}} \to U_{\mathrm{et}}$ is just restriction.
2. Since m^{-1} is restriction, we have $m^{-1}\mathcal{O}_{\mathcal{X}} = \mathcal{O}_U$ and $m^{-1}\mathcal{F} = m^*\mathcal{F}$ when $\mathcal{F}$ is a sheaf of $\mathcal{O}_{\mathcal{X}}$-modules.
3. The functor $m : \mathrm{Et}(U) \to \mathrm{Le}(\mathcal{X})$ is also continuous, and hence m^{-1} has a left adjoint by [Stacks, Tag 04BG]. Since m commutes with fibre products and equalisers, the left adjoint is exact by [Stacks, Tag 04BH]. In particular, m^{-1} preserves injectives.

Suppose we have the following commuting diagram of algebraic stacks where V and U are algebraic spaces and m is smooth:

$$\begin{tikzcd} V \arrow[r, "f'"] \arrow[d, "m'"'] & U \arrow[d, "m"] \\ X \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

4. If f is representable and $V = U \times_{\mathcal{X}} X$, then for $\mathcal{F} \in X_{\text{lis-et}}$, we have a canonical identification $f'_* m'^{-1}\mathcal{F} = m^{-1} f_*\mathcal{F}$, where $f'_* : V_{\mathrm{et}} \to U_{\mathrm{et}}$ (respectively, $f_* : X_{\text{lis-et}} \to \mathcal{X}_{\text{lis-et}}$) is the usual pushforward of étale (respectively, lisse-étale) sheaves induced by a continuous functor of sites. (Note that f_* may not have an exact left adjoint.) Indeed, if W is a scheme and $W \to U$ is étale, then we have

$$(f'_* m'^{-1}\mathcal{F})(W) = \mathcal{F}(W \times_U V) \qquad\qquad (m^{-1} f_*\mathcal{F})(W) = \mathcal{F}(W \times_{\mathcal{X}} X)$$

   but there is a natural identification of algebraic spaces $W \times_U V \simeq W \times_{\mathcal{X}} X$.
5. If f is smooth, then f_* has an exact left adjoint, and we let $f^* : \mathrm{Mod}(\mathcal{O}_{\mathcal{X}_{\text{lis-et}}}) \to \mathrm{Mod}(\mathcal{O}_{X_{\text{lis-et}}})$ be the induced pullback of $\mathcal{O}$-modules. In this case, $f'^* m'^* \mathcal{F} = m^* f^* \mathcal{F}$ for $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X}_{\text{lis-et}})$. Indeed, by part (2) above (since f is representable), the functors m^* and $m'^* f^*$ are just restriction, but f'^* is the pullback functor from $\mathrm{QCoh}(U_{\mathrm{et}})$ to $\mathrm{QCoh}(U_{X,\mathrm{et}})$. Hence the desired equality holds by the Cartesian property of $\mathcal{F}$.

A.2. The first descent theorem

In this section, we recall Lazslow-Olsson's theorem for unbounded cohomological descent for lisse-étale sheaves on an algebraic stack (Proposition A.4). To begin, we recall the following general construction (which will be used multiple times in this appendix).

Construction A.2. *Let I be a category, and let $\mathcal{C}$ be a functor from I to the 2-category of categories (see [Stacks, Tag 003N]); that is, for each $i \in I$, we have a category $\mathcal{C}_i$, and for each morphism $\phi : i \to j$ in I, we have a functor $\phi^*_{\mathcal{C}} : \mathcal{C}_i \to \mathcal{C}_j$, and these are compatible with compositions. We define a category of systems $\mathcal{C}_{total}$ whose objects are tuples $\mathcal{F} := (\mathcal{F}_i, \mathcal{F}(\phi))$ with $\mathcal{F}_i \in \mathcal{C}_i$ and $\mathcal{F}(\phi) : \phi^*_{\mathcal{C}}\mathcal{F}_i \to \mathcal{F}_j$, such that the following diagrams commute:*

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$$\begin{tikzcd} \phi_C^*\psi_C^*\mathcal{F}_\ell \arrow[rr, "\mathcal{F}(\psi\circ\phi)"] \arrow[dr, "\phi_C^*\mathcal{F}(\psi)"'] & & \mathcal{F}_n \\ & \phi_C^*\mathcal{F}_m \arrow[ur, "\mathcal{F}(\phi)"'] & \end{tikzcd}$$

A morphism from $(\mathcal{F}_i, \mathcal{F}(\phi))$ *to* $(\mathcal{G}_i, \mathcal{G}(\phi))$ *in* $\mathcal{C}_{total}$ *is a collection of morphisms* $\alpha_i : \mathcal{F}_i \to \mathcal{G}_i$ *compatible with the* $\mathcal{F}(\phi)$ *and* $\mathcal{G}(\phi)$*. The category of Cartesian systems* $\mathcal{C}_{total}^{cart}$ *is the full subcategory of* $\mathcal{C}_{total}$ *whose objects have the property that every* $\mathcal{F}(\phi)$ *is an isomorphism.*

Remark A.3. Suppose we are given two functors $\mathcal{C}, \mathcal{D}$ from I to the 2-category of categories, and suppose we have functors $\Lambda_i : \mathcal{C}_i \to \mathcal{D}_i$ such that the squares

$$\begin{tikzcd} \mathcal{C}_i \arrow[r, "\Lambda_i"] \arrow[d, "\phi_C^*"] & \mathcal{D}_i \arrow[d, "\phi_D^*"] \\ \mathcal{C}_j \arrow[r, "\Lambda_j"] & \mathcal{D}_j \end{tikzcd}$$

2-commute and the 2-morphisms respect (vertical) compositions of squares. Then we have a functor $\Lambda : \mathcal{C}_{total} \to \mathcal{D}_{total}$ given by the rule $\Lambda(\mathcal{F}_i, \mathcal{F}(\phi)) = (\Lambda_i(\mathcal{F}_i), \Lambda_j(\mathcal{F}(\phi)))$.

Let $\mathcal{X}$ be an algebraic stack, and let $U \to \mathcal{X}$ be a smooth cover by an algebraic space. Let $U_\bullet$ be the simplicial algebraic space that is equal to the 0-coskeleton of $U \to \mathcal{X}$. We apply Construction A.2 to the category $I := \Delta^+$, where Δ^+ is the subcategory of the simplicial category Δ with the same objects but only the injective morphisms. For $i \in \Delta^+$, we set $\mathcal{C}_i := U_{i,\mathrm{et}}$; and for $\phi : i \to j$, we let $\phi^* : U_{i,\mathrm{et}} \to U_{j,\mathrm{et}}$ be the usual inverse image functor for this morphism of topoi. The resulting category of systems is called the *strictly simplicial topos* in [Ols07, Sec 2.1] and [LO08, Ex 2.1.5], and we notate it $U^+_{\bullet,\mathrm{et}}$. The structure sheaves $\mathcal{O}_{U_i}$ define a distinguished ring object $\mathcal{O}_{U^+_\bullet}$ in $U^+_{\bullet,\mathrm{et}}$. A *quasi-coherent sheaf* in $U^+_{\bullet,\mathrm{et}}$ is an $\mathcal{O}_{U^+_\bullet}$-module $(\mathcal{F}_i, \mathcal{F}(\phi))$ such that each $\mathcal{F}_i$ is in $\mathrm{QCoh}(U_{i,\mathrm{et}})$ and the morphism $\phi^*\mathcal{F}_i \otimes_{\phi^*\mathcal{O}_{U_i}} \mathcal{O}_{U_j} \to \mathcal{F}_j$ induced by $\mathcal{F}(\phi^*)$ is an isomorphism. Observe that the category of quasi-coherent sheaves $\mathrm{QCoh}(U^+_{\bullet,\mathrm{et}})$ is equal to the category of Cartesian systems with $\mathcal{C}_i = \mathrm{QCoh}(U_{i,\mathrm{et}})$ and ϕ^* equal to the usual pullback of quasi-coherent sheaves.

There is a functor $\varpi^* : \mathrm{Mod}(\mathcal{O}_{\mathcal{X}_{\text{lis-et}}}) \to \mathrm{Mod}(\mathcal{O}_{U^+_{\bullet,\mathrm{et}}})$ given as follows: for $\mathcal{F} \in \mathcal{X}_{\text{lis-et}}$, set $(\varpi^*\mathcal{F})_i = m_i^{-1}\mathcal{F} \otimes_{m_i^{-1}\mathcal{O}_\mathcal{X}} \mathcal{O}_{U_i}$, where $m_i^{-1} : \mathcal{X}_{\text{lis-et}} \to U_{i,\mathrm{et}}$ is defined using the projection $U_i \to \mathcal{X}$ and Remark A.1.1, and let $\mathcal{F}(\phi)$ be the identity for each ϕ. Note that ϖ^* is exact and sends quasi-coherent sheaves to quasi-coherent sheaves. The following proposition is due to Laszo-Olsson.

Proposition A.4 (Laszlo-Olsson). *The morphism* $\varpi^* : \mathrm{QCoh}(\mathcal{X}_{\text{lis-et}}) \to \mathrm{QCoh}(U^+_{\bullet,\mathrm{et}})$ *is an exact equivalence of categories. We use* ϖ_* *to denote the quasi-inverse. Moreover,* $\varpi^* : \mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}}) \to \mathrm{D}_{\mathrm{qc}}(U^+_{\bullet,\mathrm{et}})$ *is an equivalence, and we use* $\mathsf{R}\varpi_*$ *to denote the quasi-inverse.*

Remark A.5. The equivalence of categories of quasi-coherent sheaves is proved in [Ols16, Prop 9.2.13]. The equivalence of unbounded derived categories is proved in [LO08, Ex 2.2.5] (using [Ols07, Thm 6.14]) under the assumption that $\mathcal{X}$ is quasi-separated (a standing assumption for both [Ols07] and [LO08]). This assumption is not needed for Proposition A.4. Indeed, [LO08, Thm 2.2.3] appears as [Stacks, Tag 0D7V] without the quasi-separated hypothesis, and one may check directly that the necessary portions of [Ols07] (namely Proposition 4.4, Lemma 4.5 and Lemma 4.8) do not use this hypothesis.

Remark A.6. The equivalences (ϖ^*, ϖ_*) are functorial as follows. Let $X \to \mathcal{X}$ be a smooth morphism of algebraic stacks (inducing a morphism of lisse-étale topoi), let $V \to X$ be a smooth surjective morphism from a scheme V, and let $V \to U$ be a morphism commuting with maps to $\mathcal{X}$. It follows from Remark A.1.5 that there is an identification $f^*_{\bullet,\mathrm{et}}\varpi^* \simeq \varpi^* f^*$ (where $f^*_{\bullet,\mathrm{et}}$ is given by f_i^* at level i), and since ϖ^* is an equivalence, we also have $\varpi_* f^*_{\bullet,\mathrm{et}} \simeq f^*\varpi_*$.

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A.3. The second descent theorem: hypercovers

In this section, we prove an unbounded cohomological descent theorem in the lisse-étale topology for very smooth hypercovers of algebraic stacks (Proposition A.18).

A.3.1. **Very smooth hypercovers**

Recall that if $\mathcal{U} \to \mathcal{X}$ and $\mathcal{V} \to \mathcal{X}$ are representable morphisms of algebraic stacks, then the category $\mathrm{Hom}_{\mathcal{X}}(\mathcal{U}, \mathcal{V})$ is isomorphic to a set.

Definition A.7. *The enlarged smooth site* $\mathrm{Es}(\mathcal{X})$ *of* $\mathcal{X}$ *is the category with objects given by morphisms* $f : \mathcal{U} \to \mathcal{X}$*, where* $\mathcal{U}$ *is an algebraic stack and* f *is smooth and representable, and with arrows from* $\mathcal{U} \to \mathcal{X}$ *to* $\mathcal{V} \to \mathcal{X}$ *given by the set* $\mathrm{Hom}_{\mathcal{X}}(\mathcal{U}, \mathcal{V})$*. A covering is a set of smooth maps* $\{\mathcal{U}_i \to \mathcal{U}\}_{i\in I}$ *that are jointly surjective.*

Remark A.8. The site $\mathrm{Es}(\mathcal{X})$ contains $id : \mathcal{X} \to \mathcal{X}$ as the final object.

