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TOPOLOGIES DETERMINED BY σ-IDEALS ON ω_1

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0. Introduction. *σ-ideals* (collections of sets which are closed under subset and countable union) are certainly important mathematically—consider first category sets, sets of measure zero, nonstationary sets, etc.—but aside from the observation that in certain spaces the first category σ-ideal is proper, σ-ideals have not been extensively studied by topologists. In this note we study a natural topology determined by a σ-ideal, exploiting the interplay between the set-theoretic properties of the σ-ideal and the topological properties of the associated space. For simplicity we shall restrict ourselves to studying σ-ideals on ω_1, although generalizations to large cardinals are interesting as well.

Corson [**4**] defined the space

$$\Sigma = \{f \in 2^{\omega_1} : |\{\alpha : f(\alpha) = 1\}| \leqq \aleph_0\},$$

given the topology inherited from the usual topology on the product of $\aleph_1$ copies of the two-point discrete space. Let $\mathscr{I}$ be an arbitrary σ-ideal on ω_1. Define

$$\Sigma(\mathscr{I}) = \{f \in 2^{\omega_1} : \{\alpha : f(\alpha) = 1\} \in \mathscr{I}\}.$$

The spaces $\Sigma(\mathscr{I})$ will be our object of study. One could also look at various box topologies on $\Sigma(\mathscr{I})$ but these are of less interest.

Section 1 of this note characterizes the normal $\Sigma(\mathscr{I})$'s. Section 2 deals with cardinal invariants and calibers. Section 3 is concerned with "Baireness" and the effect of various set-theoretic assumptions.

Since whenever S is an uncountable subset of ω_1, 2^S is homeomorphic to 2^{ω_1}, without loss of generality we shall assume $\mathscr{I}$ contains all countable subsets of ω_1.

1. Normality. Although we are primarily interested in cardinal functions on $\Sigma(\mathscr{I})$, in this section we give a nice characterization of the $\mathscr{I}$'s for which $\Sigma(\mathscr{I})$ is normal.

THEOREM 1. *$\Sigma(\mathscr{I})$ is normal if and only if $\mathscr{I} = \mathscr{P}(\omega_1)$ (the collection of all subsets of ω_1) or $[\omega_1]^\omega$ (the collection of all countable subsets of ω_1).*

Proof. Corson proved Σ is normal. Suppose $\mathscr{I}$ is not $[\omega_1]^\omega$ or $\mathscr{P}(\omega_1)$. We claim $\Sigma(\mathscr{I})$ is not normal. There is an uncountable $A \in \mathscr{I}$ with uncountable complement. Thus 2^A, $2^{\omega_1 - A}$, and $2^A \times 2^{\omega_1 - A}$ are all homeomorphic to $\{0, 1\}^{\omega_1}$. Let

$$h : 2^A \times 2^{\omega_1 - A} \to 2^{\omega_1}$$

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be defined by

$$h(f, g)) = f \cup g.$$

Clearly h is a homeomorphism. Let π_* be the projection map from $2^A \times 2^{\omega_1 - A}$ onto $2^{\omega_1 - A}$. Let $X = \pi_*[h^{-1}(\Sigma(\mathscr{I}))]$. Then $X = \{g \in 2^{\omega_1 - A} : g^{-1}(\{1\}) \in \mathscr{I}\}$. Thus $X \supseteq \{g \in 2^{\omega_1 - A} : |g^{-1}(\{1\})| \leqq \aleph_0\}$, but the function $g_1 : \omega_1 - A \rightarrow 2$ defined by $g_1(\beta) = 1$, for all $\beta \in \omega_1 - A$, is not in X. Note also that $2^A \times X = \pi_*^{-1}(X) = h^{-1}(\Sigma(\mathscr{I}))$, and hence $h|(2^A \times X)$ is a homeomorphism onto $\Sigma(\mathscr{I})$. But we shall show $2^A \times X$ includes a closed non-normal subspace, so it and $\Sigma(\mathscr{I})$ are not normal.