Remark A.9. The morphisms in $\mathrm{Es}(\mathcal{X})$ are all representable.

Definition A.10. *A smooth hypercover of* $\mathcal{X}$ *is a simplicial object* $X_\bullet$ *in* $\mathrm{Es}(\mathcal{X})$ *such that*

1. $X_0 \to \mathcal{X}$ *is surjective (note that it will also be smooth).*
2. $X_{n+1} \to (\mathrm{cosk}_n\mathrm{sk}_n X_\bullet)_{n+1}$ *is smooth and surjective for* $n \geq 0$.

Remark A.11. A smooth hypercover of $\mathcal{X}$ is a hypercover of the final object in $\mathrm{Es}(\mathcal{X})$ in the sense of [Stacks, Tag 01G5]. Moreover, if $\mathcal{X}$ is an algebraic space, then a smooth hypercover of $\mathcal{X}$ is also an fppf hypercover in the sense of [Stacks, Tag 0DH4].

Definition A.12. *A very smooth hypercover of* $\mathcal{X}$ *is a smooth hypercover* $X_\bullet$ *such that every degeneracy and face map* $X_i \to X_j$ *is smooth.*

If $X_\bullet$ is a smooth hypercover of $\mathcal{X}$ and $f : \mathcal{Y} \to \mathcal{X}$ is a morphism of algebraic stacks, we can pullback $X_\bullet$ to a simplicial object $Y_\bullet$ in $\mathrm{Es}(\mathcal{Y})$: define $Y_i = X_i \times_{\mathcal{X}} \mathcal{Y}$.

Remark A.13. If $\mathcal{Y} \to \mathcal{X}$ is a morphism of algebraic stacks and $X_\bullet$ is a (very) smooth hypercover of $\mathcal{X}$, then $Y_\bullet$ is a (very) smooth hypercover of $\mathcal{Y}$. This follows from [Stacks, Tag 0DAZ].

Remark A.14. From [Stacks, Tag 0DEQ] and the proof of [Stacks, Tag 0DAV], it follows that if $\mathcal{X}$ is an algebraic stack, then a very smooth hypercover of $\mathcal{X}$ exists. In fact, we may take X_i to be a disjoint union of affine schemes.

Remark A.15. Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then we can find very smooth hypercovers $X_\bullet \to \mathcal{X}$ and $Y_\bullet \to \mathcal{Y}$ with X_i and Y_i disjoint unions of affine schemes, with a morphism $X_\bullet \to Y_\bullet$ commuting with the augmentations and the given morphism $\mathcal{X} \to \mathcal{Y}$. This follows from analysing the construction of $X_\bullet$ and $Y_\bullet$ in [Stacks, 0DAV], using the fact that the functors cosk_n are finite limits and hence commute with pullback (see the proof of [Stacks, Tag 0DAZ]).

A.3.2. **The lisse-etale topos of a very smooth hypercover**

Recall that if $\mathcal{X} \to \mathcal{Y}$ is a smooth morphism of algebraic stacks, then there is a morphism of sites $\mathrm{Le}\,\mathcal{X} \to \mathrm{Le}\,\mathcal{Y}$ (see, e.g., [Stacks, Tag 00X1] and [Ols07, Sec 3.3]) and in fact a morphism of ringed topoi $(\mathcal{X}_{\text{lis-et}}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}_{\text{lis-et}}, \mathcal{O}_{\mathcal{Y}})$. We follow [Stacks, Tag 09WB] by defining the category of sites to be the category whose objects are sites and whose morphisms are morphisms of sites. If $\mathcal{C}_\bullet$ is a simplicial object in this category, then for each morphism $\varphi : [i] \to [j]$ of the simplicial category Δ, we have a morphism of sites $f_\varphi : \mathcal{C}_i \to \mathcal{C}_j$.

Definition A.16. *Let* $X_\bullet$ *be a very smooth hypercover of* $\mathcal{X}$*. We construct an associated site* $\mathrm{Le}(X_\bullet)$ *as follows: let* $\mathcal{C}_\bullet$ *be the simplicial object in the category of sites with* $\mathcal{C}_i := \mathrm{Le}(X_i)$ *and* f_φ *equal to the given morphism of sites (it is important that all the face and degeneracy maps are smooth). Define* $\mathrm{Le}(X_\bullet)$ *to be the site* $\mathcal{C}_{total}$ *in [Stacks, Tag 09WC], and use* $X_{\bullet,\text{lis-et}}$ *to denote the corresponding topos.*

Remark A.17. By [Stacks, Tag 09WF], a sheaf on $\mathrm{Le}(X_\bullet)$ is given by a system $(\mathcal{F}_i, \mathcal{F}(\varphi))$, where $\mathcal{F}_i$ is a sheaf on $\mathrm{Le}(X_i)$ and $\mathcal{F}(\varphi) : f_\varphi^{-1}\mathcal{F}_i \to \mathcal{F}_j$ are compatible morphisms.

Using Remark A.17, define a sheaf $\mathcal{O}_{X_{\bullet,\text{lis-et}}}$ on $X_{\bullet,\text{lis-et}}$ to be the sheaf equal to $\mathcal{O}_{X_i}$ on X_i with transition maps induced by the morphisms of ringed topoi already given. This makes $X_{\bullet,\text{lis-et}}$ a ringed site. An $\mathcal{O}_{X_{\bullet,\text{lis-et}}}$-module $\mathcal{F}$ on $X_{\bullet,\text{lis-et}}$ is quasi-coherent if for each i the sheaf $\mathcal{F}_i$ is a quasicoherent $\mathcal{O}_{X_i}$-module and if for each $\varphi : [i] \to [j]$ the induced maps

$$f_\varphi^{-1}\mathcal{F}_i \otimes_{f^{-1}\mathcal{O}_{X_i}} \mathcal{O}_{X_j} \to \mathcal{F}_j$$

are isomorphisms.

A.3.3. The descent theorem

For $X_\bullet$ a very smooth hypercover of $\mathcal{X}$, let $a_i : X_i \to \mathcal{X}$ denote the given (smooth) morphism of algebraic stacks.

The morphism $X_0 \to \mathcal{X}$ induces an augmentation of $\mathrm{Le}(X_\bullet)$ towards $\mathrm{Le}(\mathcal{X})$ in the sense of [Stacks, Tag 0D6Z]. By [Stacks, Tag 0D70], we get a morphism of topoi

$$a : X_{\bullet,\text{lis-et}} \to \mathcal{X}_{\text{lis-et}} \tag{A.1}$$

such that $a^{-1}\mathcal{F}$ is given by the system with $(a^{-1}\mathcal{F})_i := a_i^{-1}\mathcal{F}$ and the natural transition maps (they are all isomorphisms), and $a_*\mathcal{G}$ is given by the equaliser of the two maps $a_{0*}\mathcal{G}_0 \to a_{1*}\mathcal{G}_1$.

Using the maps $a_i^{-1}\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{X_i}$, we get a morphism $a^{-1}\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{X_{\bullet,\text{lis-et}}}$ that makes a a morphism of ringed topoi. Define $a^* : \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}) \to \mathrm{Mod}(\mathcal{O}_{X_{\bullet,\text{lis-et}}})$ by

$$a^*\mathcal{F} := a^{-1}\mathcal{F} \otimes_{a^{-1}\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{X_{\bullet,\text{lis-et}}}.$$

It is clear that a^* is exact and sends $\mathrm{QCoh}(\mathcal{X}_{\text{lis-et}})$ to $\mathrm{QCoh}(X_{\bullet,\text{lis-et}})$.

Proposition A.18. *Let $X_\bullet \to \mathcal{X}$ be a very smooth hypercover. Then*

$$a^* : \mathrm{QCoh}(\mathcal{X}_{\text{lis-et}}) \to \mathrm{QCoh}(X_{\bullet,\text{lis-et}}) \tag{A.2}$$

is an equivalence of categories with quasi-inverse a_*. Moreover, the functors $\mathsf{R}a_*$ and a^* are inverse equivalences of $\mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}})$ and $\mathrm{D}_{\mathrm{qc}}(X_{\bullet,\text{lis-et}})$.

Proof. We first show that a^* is an equivalence of categories of quasi-coherent sheaves with quasi-inverse a_*. Let $U \to \mathcal{X}$ be a smooth map from an algebraic space U, and let $U^+_{\bullet,\text{et}}$ be the strictly simplicial étale topos defined in Section A.2. We apply Construction A.2 to the category $I = \Delta \times \Delta^+$. For $(i, j) \in \Delta \times \Delta^+$, we set $\mathcal{C}_{i,j} = \mathrm{QCoh}\left((X_i \times_{\mathcal{X}} U_j)_{\text{et}}\right)$ (observe that the fibre product is an algebraic space), and we let $\phi^* : (X_i \times_{\mathcal{X}} U_j)_{\text{et}} \to (X_k \times_{\mathcal{X}} U_\ell)_{\text{et}}$ be the usual pullback of quasi-coherent sheaves. Let $\mathrm{QCoh}((X_\bullet \times_{\mathcal{X}} U^+_\bullet)_{\text{et}})$ denote the resulting category of Cartesian systems.

Let $U_{X_i} = X_i \times_{\mathcal{X}} U$. By viewing $\mathrm{QCoh}(X_{\bullet,\text{lis-et}})$ and $\mathrm{QCoh}((X_\bullet \times_{\mathcal{X}} U^+_\bullet)_{\text{et}})$ both as categories of systems with $I = \Delta$, we define functors

$$\mathrm{QCoh}(X_{\bullet,\text{lis-et}}) \underset{\varpi_{\bullet,*}}{\overset{\varpi_\bullet^*}{\rightleftarrows}} \mathrm{QCoh}((X_\bullet \times_{\mathcal{X}} U^+_\bullet)_{\text{et}})$$

induced via Remark A.3 by the inverse equivalences

$$\mathrm{QCoh}(X_{i,\text{lis-et}}) \underset{\varpi_*}{\overset{\varpi^*}{\rightleftarrows}} \mathrm{QCoh}(U^+_{X_i,\bullet,\text{et}})$$

of Proposition A.4. The rules $\varpi_\bullet^*$ and $\varpi_{*,\bullet}$ are indeed functors of categories of systems by Remark A.6, and one checks that they are inverse equivalences.

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Similarly, let $X_{U_i,\bullet}$ be the pullback of the hypercover $X_\bullet \to \mathcal{X}$ to U_i as in Remark A.13. By viewing $U^+_{\bullet,\mathrm{et}}$ and $\mathrm{QCoh}((X_\bullet \times_{\mathcal{X}} U^+_\bullet)_{\mathrm{et}})$ as categories of systems with $I = \Delta^+$, we define functors

$$\mathrm{QCoh}(U^+_{\bullet,\mathrm{et}}) \underset{a_{\bullet,*}}{\overset{a^*_\bullet}{\rightleftarrows}} \mathrm{QCoh}((X_\bullet \times_{\mathcal{X}} U^+_\bullet)_{\mathrm{et}})$$

induced via Remark A.3 by the functors

$$\mathrm{QCoh}(U_{i,\mathrm{et}}) \underset{a_*}{\overset{a^*}{\rightleftarrows}} \mathrm{QCoh}(X_{U_i,\bullet,\mathrm{et}}) \tag{A.3}$$

defined in analogy with equation (A.1) above. The functors a_* and a^* in equation (A.3) are inverse equivalences by [Stacks, Tag 0DHD].