Let $\boldsymbol{\omega_1} (\boldsymbol{\omega_1 + 1})$ denote the order topology on ω_1 $(\omega_1 + 1)$. It is easily seen that $\boldsymbol{\omega_1 + 1}$ is homeomorphically embedded in 2^{ω_1} by

$$[\phi(\alpha)](\beta) = \begin{cases} 1 \text{ if } \beta < \alpha < \omega_1, \\ 0 \text{ if } \alpha \leqq \beta < \omega_1, \\ 1 \text{ if } \alpha = \omega_1. \end{cases}$$

Since $|A| = |\omega_1 - A| = \aleph_1$, we can similarly embed $\boldsymbol{\omega_1 + 1}$ into 2^A and into $2^{\omega_1 - A}$ by homeomorphisms r and s defined as above. Thus $r(\boldsymbol{\omega_1 + 1})$ is a closed subspace of 2^A and $s(\boldsymbol{\omega_1 + 1})$ is a closed subspace of $2^{\omega_1 - A}$. Furthermore $s(\alpha) \in X$ for all $\alpha < \omega_1$, since $[s(\alpha)]^{-1}(\{1\})$ is countable, hence in $\mathscr{I}$. Since $s(\omega_1) = g_1$, we see that $s(\boldsymbol{\omega_1 + 1}) \cap X = s(\boldsymbol{\omega_1})$ is a closed subset of X. Since $s(\boldsymbol{\omega_1})$ is homeomorphic to $\boldsymbol{\omega_1}$, we have $\boldsymbol{\omega_1 + 1} \times \boldsymbol{\omega_1}$ embedded as a closed subset of $2^A \times X$. But (see e.g. [**7**, 8M4]) $\boldsymbol{\omega_1 + 1} \times \boldsymbol{\omega_1}$ is not normal. Hence $2^A \times X$ and $\Sigma(\mathscr{I})$ are not normal.

2. Cardinal invariants and calibers. We refer to [**9**] for the definitions of various cardinal functions. $\Sigma(\mathscr{I})$ is dense in 2^{ω_1} and hence satisfies the countable chain condition, indeed has *precaliber* $\aleph_1$ [**13**]. It is easy to see that any $\Sigma(\mathscr{I})$ is $\aleph_0$-*bounded* (every countable subset has compact closure) and hence countably compact, but $\Sigma(\mathscr{I})$ is separable if and only if $\mathscr{I} = \mathscr{P}(\omega_1)$. Since the weight of 2^{ω_1} is $\aleph_1$, the popular cardinal functions on $\Sigma(\mathscr{I})$ are easily determined. All $\Sigma(\mathscr{I})$ have weight $\aleph_1$ and π-weight $\aleph_1$. If $\mathscr{I} \neq \mathscr{P}(\omega_1)$, the Lindelöf number of $\Sigma(\mathscr{I})$ is $\aleph_1$. The hereditary Lindelöf number, hereditary density, and spread of all $\Sigma(\mathscr{I})$ are $\aleph_1$. Since $\Sigma(\mathscr{I})$ is dense in 2^{ω_1} which has character $\aleph_1$, so does $\Sigma(\mathscr{I})$. Indeed no point of 2^{ω_1}, hence $\Sigma(\mathscr{I})$, has countable character, so since $\Sigma(\mathscr{I})$ is countably compact, $\psi(\Sigma(\mathscr{I})) = \aleph_1$. Clearly $|\Sigma(\mathscr{I})| = 2^{\aleph_1}$ unless $\mathscr{I} = [\omega_1]^\omega$ in which case $|\Sigma(\mathscr{I})| = 2^{\aleph_0}$.

What are non-trivial and what are the heart of this note are the calibers and Baireness (see next section) of $\Sigma(\mathscr{I})$.

Definition. Let κ be an infinite cardinal. A space X has *caliber* κ (or κ is a caliber of X) if each family of κ open subsets of X includes a subfamily of power κ with non-void intersection.

See [**3**] and [**13**] for information on calibers. A problem raised in the former—to which we shall give in some sense the "natural" solution—is to find spaces

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with predetermined sets of calibers. But first we characterize those $\mathscr{I}$'s for which $\Sigma(\mathscr{I})$ has caliber $\aleph_1$.

THEOREM 2. *Let $\mathscr{I}$ be a σ-ideal on ω_1. Then $\Sigma(\mathscr{I})$ has caliber $\aleph_1$ if and only if each uncountable subset of ω_1 includes an uncountable member of $\mathscr{I}$.*

Proof. Suppose $\Sigma(\mathscr{I})$ has caliber $\aleph_1$. Let S be an uncountable subset of ω_1. For each $\alpha \in S$, let $U_\alpha = \pi_\alpha^{-1}(\{1\}) \cap \Sigma(\mathscr{I})$. Since $\Sigma(\mathscr{I})$ has caliber $\aleph_1$, there is an uncountable $S' \subseteq S$ such that $\bigcap \{U_\alpha : \alpha \in S'\} \neq \emptyset$. Let $f \in \bigcap \{U_\alpha : \alpha \in S'\}$. Then $f^{-1}(\{1\}) \supseteq S'$, so $S' \in \mathscr{I}$.

Conversely, let $\{U_\alpha\}_{\alpha<\omega_1}$ be a family of open subsets of $\Sigma(\mathscr{I})$. Without loss of generality we may assume each U_α is the intersection of $\Sigma(\mathscr{I})$ with a basic open set in 2^{ω_1}. Basic open sets in 2^{ω_1} restrict only finitely many coordinates, say U_α restricts $R_\alpha \subseteq \omega_1$. By the Δ-system lemma (see e.g. [**10**]) there is an uncountable $S \subseteq \omega_1$ and a finite $R \subseteq \omega_1$ such that for every $\alpha, \beta \in S$, $R_\alpha \cap R_\beta = R$. There is then an uncountable $S' \subseteq S$ such that $\pi_\delta(U_\alpha) = \pi_\delta(U_\beta)$ for all $\delta \in R$ and $\alpha, \beta \in S'$. Since $R_\alpha - R$ is finite, there is an $n \in \omega$ and an uncountable $S'' \subseteq S'$ such that $|R_\alpha - R| = n$ for all $\alpha \in S''$. Let $R_\alpha - R = \{a_{\alpha,1}, \ldots, a_{\alpha,n}\}$. Then $\{a_{\alpha,1} : \alpha \in S''\}$ is an uncountable subset of ω_1. By hypothesis then, there is an uncountable $S_1 \subseteq S''$ such that $\{a_{\alpha,1} : \alpha \in S_1\} \in \mathscr{I}$. Suppose $k < n$ and we have obtained uncountable $S_1, \ldots, S_k \subseteq \omega_1$ such that $S_1 \supseteq \ldots \supseteq S_k$, and each $\{a_{\alpha,i} : \alpha \in S_i\}$ is in $\mathscr{I}$. Then as before there is an uncountable $S_{k+1} \subseteq S_k$ such that $\{a_{\alpha,k+1} : \alpha \in S_{k+1}\} \in \mathscr{I}$. Continuing by induction, we obtain an uncountable S_n included in each S_k, $k \leqq n$, such that $\{a_{\alpha,k} : \alpha \in S_n\} \in \mathscr{I}$ for $1 \leqq k \leqq n$. Define $f \in 2^{\omega_1}$ by

$$f(\delta) = \begin{cases} \pi_\delta(U_\alpha) & \text{if } \delta \in R, \text{ any } \alpha \in S_n, \\ \pi_\delta(U_\beta) & \text{if } \beta \in S_n,\ \delta \in R_\beta - R, \\ 0 & \text{if } \delta \notin S_n. \end{cases}$$

Then $f \in \bigcap \{U_\alpha : \alpha \in S_n\}$ and so $\Sigma(\mathscr{I})$ has caliber $\aleph_1$.

In particular it follows immediately that Σ does not have caliber $\aleph_1$ [**13**]. This is the key to the solution of Comfort's problem. Shelah [**12**] gave an earlier solution but his spaces were not regular. Ours are products of generalizations of Σ and hence are completely regular, and—in some sense—canonical. Let λ be an infinite cardinal. Let

$$X_\lambda = \{f \in 2^\lambda : |f^{-1}(\{1\})| < \lambda\}.$$

THEOREM 3. *For every regular λ, an infinite cardinal μ is a caliber of X_λ if and only if* cf $(\mu) \neq \aleph_0$ *and* cf $(\mu) \neq \lambda$.

Proof. If μ is a caliber of X_λ, cf $(\mu) \neq \aleph_0$ since clearly $\aleph_0$ is not a caliber of X_λ and, as Shelah observes, if μ were a caliber, so would be cf (μ). A straightforward generalization of Theorem 2 yields that X_λ does not have caliber λ. Conversely, suppose cf $(\mu) \neq \aleph_0$ and cf $(\mu) \neq \lambda$. If μ is regular, $\mu < \lambda$, then again the method of Theorem 2 shows X_λ has caliber μ. If $\mu > \lambda$ is regular,

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X_λ clearly has caliber μ since it has a basis of cardinality λ. Suppose μ is singular. If cf $(\mu) > \lambda$, again X_λ trivially has caliber μ. The non-trivial question then is when cf $(\mu) < \lambda$, which breaks down into two cases depending on whether $\mu > \lambda$ or $\mu < \lambda$.