We have constructed a diagram

$$\begin{tikzcd}
\mathrm{QCoh}(X_{\bullet,\text{lis-et}}) \arrow[r, shift left, "\varpi^*_\bullet"] \arrow[d, shift right, "a_*"'] & \mathrm{QCoh}((X_\bullet \times_{\mathcal{X}} U^+_\bullet)_{\mathrm{et}}) \arrow[l, shift left, "\varpi_{\bullet,*}"] \arrow[d, shift right, "a_{\bullet,*}"'] \\
\mathrm{QCoh}(\mathcal{X}_{\text{lis-et}}) \arrow[u, shift right, "a^*"'] \arrow[r, shift left, "\varpi^*"] & \mathrm{QCoh}(U^+_{\bullet,\mathrm{et}}) \arrow[l, shift left, "\varpi_*"] \arrow[u, shift right, "a^*_\bullet"']
\end{tikzcd}$$

where three of the four pairs of morphisms are known to be inverse equivalences. It follows from Remark A.1.5 that $\varpi^*_\bullet a^* = a^*_\bullet \varpi^*$, so a^* is an equivalence with inverse $\varpi_* a_{\bullet,*} \varpi^*_\bullet$. Using Remark A.1.5 and the fact that ϖ^* is exact, one can check that $a_{\bullet,*}\varpi^*_\bullet = \varpi^* a_*$, so $a_* = \varpi_* a_{\bullet,*} \varpi^*_\bullet$. This shows that a^* and a_* are inverse equivalences of quasi-coherent sheaves.

To finish the proof of the Proposition, we use [Stacks, Tag 0D7V]. To do so, we must verify its five hypotheses. The category $\mathrm{QCoh}(X_{\bullet,\text{lis-et}})$ is a weak Serre subcategory of $\mathrm{Mod}(\mathscr{O}_{X_{\bullet,\text{lis-et}}})$, and conditions (1), (4), and (5) of [Stacks, Tag 0D7V] hold as in the proof of [Stacks, Tag 0DHF]. Condition (2) is the inverse equivalence of a^* and a_* that we just proved. The final condition, number (3), is the statement that for $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X}_{\text{lis-et}})$, the unit $\mathcal{F} \to \mathsf{R}a_*a^*\mathcal{F}$ is an isomorphism. Since we already know $\mathcal{F} \to a_*a^*\mathcal{F}$ is an isomorphism, it suffices to show $\mathsf{R}^n a_* a^* \mathcal{F} = 0$ for $n > 0$.

For any smooth map $m : U \to \mathcal{X}$ from a scheme U, let $U_\bullet \to U$ be the very smooth hypercover equal to the pullback of $X_\bullet$. We have a diagram of morphisms of topoi

$$\begin{tikzcd}
U_{\bullet,\mathrm{et}} \arrow[d, "a''"] & U_{\bullet,\text{lis-et}} \arrow[l] \arrow[r, "m_{\bullet,\text{lis-et}}"] \arrow[d, "a'"] & X_{\bullet,\text{lis-et}} \arrow[d, "a"] \\
U_{\mathrm{et}} & U_{\text{lis-et}} \arrow[l] \arrow[r, "m_{\text{lis-et}}"] & \mathcal{X}_{\text{lis-et}}
\end{tikzcd}$$

where the site $U_{\bullet,\mathrm{et}}$ is constructed with [Stacks, Tag 09WC] and a'' and a' are defined as in equation (A.1). The top horizontal morphisms come from [Stacks, Tag 0DH0]. It follows from [73, V.5.1(1)], [Ols07, Lem 3.5] and Lemma A.19 that $\mathsf{R}^n a_* a^* \mathcal{F}$ is the sheafification of the presheaf that associates to an smooth map $m : U \to \mathcal{X}$ from a scheme U the group $H^n(U_{\bullet,\text{lis-et}}, m^{-1}_{\bullet,\text{lis-et}} a^*\mathcal{F})$. Since restriction to the étale site is exact and preserves injectives, this is equal to $H^i(U_{\bullet,\mathrm{et}}, m^*_\bullet a^* \mathcal{F})$, where $m_\bullet : U_{\bullet,\mathrm{et}} \to X_{\bullet,\text{lis-et}}$ is defined as in Remark A.1. Finally, by Remark A.1.5 (since $\mathcal{F}$ is quasi-coherent), this equals $H^n(U_{\bullet,\mathrm{et}}, a''^* m^* \mathcal{F})$.

On the other hand, it follows from [73, V.5.1(1)] and Lemma A.19 that $\mathsf{R}^n a''_* a''^*(m^*\mathcal{F})$ is the sheafification of the presheaf that associates to an étale map $f : V \to U$ from a scheme V the group $H^n(V_{\bullet,\mathrm{et}}, a''^* m^* \mathcal{F})$. This group is equal to $H^i(V_{\bullet,\mathrm{et}}, a''^*_V (m \circ f)^* \mathcal{F})$, where $a''_V : V_{\bullet,\mathrm{et}} \to V_{\mathrm{et}}$ is the usual morphism in equation (A.1). It follows that if $m : U \to \mathcal{X}$ is a smooth cover by a scheme, the étale sheaves $m^*(\mathsf{R}^n a_* a^* \mathcal{F})$ and $\mathsf{R}^n a''_* a''^*(m^* \mathcal{F})$ are the sheafification of the same presheaf, hence isomorphic. But it follows from [Stacks, Tag 0DHE] that $\mathsf{R}^n a''_* a''^*(m^*\mathcal{F}) = 0$. □

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Lemma A.19. *Consider a fibre square of algebraic stacks*

$$\begin{tikzcd} U_X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "b"] \\ U \arrow[r, "m"] & \mathcal{Y} \end{tikzcd}$$

such that U is an algebraic space.

1. *If m and b are smooth and representable and $\tilde{U}$ is the sheaf represented by U on $\mathcal{X}_{\text{lis-et}}$, then $b^{-1}\tilde{U}$ is represented by U_X.*
2. *If $\mathcal{Y}$ is representable, m is étale, and b is representable, and if $\tilde{U}$ is the sheaf represented by U on $\mathcal{X}_{\text{et}}$, then the étale sheaf $b^{-1}\tilde{U}$ is represented by U_X.*

Proof. We first sketch the proof of (1). The sheaf $b^{-1}\tilde{U}$ is the sheafification of the presheaf that assigns to a scheme T with a smooth map $g : T \to \mathcal{X}$ the set $\operatorname{colim} \operatorname{Hom}_Y(W, U)$, where the colimit is taken over schemes W fitting into diagrams

$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d] & \mathcal{X} \arrow[d, "b"] \\ W \arrow[r, "\text{smooth}"] & \mathcal{Y} \end{tikzcd} \tag{A.4}$$

Composition induces a map

$$\operatorname{colim} \operatorname{Hom}_{\mathcal{Y}}(W, U) \to \operatorname{Hom}_{\mathcal{Y}}(T, U), \tag{A.5}$$

which is an isomorphism since $T \to Y$ is smooth and hence defines the final object in the category over which we take the colimit. Finally, we note that $\operatorname{Hom}_Y(T, U) = \operatorname{Hom}_Y(T, U_X)$. For (2), the map $T \to \mathcal{X}$ is now étale, and the colimit is over diagrams (A.4) with $W \to \mathcal{Y}$ étale, so $T \to \mathcal{Y}$ is not an object of the colimit category. However, the map in equation (A.5) is still surjective. It is injective as well because an element of $\operatorname{Hom}_{\mathcal{Y}}(W, U)$ must be étale, so if we have elements of $\operatorname{Hom}_{\mathcal{Y}}(W_1, U)$ and $\operatorname{Hom}_{\mathcal{Y}}(W_2, U)$ that yield the same map $T \to U$, we may compare them via the étale U-scheme $W_1 \times_U W_2$. □

Remark A.20. Let $X_\bullet \to \mathcal{X}$ and $Y_\bullet \to \mathcal{Y}$ be very smooth hypercovers of algebraic stacks $\mathcal{X}$ and $\mathcal{Y}$, and suppose we are given a morphism $f_\bullet : X_\bullet \to Y_\bullet$ of simplicial algebraic stacks and $f : \mathcal{X} \to \mathcal{Y}$ such that these maps commute with the augmentations. Then for $\mathcal{F} \in \mathsf{D}_{\text{qc}}(\mathcal{Y}_{\text{lis-et}})$, we have

$$(a^*\mathsf{L}f^*\mathcal{F})|_{X_n} = \mathsf{L}f_n^*(a^*\mathcal{F})|_{X_n},$$

and if f is concentrated and $\mathcal{G} \in \mathsf{D}_{\text{qc}}(\mathcal{X}_{\text{lis-et}})$, then

$$(a^*\mathsf{R}f_*\mathcal{G})|_{Y_n} = \mathsf{R}f_{n*}(a^*\mathcal{G})|_{Y_n},$$

where the functor $\mathsf{L}f_n^*$ (respectively, $\mathsf{R}f_{n*}$) is the usual pullback functor (respectively, direct image) between $\mathsf{D}_{\text{qc}}(Y_{n,\text{lis-et}})$ and $\mathsf{D}_{\text{qc}}(X_{n,\text{lis-et}})$. Indeed, the functor $(a^*-)|_{X_n}$ is just $a_n^*(-)$, so the desired equalities are equivalent to

$$a_n^*\mathsf{L}f^*\mathcal{F} = \mathsf{L}f_n^* a_n^*\mathcal{F} \qquad \text{and} \qquad a_n^*\mathsf{R}f_*\mathcal{G} = \mathsf{R}f_{n*}a_n^*\mathcal{G}.$$

These follow from the naturality of derived pullback and [HR17, Cor 4.13], respectively.

B. Functoriality of the Fundamental Theorem

In this section, we prove Lemmas 4.5 and 4.6.

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B.1. Categories of algebra extensions

In this section, $\mathcal{S}$ is a site with $A \to B$ a morphism of sheaves of rings on $\mathcal{S}$, and I is a sheaf of B-modules.

B.1.1. Categories

The Picard category $\underline{\mathrm{Exal}}_A(B, I)$ was defined in [Ill71, Sec III.1.1.2.3]: an object is a surjective A-algebra map $E \to B$ whose kernel is (1) square-zero as an ideal of E, and (2) isomorphic to I as a B-module. We write these objects as short exact sequences of abelian sheaves

$$0 \to I \to E \to B \to 0. \tag{B.1}$$

A morphism in $\underline{\mathrm{Exal}}_A(B, I)$ is a commuting diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "g"] & B \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & I \arrow[r] & E' \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

where f is a morphism of B-modules and g, h are morphisms of A-algebras.

B.1.2. Functors

If $I \to I'$ is a morphism of B-modules, $B' \to B$ is a morphism of A-algebras and $A' \to A$ is a morphism of rings, then we have natural functors

$$\underline{\mathrm{Exal}}_A(B, I) \to \underline{\mathrm{Exal}}_A(B, I') \tag{B.2}$$

$$\underline{\mathrm{Exal}}_A(B, I) \to \underline{\mathrm{Exal}}_A(B', I_{B'}) \tag{B.3}$$

$$\underline{\mathrm{Exal}}_A(B, I) \to \underline{\mathrm{Exal}}_{A'}(B, I) \tag{B.4}$$

defined in [Ill71, Equ III.1.1.5.2], [Ill71, Equ III.1.1.5.3] and [Ill71, Equ III.1.1.5.4], respectively. Here, $I_{B'}$ denotes the sheaf I considered as a B'-module. Let $\mathcal{S}' \to \mathcal{S}$ be a continuous morphism of sites inducing a morphism of topoi (p^{-1}, p_*). Then we have an induced morphism

$$\underline{\mathrm{Exal}}_A(B, I) \to \underline{\mathrm{Exal}}_{p^{-1}A}(p^{-1}B, p^{-1}I) \tag{B.5}$$

sending equation (B.1) to its image under p^{-1}. We are using that p^{-1} is an exact functor.

Lemma B.1. *The morphisms in equations (B.2), (B.3), (B.4) and (B.5) commute pairwise.*

Proof. The most involved pair to check is equations (B.2) and (B.3). We work it out in detail and offer a few words about the remaining pairs at the end of the proof. When we say equations (B.2) and (B.3) commute, we mean if $B' \to B$ is a morphism of rings and $I \to I'$ is a morphism of B-modules; then the diagram

$$\begin{tikzcd} \underline{\mathrm{Exal}}_A(B, I) \arrow[r, "(B.3)"] \arrow[d, "(B.2)"] & \underline{\mathrm{Exal}}_A(B', I_{B'}) \arrow[d, "(B.2)"] \\ \underline{\mathrm{Exal}}_A(B, I') \arrow[r, "(B.3)"] & \underline{\mathrm{Exal}}_A(B', I'_{B'}) \end{tikzcd} \tag{B.6}$$

commutes up to a natural transformation.

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Given an element (B.1) of $\underline{\mathrm{Exal}}_A(B, I)$, we have a diagram

$$\begin{tikzcd}
 & I' \arrow[r, "a'"] & P \arrow[dr, dashed, "{(0,b)}"] & & \\
0 \arrow[r] & I \arrow[u] \arrow[r, "a"] \arrow[dr, dashed, "{(a,0)}"'] & E \arrow[u, "\iota_E"] \arrow[r, "b"] & B \arrow[r] & 0 \\
 & & F \arrow[u, "p_E"'] \arrow[r, "b'"] & B' \arrow[u] &
\end{tikzcd} \tag{B.7}$$

where $P = I' \oplus_I E$ and $F = E \times_B B'$: as abelian groups, P and F are the colimit and limit of the usual diagrams, while the ring structures are described in [GD67, 0_{IV}.18.2.8] and [GD67, 0_{IV}.18.1.5]. Set $Q = I' \oplus_I F$ and $G = P \times_B B'$. Then

$$0 \to I' \to Q \to B' \to 0$$

is the image of equation (B.1) under the composition $^{\rightarrow} \downarrow$ in diagram (B.6), and likewise G defines the image under the composition $\downarrow_{\rightarrow}$. An arrow from Q to G in the groupoid $\underline{\mathrm{Exal}}_A(B', I')$ is given by four dashed arrows so that this diagram commutes:

$$\begin{tikzcd}
I \arrow[r] \arrow[dr] & I' \arrow[r, dashed] \arrow[dr, dashed] & P \arrow[dr] & \\
 & F \arrow[r, dashed] \arrow[ur, dashed] & B' \arrow[r] & B
\end{tikzcd} \tag{B.8}$$

(To check commutativity, it suffices to check that the quadrilaterals $I'IFP$ and $I'IFB'$ and the perimeter commute.) The required collection of dotted arrows is given by $a' : I' \to P$, $0 : I' \to B'$, $\iota_E \circ p_E : F \to P$, and $b' : F \to B'$.

To show that the resulting arrows in $\underline{\mathrm{Exal}}_A(B', I')$ define a natural transformation (in this groupoid), suppose we are given an arrow

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "a_1"] \arrow[dr, "a_2"'] & E_1 \arrow[r, "b_1"] \arrow[d, "f"] & B \arrow[r] & 0 \\
 & & E_2 \arrow[ur, "b_2"'] & &
\end{tikzcd}$$

in $\underline{\mathrm{Exal}}_A(B, I)$. Let $f_P : P_1 \to P_2$ and $f_F : F_1 \to F_2$ be the maps induced by f, where P_i and F_i are defined as in equation (B.7). Likewise let Q_i and G_i be the images of E_i in $\underline{\mathrm{Exal}}_A(B', I')$ under the maps in diagram (B.6). We must compare two maps from Q_1 to G_2 in $\underline{\mathrm{Exal}}_A(B', I')$. Such maps are given by diagrams of the form in diagram (B.8), with F replaced by F_1 and P replaced by P_2. In the situation at hand, one of the maps from Q_1 to G_2 is given by the diagram

$$\begin{tikzcd}
I' \arrow[r, dashed, "f_P \circ a'_1"] \arrow[dr, dashed, "0"' near start] & P_2 \\
F_1 \arrow[r, dashed, "b'_1"'] \arrow[ur, dashed, "f_P \circ \iota_{E_1} \circ p_{E_1}" near start] & B'
\end{tikzcd}$$

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and the other is given by the diagram

$$\begin{tikzcd} I' \arrow[r, dashed, "a'_2"] \arrow[dr, dashed, "0" near start] & P_2 \\ F_1 \arrow[ur, dashed, "\iota_{E_2}\circ p_{E_2}\circ f_F" near start] \arrow[r, dashed, "b'_2\circ f_F"'] & B' \end{tikzcd}$$

These are easily seen to consist of the same morphisms.

This completes the proof that equations (B.3) and (B.2) commute. Of the remaining pairs, most of the checks are trivial (in particular, the analogue of diagram (B.6) is strictly commutative). Only the pairs (equations (B.2), (B.5)) and (equations (B.3), (B.5)) are nontrivial. For these, one uses that p^{-1} is exact and hence preserves finite limits and colimits. □

B.2. Illusie's theorem

B.2.1. Statement

In this section, $\mathscr{S}$ is a site with $A \longrightarrow B$ a morphism of sheaves of rings on $\mathscr{S}$, and I is a sheaf of B-modules.

Theorem B.2 [Ols06, Thm A.7], [Ill71, Sec III.1.2.2]. *There is a canonical isomorphism*

$$\beta : \underline{\mathrm{Exal}}_A(B, I) \to \mathrm{Ext}_B^{0/-1}(\mathbb{L}_{B/A}, I[1]), \tag{B.9}$$

where the right-hand side was defined in equation (4.1).

Proof. Since the isomophism in [Ols06, Thm A.7] is defined on groupoid fibres, we may use the same definition for our morphism in equation (B.9) (written out in the proof of Lemma B.5), and the argument in [Ols06, Thm A.7] shows that it is an isomorphism. Note that when $\mathscr{S}$ has a final object S, the map in equation (B.9) is the value on S of the isomorphism in [Ols06, Thm A.7]. □

B.2.2. Functoriality

We will show that equation (B.9) is compatible with the functors defined in Section B.1.2. We will use the following instances of Situation 2.1.

Example B.3. The following is an example of Situation 2.1. Let $\mathscr{S}$ be a site and B a sheaf of rings on $\mathscr{S}$. Then $\mathrm{D}(B)$ is a closed symmetric monoidal category with product $\overset{\mathsf{L}}{\otimes}_B$ and internal hom $\mathsf{R}\mathcal{H}om_B$. If $B' \to B$ is a flat morphism of sheaves of rings, then extension of scalars $-\otimes_{B'} B : B'\text{−mod} \to B\text{−mod}$ is strong monoidal and exact, with an exact right adjoint $(-)_{B'}$ given by restriction of scalars.

Let $\mathscr{C} = \mathrm{D}(B')$, and let $\mathscr{D} = \mathrm{D}(B)$. By [Stacks, Tag 0DVC], the functors $-\otimes_{B'}$ and $(-)_{B'}$ extend to an adjoint pair for $\mathrm{D}(B')$ and $\mathrm{D}(B)$, and by [Stacks, Tags 07A4, 08I6], the functor $-\otimes_{B'} B : \mathrm{D}(B') \to \mathrm{D}(B)$ is still strong monoidal.

In addition, it follows from [Stacks, Tags 08J9, 0A90, 0A5Y] that if $M^\bullet$ is K-flat and $N^\bullet$ is injective, the counit $\mathsf{R}\mathcal{H}om_{B'}(M, N) \overset{\mathsf{L}}{\otimes}_{B'} M \to N$ is given in degree n by a product over $p + q + r = n$ of the sheaf maps

$$\mathcal{H}om_{B'}(M^{-p}, N^q) \otimes_{B'} M^r \to N^n,$$

where this map is equal to the usual evaluation map if $q = n$ and it is zero otherwise. We will give an explicit description of equation (2.4) in the proof of Lemma B.6.

Example B.4. If $p : (\mathscr{C}, \mathscr{O}_\mathscr{C}) \to (\mathscr{D}, \mathscr{O}_\mathscr{D})$ is a flat morphism of ringed topoi given by an adjoint pair (p^{-1}, p_*), then p^* is exact and hence defines a strong monoidal functor $\mathrm{D}(\mathscr{O}_\mathscr{D}) \to \mathrm{D}(\mathscr{O}_\mathscr{C})$ with a right adjoint $\mathsf{R}p_*$. We will give an explicit description of equation (2.5) in the proof of Lemma B.7.

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Lemma B.5. *The isomorphism in (B.9) is functorial as follows:*

1. *Let $A \to B$ be a map of sheaves of rings on $\mathcal{S}$. If $I \to J$ is a morphism of B-modules, there is a commuting diagram*

$$\begin{tikzcd} \underline{\mathrm{Exal}}_A(B,I) \arrow[r, "\beta"] \arrow[d, "(B.2)"] & \mathrm{Ext}_B^{0/-1}(\mathbb{L}_{B/A}, I[1]) \arrow[d] \\ \underline{\mathrm{Exal}}_A(B,J) \arrow[r, "\beta"] & \mathrm{Ext}_B^{0/-1}(\mathbb{L}_{B/A}, J[1]) \end{tikzcd}$$

2. *If there is a commuting square of rings with $B' \to B$ flat*

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$

then the canonical map $\mathbb{L}_{B'/A'} \overset{\mathsf{L}}{\otimes}_{B'} B \to \mathbb{L}_{B/A}$ induces a commuting square

$$\begin{tikzcd} \underline{\mathrm{Exal}}_A(B,I) \arrow[rr, "\beta"] \arrow[d, "(B.3)\circ(B.4)"] & & \mathrm{Ext}_B^{0/-1}(\mathbb{L}_{B/A}, I[1]) \arrow[d] \\ \underline{\mathrm{Exal}}_{A'}(B', I_{B'}) \arrow[r, "\beta"] & \mathrm{Ext}_{B'}^{0/-1}(\mathbb{L}_{B'/A'}, I_{B'}[1]) \arrow[r, "(2.3)"] & \mathrm{Ext}_B^{0/-1}(\mathbb{L}_{B'/A'} \otimes_{B'} B, I[1]) \end{tikzcd} \tag{B.10}$$

where equation (2.3) is defined in the context of Example B.3.

3. *Let $(\mathcal{S}, \mathcal{O}_{\mathcal{S}}) \to (\mathcal{S}', \mathcal{O}_{\mathcal{S}'})$ be a continuous morphism of ringed sites inducing a flat morphism of topoi (p^{-1}, p_*). Let A and $B = \mathcal{O}_{\mathcal{S}}$ be sheaves of rings on $\mathcal{S}$. Then if I is a sheaf of B-modules, there is a commuting diagram*

$$\begin{tikzcd} \underline{\mathrm{Exal}}_A(B,I) \arrow[rr, "\beta"] \arrow[d, "(B.5)"] & & \mathrm{Ext}_B^{0/-1}(\mathbb{L}_{B/A}, I[1]) \arrow[d, "(2.5)"] \\ \underline{\mathrm{Exal}}_{p^{-1}A}(p^{-1}B, p^{-1}I) \arrow[r, "\beta"] & \mathrm{Ext}_{p^{-1}B}^{0/-1}(p^{-1}\mathbb{L}_{B/A}, p^{-1}I[1]) \arrow[r, "(2.5)"] & \mathrm{Ext}_{p^*B}^{0/-1}(p^*\mathbb{L}_{B/A}, p^*I[1]) \end{tikzcd} \tag{B.11}$$

where the horizontal instance of equation (2.5) is defined in the context of Example B.3 and the vertical instance of equation (2.5) is defined in the context of Example B.4, and we have suppressed an isomorphism induced by $\mathbb{L}_{p^{-1}B/p^{-1}A} \simeq p^{-1}\mathbb{L}_{B/A}$.