Case 1. Suppose $\mu > \lambda$. There is a strictly increasing sequence $\{\mu_\alpha\}_{\alpha<\mathrm{cf}(\mu)}$ of regular cardinals such that $\mu = \bigcup\{\mu_\alpha : \alpha < \mathrm{cf}\ (\mu)\}$ and each $\mu_\alpha > \lambda$. Let $\mathscr{U} = \{U_i : i < \mu\}$ be a family of basic open sets in X_λ. For each $\alpha < \mathrm{cf}\ (\mu)$ there is a basic open set B_α in X_λ such that $|\{i \in \mu_\alpha : B_\alpha \subseteq U_i\}| = \mu_\alpha$, since the weight of X_λ is λ. Since cf (μ) is a caliber of X_λ, there is then an $A \subseteq \mathrm{cf}\ (\mu)$, $|A| = \mathrm{cf}\ (\mu)$ such that $\bigcap\{B_\alpha : \alpha \in A\} \neq \emptyset$. Thus

$$\bigcap\{\bigcap\{U_i : i \in \mu_\alpha\} : \alpha \in A\} \supseteq \bigcap\{B_\alpha : \alpha \in A\} \neq \emptyset.$$

Hence X_λ has caliber μ.

Case 2. Let $\mathscr{U} = \{U_i : i < \mu\}$ be a family of basic open sets in X_λ. Let R_i be the set of coordinates restricted by U_i. R_i is finite and so $|\bigcup\{R_i : i < \mu\}| \leqq \mu$. Let $R = \bigcup\{R_i : i < \mu\}$. Let U_i^R be the trace of U_i in 2^R. Shelah [**12**] proved that even singular calibers are preserved by arbitrary products, so 2^R has caliber μ. Hence there is an $S \subseteq \mu$, $|S| = \mu$, such that $\bigcap \{U_i^R : i \in S\} \neq \emptyset$. But then in X_λ, $\bigcap\{U_i : i \in S\} \neq \emptyset$ since $|R| < \lambda$. Thus X_λ has caliber μ.

THEOREM 4. *For singular* λ *with* cf $(\lambda) > \aleph_0$, μ *is a caliber of* X_λ *if and only if* cf $(\mu) > \aleph_0$ *and* $\mu \neq \lambda$.

Proof. If μ is a caliber, then cf $(\mu) > \aleph_0$. Also, λ is not a caliber of X_λ by the usual argument. Suppose cf $(\mu) > \aleph_0$ and $\mu \neq \lambda$. If μ is regular then X_λ has caliber μ as before. If μ is singular and cf $(\mu) > \lambda$, again μ is a caliber as in the previous Theorem. If $\mu < \lambda$, then μ is a caliber of X_λ as in Case 2 above. If $\mu > \lambda$ but cf $(\mu) < \lambda$, as in Case 1 μ is a caliber of X_λ.

COROLLARY 5. *For any set* Γ *of infinite cardinals, there is a* $T_{3\frac{1}{2}}$ *space* X_Γ *such that* λ *is a caliber of* X_Γ *if and only if* cf $(\lambda) \neq \aleph_0$, $\lambda \notin \Gamma$, *and* cf $(\lambda) \notin \Gamma$.

Proof. If Γ is empty, let X_Γ be, for example, 2^ω. Otherwise let

$$X_\Gamma = \prod \{X_\lambda : \lambda \in \Gamma\}.$$

This solution to Comfort's problem is natural in that the X_λ's are easily defined from the λ's. The only improvement one could ask for is to get a *compact* space with predetermined calibers. One cannot just take compactification of our spaces since this will reintroduce the calibers we have omitted. This is because the X_λ's are dense in powers of 2 and hence inherit *precalibers* [**13**].

3. Baireness and set theory. We now consider the "Baireness" of the $\Sigma(\mathscr{I})$'s.

Definition. Let κ be an infinite cardinal. A space X is κ-*Baire* if the intersection of $\leqq \kappa$ dense open sets is always dense. A *Baire* space is an $\aleph_0$-Baire space.

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Every $\Sigma(\mathscr{I})$ is countably compact and hence Baire. Since $\Sigma(\mathscr{I})$ is dense in 2^{ω_1}, if 2^{ω_1} is not κ-Baire for some κ, then $\Sigma(\mathscr{I})$ is not κ-Baire. We are interested in the $\aleph_1$-Baireness of the $\Sigma(\mathscr{I})$'s because of the sophisticated set theory involved.