Proof. We summarise the definition of β; see [Ols06, Thm A.7] for more details. Let $P_\bullet$ be the simplicial A-algebra given by the standard free resolution of the A-algebra B [Stacks, Tag 08SR]. The morphism β is defined to be the composition

$$\underline{\mathrm{Exal}}_A(B,I) \xrightarrow{\beta_1} \underline{\mathrm{Exal}}_A(P_\bullet, I) \xrightarrow{\beta_2} \underline{\mathrm{Ext}}(\Omega_{P_\bullet/A}, I) \xrightarrow{\beta_3} \underline{\mathrm{Ext}}(\Omega_{P_\bullet/A} \otimes B, I) \xrightarrow{\beta_4} \mathrm{Ext}^{0/-1}(\mathbb{L}_{B/A}, I[1])$$

Here, $\underline{\mathrm{Ext}}(\Omega_\bullet, I)$ denotes the Picard category of simplicial $\mathcal{O}_{\mathcal{C}}$-module extensions of $\Omega_\bullet$ by I (viewed as a simplicial module); see [Ols06, Sec A.1]. The map β_1 is given by the map in equation (B.3) applied to the augmentation $P_\bullet \to B$, the morphism β_2 is given by taking differentials, β_3 is given by tensoring with B, and β_4 is the functorial isomorphism in [Ols06, Prop A.3].

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Proof of (1). The desired functoriality follows from a commuting diagram

$$\begin{tikzcd}
\underline{\mathrm{Exal}}_A(B, I) \arrow[r, "\beta_1"] \arrow[d, "(B.2)"] & \underline{\mathrm{Exal}}_A(P_\bullet, I) \arrow[r, "\beta_2"] \arrow[d, "(B.2)"] & \underline{\mathrm{Ext}}(\Omega_{P_\bullet/A}, I) \arrow[r, "\beta_3"] \arrow[d] & \underline{\mathrm{Ext}}(\Omega_{P_\bullet/A} \otimes B, I) \arrow[r, "\beta_4"] \arrow[d] & \mathrm{Ext}^{0/-1}(\mathbb{L}_{B/A}, I[1]) \arrow[d] \\
\underline{\mathrm{Exal}}_A(B, J) \arrow[r, "\beta_1"] & \underline{\mathrm{Exal}}_A(P_\bullet, J) \arrow[r, "\beta_2"] & \underline{\mathrm{Ext}}(\Omega_{P_\bullet/A}, J) \arrow[r, "\beta_3"] & \underline{\mathrm{Ext}}(\Omega_{P_\bullet/A} \otimes B, J) \arrow[r, "\beta_4"] & \mathrm{Ext}^{0/-1}(\mathbb{L}_{B/A}, J[1])
\end{tikzcd}$$

The square with β_1 commutes by Lemma B.1. The square with β_2 commutes because differentials commute with colimits [Stacks, Tag 031G]. The square with β_3 commutes because the tensor product is a left adjoint and so commutes with colimits, and the square with β_4 commutes by the naturality in [Ols06, Prop A.3].

Proof of (2). The desired functoriality follows from two commuting diagrams. First we have

$$\begin{tikzcd}
\underline{\mathrm{Exal}}_A(B, I) \arrow[r, "\beta_1"] \arrow[d] & \underline{\mathrm{Exal}}_A(P_\bullet, I) \arrow[r, "\beta_2"] \arrow[d] & \underline{\mathrm{Ext}}(\Omega_{P_\bullet/A}, I) \arrow[r, "\beta_3"] \arrow[d] & \underline{\mathrm{Ext}}(\Omega_{P_\bullet/A} \otimes B, I) \arrow[d] \\
\underline{\mathrm{Exal}}_{A'}(B', I_{B'}) \arrow[r, "\beta_1"] & \underline{\mathrm{Exal}}_{A'}(P'_\bullet, I_{B'}) \arrow[r, "\beta_2"] & \underline{\mathrm{Ext}}(\Omega_{P'_\bullet/A'}, I_{P'_\bullet}) \arrow[r, "\beta_3"] & \underline{\mathrm{Ext}}(\Omega_{P'_\bullet/A'} \otimes B', I_{B'})
\end{tikzcd} \tag{B.12}$$

which we claim commutes. Here $P'_\bullet$ is the simplicial A'-algebra that is the standard resolution of B'. The two left vertical arrows are given by equations (B.4) and (B.3); the next two vertical arrows are given by the analogue of equation (B.3) for the $\underline{\mathrm{Ext}}$ categories. The first square commutes by Lemma B.1. The commutativity of the squares with β_2 and β_3 may be checked with the same type of computation used in Lemma B.1, and we will be brief here.

For the square with β_2, if $0 \to I \to E_\bullet \to P_\bullet \to 0$ is an object of $\underline{\mathrm{Exal}}_A(P_\bullet, I)$, then the natural transformation is given on this object by the (iso)morphism

$$\Omega_{E_\bullet \times_{P_\bullet} P'_\bullet/A'} \to \Omega_{E_\bullet/A} \times_{\Omega_{P_\bullet/A}} \Omega_{P'_\bullet/A'}$$

induced by the commuting cube

$$\begin{tikzcd}
 & E_\bullet \arrow[rr] & & P_\bullet \\
E_\bullet \times_{P_\bullet} P'_\bullet \arrow[ur] \arrow[rr] & & P'_\bullet \arrow[ur] & \\
 & A \arrow[uu] \arrow[rr, equal] & & A \arrow[uu] \\
A' \arrow[uu] \arrow[ur] \arrow[rr, equal] & & A' \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

For the square with β_3, if $0 \to I \to E_\bullet \to \Omega_{P_\bullet/A} \to 0$ is an object of $\underline{\mathrm{Ext}}_{P_\bullet}(\Omega_{P_\bullet/A}, I)$, then the natural transformation is given on this object by the (iso)morphism of B'-modules

$$(E_\bullet \times_{\Omega_{P_\bullet/A}} \Omega_{P'_\bullet/A'}) \otimes_{P'_\bullet} B' \to (X \otimes_P B) \times_{\Omega_{P_\bullet/A} \otimes_{P_\bullet} B} (\Omega_{P'_\bullet/A'} \otimes_{P'_\bullet} B')$$

induced by the natural map of $P'_\bullet$-modules

$$E_\bullet \times_{\Omega_{P_\bullet/A}} \Omega_{P'_\bullet/A'} \to (E_\bullet \otimes_{P_\bullet} B) \times_{\Omega_{P_\bullet/A} \otimes_{P_\bullet} B} (\Omega_{P'_\bullet/A'} \otimes_{P'_\bullet} B').$$

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The second diagram comprising diagram (B.10) is as follows:

$$
\begin{tikzcd}
\underline{\mathrm{Ext}}_B(\Omega_{P_\bullet/A}\otimes B, I) \arrow[r, "\beta_4"] \arrow[d, "\rho"] & \mathrm{Ext}_B^{0/-1}(\mathbb{L}_{B/A}, I[1]) \arrow[d, "(2.4)"] \arrow[dr, "\epsilon_{\mathbb{L}_{B/A}}"] & \\
\underline{\mathrm{Ext}}_{B'}((\Omega_{P_\bullet/A}\otimes B)_{B'}, I_{B'}) \arrow[r, "\beta_4"] \arrow[d, "(B.3)"] & \mathrm{Ext}_{B'}^{0/-1}((\mathbb{L}_{B/A})_{B'}, I_{B'}[1]) \arrow[r, equal, "(2.3)"] \arrow[d] & \mathrm{Ext}_B^{0/-1}((\mathbb{L}_{B/A})_{B'}\otimes_{B'} B, I[1]) \arrow[d] \\
\underline{\mathrm{Ext}}_{B'}(\Omega_{(P_\bullet)'/A'}\otimes B', I_{B'}) \arrow[r, "\beta_4"] & \mathrm{Ext}_{B'}^{0/-1}(\mathbb{L}_{B'/A'}, I_{B'}[1]) \arrow[r, equal, "(2.3)"] & \mathrm{Ext}_B^{0/-1}(\mathbb{L}_{B'/A'}\otimes_{B'} B, I[1])
\end{tikzcd}
$$

The arrow labelled ρ sends an extension of B-modules to the extension of B'-modules obtained by restriction of scalars (an exact functor). One may check directly that the composition of the left vertical arrows is equal to the right vertical arrow in diagram (B.12). The map labelled equation (2.4) is in the context of Example B.3, and the triangle commutes by definition, while the top-left square commutes by Lemma B.6 below. The unlabelled vertical maps are induced by the canonical map $(\mathbb{L}_{B/A})_{B'} \to \mathbb{L}_{B'/A'}$, so the bottom squares commute by functoriality of β_4 and equation (2.3).

Proof of (3). Let $A' = p^{-1}A$, let $B' = p^{-1}B$, and let $P'_\bullet$ denote the standard resolution of B' as an A'-algebra. The desired commuting square comes from two commuting diagrams. First, we have

$$
\begin{tikzcd}
\underline{\mathrm{Exal}}_A(B, I) \arrow[r, "\beta_1"] \arrow[d, "p^{-1}"] & \underline{\mathrm{Exal}}_A(P_\bullet, I) \arrow[r, "\beta_2"] \arrow[d, "p^{-1}"] & \underline{\mathrm{Ext}}(\Omega_{P_\bullet/A}, I) \arrow[r, "\beta_3"] \arrow[d] & \underline{\mathrm{Ext}}(\Omega_{P_\bullet/A}\otimes B, I) \arrow[d] \\
\underline{\mathrm{Exal}}_{A'}(B', a^{-1}I) \arrow[r, "\beta_1"] & \underline{\mathrm{Exal}}_{A'}(P'_\bullet, p^{-1}I) \arrow[r, "\beta_2"] & \underline{\mathrm{Ext}}(\Omega_{P'_\bullet/A'}, p^{-1}I) \arrow[r, "\beta_3"] & \underline{\mathrm{Ext}}(\Omega_{P'_\bullet/A'}\otimes B', p^{-1}I)
\end{tikzcd} \tag{B.13}
$$

The first square commutes by Lemma B.1. The third vertical map is induced by p^{-1} and the isomorphism $p^{-1}\Omega_{P_\bullet/A} \simeq \Omega_{p^{-1}P_\bullet/p^{-1}A}$ ([Stacks, Tag 08TQ]), the fourth is induced by p^{-1} and the isomorphism $p^{-1}(\Omega_{P_\bullet/A}\otimes B) \simeq \Omega_{p^{-1}P_\bullet/p^{-1}A}\otimes p^{-1}B$ ([Stacks, Tag 03EL]), and the squares commute by functoriality of the same isomorphisms. Second, we have

$$
\begin{tikzcd}
\underline{\mathrm{Ext}}(\Omega_{P_\bullet/A}\otimes B, I) \arrow[r, "\beta_4"] \arrow[d, "p^{-1}"] & \mathrm{Ext}_B^{0/-1}(\mathbb{L}_{B/A}, I[1]) \arrow[d, "(2.5)"] \arrow[dr, "(2.5)"] & \\
\underline{\mathrm{Ext}}(p^{-1}(\Omega_{P_\bullet/A}\otimes B), p^{-1}I) \arrow[r] \arrow[d, equal] & \mathrm{Ext}_{p^{-1}B}^{0/-1}(p^{-1}\mathbb{L}_{B/A}, p^{-1}I[1]) \arrow[r, "(2.5)"] \arrow[d, equal] & \mathrm{Ext}_{p^*B}^{0/-1}(p^*\mathbb{L}_{B/A}, p^*I[1]) \\
\underline{\mathrm{Ext}}(\Omega_{P'_\bullet/A'}\otimes B', p^{-1}I) \arrow[r, "\beta_4"] & p_*\mathrm{Ext}_{p^{-1}B}^{0/-1}(\mathbb{L}_{B'/A'}, p^{-1}I[1]) &
\end{tikzcd}
$$