Since the Cantor set 2^ω is a direct factor of 2^{ω_1}, if the Cantor set is not $\aleph_1$-Baire, neither is 2^{ω_1} or any other $\Sigma(\mathscr{I})$. Assuming the continuum hypothesis (*CH*) then, no $\Sigma(\mathscr{I})$ is $\aleph_1$-Baire. This seems to be the theme of our results; we can show under various hypotheses how to get all or some $\Sigma(\mathscr{I})$'s not $\aleph_1$-Baire, but we have not been able to find a condition which implies a non-trivial class of $\Sigma(\mathscr{I})$'s are $\aleph_1$-Baire.

It is set-theoretic folklore that it is consistent with the negation of *CH* that the Cantor set be not $\aleph_1$-Baire (add Cohen reals). Thus we see it is consistent with either *CH* or $\sim CH$ that no $\Sigma(\mathscr{I})$ is $\aleph_1$-Baire.

Under the assumption of Martin's Axiom (*MA*) plus $\sim CH$, 2^{ω_1} is $\aleph_1$-Baire. This is the only example we have of a $\Sigma(\mathscr{I})$ even consistently $\aleph_1$-Baire. The following result gives a useful condition under which $\Sigma(\mathscr{I})$ is not $\aleph_1$-Baire.

THEOREM 6. *Let $\mathscr{I}$ be a σ-ideal on ω_1. Suppose there is a family $\{A_\alpha\}_{\alpha<\omega_1}$ of infinite subsets of ω_1 such that for each $I \in \mathscr{I}$, there is an A_α such that $I \cap A_\alpha = \emptyset$. Then $\Sigma(\mathscr{I})$ is not $\aleph_1$-Baire.*

Proof. Let $U_\alpha = \{f \in \Sigma(\mathscr{I}) : f^{-1}(\{1\}) \cap A_\alpha \neq \emptyset\}$. Each U_α is a dense open subset of $\Sigma(\mathscr{I})$. If $f \in \bigcap\{U_\alpha : \alpha < \omega_1\}$, then $f^{-1}(\{1\})$ meets each A_α so $f \notin \Sigma(\mathscr{I})$.

It is not difficult to see that disjointness can be weakened to finite intersection in this result.

Definition [**14**]. An uncountable subset L of ω_1 is said to be *$\mathscr{I}$-Lusin* for a σ-ideal $\mathscr{I}$ if $L \cap I$ is countable for all $I \in \mathscr{I}$.

COROLLARY 7. *If there is an $\mathscr{I}$-Lusin set, then $\Sigma(\mathscr{I})$ is not $\aleph_1$-Baire.*

Proof. Let $L = \{a_\beta : \beta < \omega_1\}$. Let $A_\alpha = \{a_\beta : \beta \geqq \alpha\}$.

Observe that the condition in Corollary 7 is just the negation of that in Theorem 2. Thus if $\Sigma(\mathscr{I})$ is $\aleph_1$-Baire, $\Sigma(\mathscr{I})$ has caliber $\aleph_1$. Indeed Tall [**13**] shows more generally that any countable chain condition $\aleph_1$-Baire space has caliber $\aleph_1$. Note that any uncountable subset of ω_1 satisfies the condition in Corollary 7 for the space Σ, showing that that space is not $\aleph_1$-Baire. This result is due to Solomon [**13**].

Definition. A σ-ideal $\mathscr{I}$ is *κ-generated* if there is a set $\mathscr{I}' \subseteq \mathscr{I}$, $|\mathscr{I}'| \leqq \kappa$, such that each member of $\mathscr{I}$ is included in a countable union of members of $\mathscr{I}'$.

THEOREM 8 [**14**]. *If $\mathscr{I}$ is $\aleph_1$-generated, then there is an $\mathscr{I}$-Lusin set.*

There is a model of set theory (namely that for Baumgartner's version of

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Generalized Martin's Axiom) in which $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} > \aleph_2$, and for each $\mathscr{I}$ which is κ-generated for some $\kappa < 2^{\aleph_1}$ there is an $\mathscr{I}$-Lusin set.

It is thus easy to generate σ-ideals which—assuming some extra set-theoretic axioms if necessary—fail to have caliber $\aleph_1$ and hence are not $\aleph_1$-Baire.

Our next combinatorial notion seems to have a more felicitous formulation in terms of filters rather than ideals. Given a σ-ideal $\mathscr{I}$, let $\mathscr{I}^*$ be the dual countably complete filter, i.e. $\mathscr{I}^* = \{S \subseteq \omega_1 : \omega_1 - S \in \mathscr{I}\}$.