The composition of the left vertical arrows is equal to the right vertical arrow in diagram (B.13). The middle horizontal arrow comprises β_4 and an isomorphism (see Lemma B.7), and the top-left square commutes by Lemma B.7. The bottom square commutes by functoriality of β_4, and the triangle commutes by functoriality of equation (2.5) in the functors. ◻

Lemma B.6. *Let $(\mathcal{S}, \mathcal{O}_\mathcal{S})$ be a ringed site, let $\Omega_\bullet$ be a simplicial $\Omega_\mathcal{S}$-module, and let I be an $\Omega_\mathcal{S}$-module. Let $\mathcal{O}'_\mathcal{S} \to \mathcal{O}_\mathcal{S}$ be a flat morphism of rings. There is a commuting diagram*

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$$\begin{tikzcd}
\underline{\mathrm{Ext}}_{\mathcal{O}_{\mathcal{S}}}(\Omega_\bullet, I) \arrow[rr, "\beta_4"] \arrow[d, "\rho"] & & \mathrm{Ext}^{0/-1}_{\mathcal{O}_{\mathcal{S}}}(N(\Omega_\bullet), I[1]) \arrow[d, "(2.4)"] \\
\underline{\mathrm{Ext}}_{\mathcal{O}'_{\mathcal{S}}}((\Omega_\bullet)_{\mathcal{O}'_{\mathcal{S}}}, I_{\mathcal{O}'_{\mathcal{S}}}) \arrow[r, "\beta_4"] & \mathrm{Ext}^{0/-1}_{\mathcal{O}'_{\mathcal{S}}}(N((\Omega_\bullet)_{\mathcal{O}'_{\mathcal{S}}}), (I)_{\mathcal{O}'_{\mathcal{S}}}[1]) \arrow[r, equal] & \mathrm{Ext}^{0/-1}_{\mathcal{O}'_{\mathcal{S}}}((N(\Omega_\bullet))_{\mathcal{O}'_{\mathcal{S}}}, (I)_{\mathcal{O}'_{\mathcal{S}}}[1])
\end{tikzcd} \tag{B.14}$$

where $N(\Omega_\bullet)$ is the normalisation of the Moore complex associated to $\Omega_\bullet$ (see [Stacks, Tag 0194]), β_4 is the isomorphism of [Ols07, Prop A.3], ρ applies restriction of scalars to an exact sequence, and equation (2.4) is in the context of Example B.3.

Proof. By [Stacks, Tag 05NI, 05T7], there is a quasi-isomorphism $N \to N(\Omega_\bullet)$ from a complex $N \in D^{[-\infty,0]}(\mathcal{O}_{\mathcal{S}})$ of flat $\mathcal{O}_{\mathcal{S}}$-modules. We enlarge diagram (B.14) on its right side by composing with the square induced by $N \to N(\Omega_\bullet)$ and show that the perimeter of the new diagram commutes. From the definition of β_4, we may assume I is injective. To simplify notation, let $B = \mathcal{O}_{\mathcal{S}}$ and $B' = \mathcal{O}'_{\mathcal{S}}$.

Most of the work is to describe equation (2.4) explicitly. To this end, we first recall the definition of equation (2.3): in the notation of Section 2.1, it is the image of the composition

$$f^*\operatorname{Hom}(X, f_*Y) \otimes f^*X \xrightarrow[\sim]{(2.1)} f^*(\operatorname{Hom}(X, f_*Y) \otimes X) \xrightarrow{f^*(\epsilon^{\otimes}_{f_*Y})} f^*f_*Y \xrightarrow{\epsilon^{f^*}_Y} Y \tag{B.15}$$

under the $(\otimes, \operatorname{Hom})$ adjunction and then the (f^*, f_*) adjunction. One sees using the description of $\epsilon^{\otimes}$ in Example B.3 that with $M \in \mathrm{D}^{[-\infty,0]}(B')$ a complex of flat B'-modules and I as in the previous paragraph, the morphism $(\mathsf{R}\mathcal{H}om_{B'}(M, (I)_{B'}[1]) \otimes_{B'} B) \overset{\mathsf{L}}{\otimes}_B (M \otimes_{B'} B) \to I[1]$ of equation (B.15) is given by a product over $p + r = 0$ of the canonical sheaf maps

$$(\mathcal{H}om_{B'}(M^{-p}, (I)_{B'}) \otimes_{B'} B) \otimes_B (M^r \otimes_{B'} B) \to I.$$

We are using the fact that $(-)_{B'}$ preserves injectives (since it has an exact left adjoint) and hence $(I)_{B'}$ is injective. We see that morphism (2.3), a morphism $\mathsf{R}\mathcal{H}om_{B'}(M, (I)_{B'}[1]) \to (\mathsf{R}\mathcal{H}om_B(M \otimes_{B'} B, I[1]))_{B'}$ is given in degree n by the canonical sheaf map

$$\mathcal{H}om_{B'}(M^{-n-1}, (I)_{B'}) \to (\mathcal{H}om_B(M^{-n-1} \otimes_{B'} B, I))_{B'},$$

To compute equation (2.4), given N as at the beginning of this proof, we note that $(N)_{B'} \in \mathrm{D}^{[-\infty,0]}$ is a complex of flat B'-modules by [Stacks, Tag 00HC], so our previous description of equation (2.3) applies with $M = (N)_{B'}$. From the definition of

$$(2.4) : (\mathsf{R}\mathcal{H}om_B(N, I[1]))_{B'} \to \mathsf{R}\mathcal{H}om_{B'}((N)_{B'}, (I)_{B'}[1]), \tag{B.16}$$

we see that it is given in degree n by the usual sheaf map

$$\mathcal{H}om_B(N^{-n-1}, I) \to \mathcal{H}om_{B'}((N^{n-1})_{B'}, (I)_{B'}).$$

We are interested in the cases $n = 0$ and $n = -1$. Given $U \in \mathcal{S}$ and a section $f : N^{n-1}|_U \to I|_U$ of the left-hand side—that is, a morphism of B-modules—this map sends f to the corresponding morphism of B'-modules. (One may verify this claim by unwinding the definitions and ultimately appealing to Example 2.2.) Now applying $\mathsf{R}\Gamma$ to equation (B.16) is straightforward since both complexes are complexes of injectives by [Stacks, Tag 0A96].

This gives a completely explicit description of the morphism labelled (2.4) in diagram (B.14). With this in hand, it is easy to check that diagram (B.14) commutes. ◻

Lemma B.7. *Let $(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ be a ringed site, let $\Omega_\bullet$ be a simplicial $\Omega_{\mathcal{S}}$-module, and let I be an $\Omega_{\mathcal{S}}$-module. Let $(\mathcal{S}, \mathcal{O}_{\mathcal{S}}) \to (\mathcal{S}', \mathcal{O}_{\mathcal{S}'})$ be a continuous morphism of ringed sites inducing a flat morphism of topoi (p^{-1}, p_*) such that $p^{-1}\mathcal{O}_{\mathcal{S}} = \mathcal{O}_{\mathcal{S}'}$. Then there is a commuting diagram*

$$\begin{tikzcd}
\underline{\mathrm{Ext}}_{\mathcal{O}_{\mathcal{S}}}(\Omega_\bullet, I) \arrow[rr, "\beta_4"] \arrow[d] & & \mathrm{Ext}^{0/-1}_{\mathcal{O}_{\mathcal{S}}}(N(\Omega_\bullet), I[1]) \arrow[d, "(2.5)"] \\
\underline{\mathrm{Ext}}_{\mathcal{O}_{\mathcal{S}'}}(p^{-1}(\Omega_\bullet), p^{-1}(I)) \arrow[r, "\beta_4"] & \mathrm{Ext}^{0/-1}_{\mathcal{O}_{\mathcal{S}'}}(N(p^{-1}\Omega_\bullet), p^{-1}I[1]) \arrow[r, equal] & \mathrm{Ext}^{0/-1}_{\mathcal{O}_{\mathcal{S}'}}(p^{-1}N(\Omega_\bullet), p^{-1}I[1])
\end{tikzcd} \tag{B.17}$$

where $N(\Omega_\bullet)$ is the normalisation of the simplical module, β_4 is the isomorphism of [Ols07, Prop A.3], equation (2.5) is in the context of Example B.4, and the left vertical arrow applies p^{-1} to an exact sequence.

Proof. By [Stacks, Tag 05NI, 05T7], there is a quasi-isomorphism $N \to N(\Omega_\bullet)$ from a complex $N \in D^{[-\infty,0]}(\mathcal{O}_{\mathcal{S}})$ of flat $\mathcal{O}_{\mathcal{S}}$-modules. We enlarge diagram (B.17) on its right side by composing with the square induced by $N \to N(\Omega_\bullet)$ and show that the perimeter of the new diagram commutes. From the definition of β_4, we may assume I is injective. To simplify notation, let $B = \mathcal{O}_{\mathcal{S}}$ and $B' = \mathcal{O}'_{\mathcal{S}}$.

Most of the work is to describe equation (2.5) explicitly. To this end, we first note that (in the notation of Section 2.1) equation (2.5) is equal to the image of the composition

$$f^* \operatorname{Hom}(X,Y) \otimes f^*X \xrightarrow[\sim]{(2.1)} f^*(\operatorname{Hom}(X,Y) \otimes X) \xrightarrow{f^*(\epsilon_Y^{\otimes})} f^*Y \tag{B.18}$$

under the $(\otimes, \operatorname{Hom})$ adjunction and the (f^*, f_*) adjunction. (To see this, use equation (B.15) and the triangle identity $\epsilon^{f^*}_{f^*Y} \circ f^*\eta^{f^*}_Y = 1_{f^*Y}$.) One sees using the description of $\epsilon^{\otimes}$ in Example B.3 that with N and I as in the previous paragraph, the morphism in equation (B.18), $p^{-1}\mathsf{R}\mathcal{H}om_B(N, I[1]) \overset{\mathsf{L}}{\otimes}_{B'} p^{-1}N \to p^{-1}I[1]$, is given by the product over $r \in \mathbb{Z}$ of the usual sheaf maps

$$p^{-1}\mathcal{H}om_B(N^r, I) \otimes_{B'} p^{-1}N^r \to p^{-1}I.$$

To compute the $(\otimes, \operatorname{Hom})$ adjunction, we must take an injective resolution $p^{-1}I[1] \to J$ of $p^{-1}I[1]$. Given this, one checks that

$$(2.5) : \mathsf{R}\mathcal{H}om_B(N, I[1]) \to \mathsf{R}p_*\mathsf{R}\mathcal{H}om_{B'}(p^{-1}N, J) \tag{B.19}$$

is given in degree n by the composition of the usual sheaf maps

$$\mathcal{H}om_B(N^{-n-1}, I) \to p_*\mathcal{H}om'_B(p^{-1}N^{-n-1}, p^{-1}I) \to p_*\mathcal{H}om_{B'}(p^{-1}N^{-n-1}, J^{-1}).$$

We have used [Stacks, Tag 0A96] to conclude that $\mathsf{R}\mathcal{H}om_{B'}(p^{-1}N, J)$ is a complex of injectives so its pushforward can be computed termwise. We are interested in the cases $n = 0, -1$. Given $U \in \mathcal{S}$ and a section $f : N^{-n-1}|_U \to I|_U$ of the left-hand side, this map sends f to $p^{-1}f : p^{-1}N^{-n-1}|_U \to p^{-1}I|_U \to J^{-1}|_U$ (it is an exercise to check that the 'usual sheaf map' does this). Now applying $\mathsf{R}\Gamma$ to equation (B.19) is straightforward since both complexes are complexes of injectives by [Stacks, Tag 0730, 0A96].

This gives a completely explicit description of equation (2.5) in diagram (B.17). With this in hand, one may check directly that diagram (B.17) commutes. □

B.3. Description of equation (4.4)

To define equation (4.4), we use another example of Situation 2.1.

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Example B.8. Let $\mathcal{X}$ be an algebraic stack, and let $X \to \mathcal{X}$ be a smooth cover by an algebraic space X. By Proposition A.4, the morphism $\varpi^* : \mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}}) \to \mathrm{D}_{\mathrm{qc}}(U^+_{\bullet,\mathrm{et}})$ is an equivalence of categories. In fact, it follows from the construction of ϖ^* that it is a strong monoidal equivalence of symmetric monoidal categories. A standard argument shows that the inverse equivalence $\mathsf{R}\varpi_*$ is also strong monoidal.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable morphism of algebraic stacks. Let $Y \to \mathcal{Y}$ be a smooth cover by a scheme with $Y^+_\bullet \to \mathcal{Y}$ the associated strictly simplicial algebraic space and $\varpi : X^+_\bullet \to \mathcal{X}$ its pullback to $\mathcal{X}$. We will use ϖ^* and $\mathsf{R}\varpi_*$ to denote the functors in Example B.8. We recall that the cotangent complex $\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ is defined to be the object in $\mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}})$ corresponding, under the equivalence $\mathsf{R}\varpi_*$ of Example B.8, to the cotangent complex of the morphism of topoi $X^+_{\bullet,\mathrm{et}} \to Y^+_{\bullet,\mathrm{et}}$.

Definition B.9 [Ols06]. *Let $I \in \mathrm{QCoh}(\mathcal{X}_{\text{lis-et}})$. The isomorphism in equation (4.4) is defined to be the following composition of morphisms of Picard categories on $\mathcal{X}_{\text{lis-et}}$:*

$$\underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{X}, I) \xrightarrow{\alpha} \underline{\mathrm{Exal}}_{f^{-1}\mathcal{O}_{Y^+_\bullet}}(\mathcal{O}_{X^+_\bullet}, \varpi^* I) \xrightarrow{\varpi_*\beta} \mathrm{Ext}^{0/-1}(\mathbb{L}_{X^+_\bullet/Y^+_\bullet}, \varpi^* I[1]) \xleftarrow{\gamma} \mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}, I[1]).$$

The objects and maps in this composition are defined as follows:

- *The Picard category $\underline{\mathrm{Exal}}_{f^{-1}\mathcal{O}_{Y^+_\bullet}}(\mathcal{O}_{X^+_\bullet}, \varpi^* I)$ is defined as in Section B.2 on the site $X^+_{\bullet,\mathrm{et}}$.*
- *The map α is the composition of [Ols06, (2.8.1), (2.20.1)]: it sends an extension $\mathcal{X} \to \mathcal{X}'$ by I to the exact sequence of $f^{-1}\mathcal{O}_{Y^+_\bullet}$-modules*

$$0 \to \varpi^* I \to \mathcal{O}_{\mathcal{X}'^+_\bullet} \to \mathcal{O}_{\mathcal{X}^+_\bullet} \to 0.$$

  It is an isomorphism by [Ols06, Prop 2.9, Lem 2.21].
- *The map β was defined in equation (B.9).*
- *The arrow γ is induced by applying $ch \circ \tau_{\leq 0} \circ \mathsf{R}\Gamma$ to equation (2.5) in the context of Example B.8. It is an isomorphism since ϖ^* is fully faithful (in fact, an equivalence of categories).*

Remark B.10. The proof of Lemma 4.5 shows that equation (4.4) is independent of the choice of cover $Y \to \mathcal{Y}$.

B.4. Proofs of Lemmas 4.5 and 4.6

We describe an amalgamation diagram (B.20) of the three diagrams in Lemma B.5 that will be used to prove both functoriality lemmas. Let $(\mathcal{S}, \mathcal{O}_{\mathcal{S}}) \to (\mathcal{S}', \mathcal{O}_{\mathcal{S}'})$ be a continuous morphism of ringed sites inducing a flat morphism of topoi (p^{-1}, p_*). Let A and $B := \mathcal{O}_{\mathcal{S}}$ be sheaves of rings on $\mathcal{S}$, let I be a sheaf of B-modules, and let A' be a sheaf of rings on $\mathcal{S}'$ such that there is a commuting diagram as follows (note that $p^* B = \mathcal{O}_{\mathcal{S}'}$):

$$\begin{tikzcd} p^{-1}A \arrow[r] \arrow[d] & A' \arrow[d] \\ p^{-1}B \arrow[r] & p^*B \end{tikzcd}$$

Then we obtain the following commuting diagram:

$$
\begin{tikzcd}[column sep=small]
\mathrm{Ext}^{0/-1}_{B}(\mathbb{L}_{B/A}, I[1]) \arrow[rrr] & & & \mathrm{Ext}^{0/-1}_{p^*B}(p^*\mathbb{L}_{B/A}, p^*I[1]) & & & \mathrm{Ext}^{0/-1}_{p^*B}(\mathbb{L}_{p^*B/A'}, p^*I[1]) \arrow[lll] \\
& & \mathrm{Ext}^{0/-1}_{p^{-1}B}(p^{-1}\mathbb{L}_{B/A}, p^{-1}I[1]) \arrow[ur, "(2.5)"] \arrow[rr] & & \mathrm{Ext}^{0/-1}_{p^{-1}B}(p^{-1}\mathbb{L}_{B/A}, (p^*I[1])_{p^{-1}B}) \arrow[ul, "(2.3)"', "\sim"] & & \\
\underline{\mathrm{Exal}}_A(B, I) \arrow[uu, "\beta"] \arrow[rr] & & \underline{\mathrm{Exal}}_{p^{-1}A}(p^{-1}B, p^{-1}I) \arrow[u, "\beta"] \arrow[rr] & & \underline{\mathrm{Exal}}_{p^{-1}A}(p^{-1}B, (p^*I)_{p^{-1}B}) \arrow[u, "\beta"] & & \underline{\mathrm{Exal}}_{A'}(p^*B, p^*I) \arrow[uu, "\beta"'] \arrow[ll]
\end{tikzcd}
\tag{B.20}
$$

Here, the left square is Lemma B.5 (3) and the right square is Lemma B.5 (2). The middle square is Lemma B.5 (1), using the unit $p^{-1}I \to (p^*I)_{p^{-1}B}$ of the $(\otimes, \mathrm{Hom})$ adjunction. The triangle commutes by definition of the maps involved.

Proof of Lemma 4.5. Construct a diagram

$$
\begin{tikzcd}
U \arrow[r, "r"] \arrow[d, "\rho"] & V \arrow[d] & \\
Z \arrow[r] \arrow[d, "\varpi"] & \mathcal{W} \times_{\mathcal{Y}} Y \arrow[r] \arrow[d] & Y \arrow[d] \\
\mathcal{Z} \arrow[r] & \mathcal{W} \arrow[r] & \mathcal{Y}
\end{tikzcd}
$$

where U, V, Z, and Y are algebraic spaces with $Y \to \mathcal{Y}$ and $V \to \mathcal{W} \times_{\mathcal{Y}} Y$ smooth and surjective and all squares are fibred. Let q denote the map $Z \to Y$, and let $\varpi' = \varpi \circ \rho$, and use the same letters to denote induced morphisms of (simplicial) topoi. Then commutativity of equation (4.5) is equivalent to commutativity of the following diagram:

$$
\begin{tikzcd}
\mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{Z}/\mathcal{W}}, I[1]) \arrow[rr, "A"] \arrow[d, "\gamma", "\sim"'] & & \mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{Z}/\mathcal{Y}}, I[1]) \arrow[d, "\gamma", "\sim"'] \arrow[dl] \\
\mathrm{Ext}^{0/-1}(\mathbb{L}_{U^+_\bullet/V^+_\bullet}, \varpi'^* I[1]) \arrow[r] & \mathrm{Ext}^{0/-1}(\rho^*\mathbb{L}_{Z^+_\bullet/Y^+_\bullet}, \varpi'^* I[1]) & \mathrm{Ext}^{0/-1}(\mathbb{L}_{Z^+_\bullet/Y^+_\bullet}, \varpi^* I[1]) \arrow[l, "\sim"'] \\
\underline{\mathrm{Exal}}_{r^{-1}\mathcal{O}_{V^+_\bullet}}(\mathcal{O}_{U^+_\bullet}, \varpi'^* I) \arrow[u, "\beta"] \arrow[r] & \underline{\mathrm{Exal}}_{\rho^{-1}q^{-1}\mathcal{O}_{Y^+_\bullet}}(\rho^{-1}\mathcal{O}_{Z^+_\bullet}, \varpi'^* I) \arrow[u] & \underline{\mathrm{Exal}}_{q^{-1}\mathcal{O}_{Y^+_\bullet}}(\mathcal{O}_{Z^+_\bullet}, \varpi^* I) \arrow[u, "\beta"] \arrow[l] \\
\underline{\mathrm{Exal}}_{\mathcal{W}}(\mathcal{Z}, I) \arrow[u, "\alpha"] \arrow[rr, "B"] & & \underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{Z}, I) \arrow[u, "\alpha"]
\end{tikzcd}
\tag{B.21}
$$

In the triangle, all of the maps are equal to equation (2.5), and the triangle commutes by the functoriality of equation (2.5) in the adjoint pair. The arrow

$$
\mathrm{Ext}^{0/-1}(\rho^*\mathbb{L}_{Z^\bullet/Y^\bullet}, \varpi'^* I[1]) \leftarrow \mathrm{Ext}^{0/-1}(\mathbb{L}_{Z^\bullet/Y^\bullet}, \varpi^* I[1])
$$

is an equivalence (as claimed in the diagram) because $\rho^* : \mathrm{D}_{\mathrm{qc}}(Z^+_{\bullet,\mathrm{et}}) \to \mathrm{D}_{\mathrm{qc}}(U^+_{\bullet,\mathrm{et}})$ is fully faithful. The trapezoid commutes by definition of the canonical map $\mathbb{L}_{\mathcal{Z}/\mathcal{Y}} \to \mathbb{L}_{\mathcal{Z}/\mathcal{W}}$ (one can produce an explicit description for γ by the same argument as was used in the proof of Lemma B.7). The commutativity of the middle square is diagram (B.20), reflected left-to-right, with $p = \rho$, $A = q^{-1}\mathcal{O}_{Y^+_\bullet}$, $A' = r^{-1}\mathcal{O}_{V^+_\bullet}$ and $B = \mathcal{O}_{Z^+_\bullet}$.

It remains to check that the bottom square commutes. We do this by direct computation. Let $i : \mathcal{Z} \hookrightarrow \mathcal{Z}'$ be an element of $\underline{\mathrm{Exal}}_{\mathcal{W}}(\mathcal{Z}, I)$. We have the following commuting diagram, where all squares are fibred:

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d, "\rho"] & U' \arrow[r] \arrow[d, "\rho'"] & V \arrow[d] & \\ Z \arrow[r, "i"] \arrow[d, "\varpi"] & Z' \arrow[r] \arrow[d] & \mathcal{W} \times_{\mathcal{Y}} Y \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathcal{Z} \arrow[r, "i"] & \mathcal{Z}' \arrow[r] & \mathcal{W} \arrow[r] & \mathcal{Y} \end{tikzcd} \tag{B.22}$$

The map α sends $i : \mathcal{Z} \hookrightarrow \mathcal{Z}'$ to the extension

$$0 \to \varpi'^* I \xrightarrow{m} i^{-1}\mathcal{O}_{U'^+_\bullet} \to \mathcal{O}_{U^+_\bullet} \to 0$$

of $r^{-1}\mathcal{O}_{V^+_\bullet}$-modules, and the maps in equations (B.4) and (B.3) send this to the extension

$$0 \to \varpi'^* I \xrightarrow{(m,0)} i^{-1}\mathcal{O}_{U'^+_\bullet} \times_{\mathcal{O}_{U^+_\bullet}} \rho^{-1}\mathcal{O}_{Z^+_\bullet} \to \rho^{-1}\mathcal{O}_{Z^+_\bullet} \to 0 \tag{B.23}$$

of $\rho^{-1}q^{-1}\mathcal{O}_{Y^+_\bullet}$-modules.

On the other hand, the map B sends $i : \mathcal{Z} \hookrightarrow \mathcal{Z}'$ to the same extension, now as an element of $\underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{Z}, I)$. The image of this under $\rho^{-1} \circ \alpha$ is

$$0 \to \rho^{-1}\varpi^* I \to \rho^{-1}i^{-1}\mathcal{O}_{Z'^+_\bullet} \xrightarrow{n} \rho^{-1}\mathcal{O}_{Z^+_\bullet} \to 0,$$

an extension of $\rho^{-1}q^{-1}\mathcal{O}_{Y^\bullet}$-modules. Here n is part of the data of the morphism of ringed topoi associated to $i : Z \to Z'$. Finally, the map in equation (B.2) sends this extension to

$$0 \to \varpi'^* I \to \varpi'^* I \oplus_{\rho^{-1}\varpi^* I} \rho^{-1}i^{-1}\mathcal{O}_{Z'^+_\bullet} \xrightarrow{(0,n)} \rho^{-1}\mathcal{O}_{Z^+_\bullet} \to 0, \tag{B.24}$$

also an extension of $\rho^{-1}q^{-1}\mathcal{O}_{Y^+_\bullet}$-modules.

A morphism from equation (B.24) to equation (B.23) in the groupoid $\underline{\mathrm{Exal}}_{\rho^{-1}q^{-1}\mathcal{O}_{Y^+_\bullet}}(\rho^{-1}\mathcal{O}_{Z^+_\bullet}, \varpi'^* I)$ is given by a collection of dotted arrows, making the following diagram commute:

$$\begin{tikzcd} \rho^{-1}\varpi^* I \arrow[r] \arrow[dr] & \varpi'^* I \arrow[r, dashed] \arrow[dr, dashed] & i^{-1}\mathcal{O}_{U'^+_\bullet} \arrow[dr] & \\ & \rho^{-1}i^{-1}\mathcal{O}_{Z'^+_\bullet} \arrow[r, dashed] \arrow[ur, dashed] & \rho^{-1}\mathcal{O}_{Z^+_\bullet} \arrow[r] & \mathcal{O}_{U^+_\bullet} \end{tikzcd} \tag{B.25}$$

We choose arrows as follows (note that they are compatible with restriction)

$$\begin{aligned} m &: \varpi'^* I \to i^{-1}\mathcal{O}_{U'^+_\bullet} & \qquad n &: \rho^{-1}i^{-1}\mathcal{O}_{Z'^+_\bullet} \to \rho^{-1}\mathcal{O}_{Z^+_\bullet} \\ 0 &: \varpi'^* I \to \rho^{-1}\mathcal{O}_{Z^+_\bullet} & \qquad k &: \rho^{-1}i^{-1}\mathcal{O}_{Z'^+_\bullet} \to i^{-1}\mathcal{O}_{U'^+_\bullet}, \end{aligned}$$

where k is equal to i^{-1} applied to the canonical morphism $\rho^{-1}\mathcal{O}_{Z'^+_\bullet} \to \mathcal{O}_{U'^+_\bullet}$. Commutativity of the resulting diagram follows from commutativity of diagram (B.22).

We claim that this morphism is natural for arrows coming from $\underline{\mathrm{Exal}}_{\mathcal{W}}(\mathcal{Z}, I)$. If we are given an arrow f from $i_1 : \mathcal{Z} \to \mathcal{Z}_1$ to $i_2 : \mathcal{Z} \to \mathcal{Z}_2$ inducing maps $f_U : i_1^{-1}\mathcal{O}_{U^+_{1,\bullet}} \to i_2^{-1}\mathcal{O}_{U^+_{2,\bullet}}$ and $f_Z : \rho^{-1}i_1^{-1}\mathcal{O}_{Z^+_{1,\bullet}} \to \rho^{-1}i_1^{-1}\mathcal{O}_{Z^+_{2,\bullet}}$, then this naturality is equivalent to the fact that the maps in the following two criss-cross diagrams coincide:

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$$
\begin{tikzcd}
\varpi'^* I \arrow[r, dashed, "m_2"] \arrow[dr, dashed, "0" near start] & i_2^{-1}\mathcal{O}_{U^+_{2,\bullet}} \\
\rho^{-1} i_1^{-1}\mathcal{O}_{Z^+_{1,\bullet}} \arrow[r, dashed, "n_2 \circ f_Z"'] \arrow[ur, dashed, "k_2 \circ f_Z" near start] & \rho^{-1}\mathcal{O}_{Z^+_\bullet}
\end{tikzcd}
\qquad
\begin{tikzcd}
\varpi'^* I \arrow[r, dashed, "f_U \circ m_1"] \arrow[dr, dashed, "0" near start] & i_2^{-1}\mathcal{O}_{U^+_{2,\bullet}} \\
\rho^{-1} i_1^{-1}\mathcal{O}_{Z^+_{1,\bullet}} \arrow[r, dashed, "n_1"'] \arrow[ur, dashed, "f_U \circ k_1" near start] & \rho^{-1}\mathcal{O}_{Z^+_\bullet}
\end{tikzcd}
$$

□

Proof of Lemma 4.6. Construct a fibre diagram

$$
\begin{tikzcd}
Z \arrow[r] \arrow[d, "r"] & X \times_{\mathcal{X}} \mathcal{Z} \arrow[r] \arrow[d] & X \arrow[d, "q"] \\
W \arrow[r] & Y \times_{\mathcal{Y}} \mathcal{W} \arrow[r] & Y
\end{tikzcd}
$$

where $Y \to \mathcal{Y}$ is a smooth cover by a scheme, $X = Y \times_{\mathcal{Y}} \mathcal{X}$, and $W \to Y \times_{\mathcal{Y}} \mathcal{W}$ is a smooth cover by a scheme. Use p to denote the map $Z \to X$. Then commutativity of diagram (4.8) is equivalent to commutativity of the diagram below:

$$
\begin{tikzcd}
\mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{Z}/\mathcal{W}}, I[1]) \arrow[r, "C", "(2.5)"'] \arrow[d, "\gamma"', "\sim"] & \mathrm{Ext}^{0/-1}(p^*\mathbb{L}_{\mathcal{X}/\mathcal{Y}}, p^*I[1]) \arrow[d, "(2.5)"', "\sim"] & \mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{Z}/\mathcal{W}}, p^*I[1]) \arrow[l, "D"', "\sim"] \arrow[d, "\gamma"', "\sim"] \\
\mathrm{Ext}^{0/-1}_{\mathcal{O}_{X^+_\bullet}}(\mathbb{L}_{X^+_\bullet/Y^+_\bullet}, \varpi^* I[1]) \arrow[r, "(2.5)"] & \mathrm{Ext}^{0/-1}_{p^*\mathcal{O}_{X^+_\bullet}}(p^*\mathbb{L}_{X^+_\bullet/Y^+_\bullet}, p^*\varpi^* I[1]) & \mathrm{Ext}^{0/-1}_{\mathcal{O}_{Z^+_\bullet}}(\mathbb{L}_{Z^+_\bullet/W^+_\bullet}, \varpi^* p^* I[1]) \arrow[l, "\sim"'] \\
\underline{\mathrm{Exal}}_{q^{-1}\mathcal{O}_{Y^+_\bullet}}(\mathcal{O}_{X^+_\bullet}, \varpi^* I) \arrow[rr] \arrow[u, "\beta"] & & \underline{\mathrm{Exal}}_{r^{-1}\mathcal{O}_{W^+_\bullet}}(\mathcal{O}_{Z^+_\bullet}, \varpi^* p^* I) \arrow[u, "\beta"] \\
\underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{X}, I) \arrow[rr, "E"] \arrow[u, "\alpha"] & & \underline{\mathrm{Exal}}_{\mathcal{W}}(\mathcal{Z}, p^* I) \arrow[u, "\alpha"]
\end{tikzcd}
\tag{B.26}
$$

The vertical instance of equation (2.5) is an isomorphism since ϖ^* is fully faithful. This implies that the unnamed arrow in the top-right square of diagram (B.26) is an isomorphism (it is already labelled as such) since the other three maps in the square are. The commutativity of the top-left square uses the functoriality of equation (2.5) in the adjoint functors. The top right square commutes by definition of the canonical map of cotangent complexes. The middle rectangle is diagram (B.20) with p the map $Z \to X$, $A = q^{-1}\mathcal{O}_{Y^+_\bullet}$, $B = \mathcal{O}_{X^+_\bullet}$, and $A' = r^{-1}\mathcal{O}_{W^+_\bullet}$. We have suppressed various squares commuting the maps p and ϖ.

It remains to check that the bottom square of diagram (B.26) commutes. This we do by direct computation, using diagram (B.20) to factor the map

$$
\underline{\mathrm{Exal}}_{q^{-1}\mathcal{O}_{Y^+_\bullet}}(\mathcal{O}_{X^+_\bullet}, \varpi^* I) \to \underline{\mathrm{Exal}}_{r^{-1}\mathcal{O}_{W^+_\bullet}}(\mathcal{O}_{Z^+_\bullet}, \varpi^* p^* I).
$$

Let $i : \mathcal{X} \hookrightarrow \mathcal{X}'$ be an element of $\underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{X}, I)$. Then we have a commuting diagram

$$
\begin{tikzcd}[row sep=small, column sep=small]
Z \arrow[rr] \arrow[dr] \arrow[dd] & & Z' \arrow[rr] \arrow[dr] \arrow[dd] & & W \arrow[dr] \arrow[dd] & \\
& X \arrow[rr] \arrow[dd] & & X' \arrow[rr] \arrow[dd] & & Y' \arrow[dd] \\
\mathcal{Z} \arrow[rr] \arrow[dr] & & \mathcal{Z}' \arrow[rr] \arrow[dr] & & \mathcal{W} \arrow[dr] & \\
& \mathcal{X} \arrow[rr] & & \mathcal{X}' \arrow[rr] & & \mathcal{Y}
\end{tikzcd}
$$

where the front, bottom and back squares are fibred (six squares in all). The map $p^{-1} \circ \alpha$ sends $i : \mathcal{X} \hookrightarrow \mathcal{X}'$ to the extension

$$0 \to p^{-1}\varpi^* I \to p^{-1}i^{-1}\mathscr{O}_{X'^+_\bullet} \to p^{-1}\mathscr{O}_{X^+_\bullet} \to 0$$

of $p^{-1}q^{-1}\mathscr{O}_{Y^+_\bullet}$-algebras, and the map in equation (B.2) sends this extension to the extension

$$0 \to p^*\varpi^* I \to p^*\varpi^* I \oplus_{z^{-1}\varpi^* I} p^{-1}i^{-1}\mathscr{O}_{X^+_\bullet} \to p^{-1}\mathscr{O}_{X^+_\bullet} \to 0. \tag{B.27}$$

On the other hand, the map E sends $i : \mathcal{X} \hookrightarrow \mathcal{X}'$ to $\mathcal{Z} \hookrightarrow \mathcal{Z}'$, which under α corresponds to the extension

$$0 \to \varpi^* p^* I \to i^{-1}\mathscr{O}_{Z'^+_\bullet} \to \mathscr{O}_{Z^+_\bullet} \to 0$$

of $r^{-1}\mathscr{O}_{W^+_\bullet}$-algebras. After applying $\varpi^* p^* = p^*\varpi^*$ and the morphisms in equations (B.4) and (B.3), this becomes the extension

$$0 \to p^*\varpi^* \to i^{-1}\mathscr{O}_{Z'^+_\bullet} \times_{\mathscr{O}_{Z^+_\bullet}} p^{-1}\mathscr{O}_{X^+_\bullet} \to p^{-1}\mathscr{O}_{X^+_\bullet} \to 0 \tag{B.28}$$

of $p^{-1}q^{-1}\mathscr{O}_{Y^+_\bullet}$-algebras. As in the proof of Lemma 4.5, one can write down a functorial (necessarily iso)morphism between equations (B.27) and (B.28) and check that it is compatible with restrictions. □

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Conflict of Interests. None

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