Definition. ⫯$_{\mathscr{I}}$ is the assertion that there exist infinite subsets of $\omega_1 \{A_\alpha\}_{\alpha<\omega_1}$ such that each uncountable member of $\mathscr{I}^*$ includes some A_α. ⫯ is ⫯$_{\mathscr{P}(\omega_1)}$.

By Theorem 6, ⫯$_{\mathscr{I}}$ implies $\mathscr{I}$ is not $\aleph_1$-Baire *if $\mathscr{I}$ is proper*. It also implies 2^{ω_1} is not $\aleph_1$-Baire, for let $\{A_\alpha\}_{\alpha<\omega_1}$ witness ⫯. Let $T_\alpha = \{f \in 2^{\omega_1} : f(\xi) = 1$ for all $\xi \in A_\alpha\}$. Then $f \notin \bigcup \{T_\alpha : \alpha < \omega_1\}$ implies $|\{\xi : f(\xi) = 1\}| \leqq \aleph_0$. Let $S_\alpha = \{f \in 2^{\omega_1} : f(\xi) = 0$ for all $\xi > \alpha\}$. Then

$$2^{\omega_1} = \bigcup \{T_\alpha : \alpha < \omega_1\} \cup \{S_\alpha : \alpha < \omega_1\}.$$

But each S_α and T_α is nowhere dense. It follows that $MA + \sim CH$ refutes ⫯. Baumgartner had also shown this directly.

Clearly ⫯ implies ⫯$_{\mathscr{I}}$ for all $\mathscr{I}$. CH trivially implies ⫯ as does

$\clubsuit$: there exists $\{S_\lambda : \lambda$ countable limit ordinal$\}$ such that $S_\lambda \subseteq \lambda$, the order-type of S_λ in the natural order is ω, $\bigcup S_\lambda = \lambda$, and every uncountable subset of ω_1 includes some S_λ.

$\clubsuit$ was introduced in [**11**] and is called "club". ⫯, being weaker, is here called "stick". This proposition was considered and generalized in [**2**]. CH does not imply $\clubsuit$ since $\clubsuit + CH = \Diamond$ [**5**] and $CH \not\rightarrow \Diamond$ (Jensen [**6**]). Hence ⫯ $\not\rightarrow \clubsuit$. Shelah has recently shown that $\clubsuit$ does not imply CH. It follows (as Baumgartner [**2**] had already shown) that ⫯ $\not\rightarrow CH$. Baumgartner also obtains models in which ⫯ fails, for example by adjoining $\aleph_2$ Cohen reals. These models of Baumgartner, we note for future use, are obtained by CCC extensions.

Although we cannot prove that no $\Sigma(\mathscr{I})$, $\mathscr{I}$ proper, is $\aleph_1$-Baire, we can show that $(\Sigma(\mathscr{I}))^\omega$ cannot be $\aleph_1$-Baire for any proper $\mathscr{I}$. This follows from the fact that no such $\Sigma(\mathscr{I})$ is separable, but all have π-weight $\aleph_1$. As Juhász observed [**8**; **13**], if X^ω is $\aleph_1$-Baire and $\pi(X) \leqq \aleph_1$, then X is separable. It follows that, assuming $MA + \sim CH$, $\Sigma(\mathscr{I})$ has no "reasonable" completeness property (else $(\Sigma(\mathscr{I}))^\omega$ would be $\aleph_1$-Baire [**13**]).

In contrast to the $\aleph_1$-Baire question, we do have two interesting examples of $\mathscr{I}$'s for which $\Sigma(\mathscr{I})$ has caliber $\aleph_1$. It follows from Theorem 2 that $\Sigma(\mathscr{P}(\omega_1))$ has caliber $\aleph_1$. Another example can be obtained by generating a σ-ideal from the elements of a maximal almost disjoint collection of subsets of ω_1. Most interesting is $\Sigma(\mathscr{J})$ for the σ-ideal $\mathscr{J}$ of *nonstationary* sets. (A subset of ω_1 is nonstationary if its complement includes a closed unbounded set. See [**10**] for discussion and proof that $\mathscr{J}$ is a σ-ideal.) It is an easy exercise to show that every uncountable subset of ω_1 includes an uncountable nonstationary set, so by Theorem 2, $\Sigma(\mathscr{J})$ has caliber $\aleph_1$.

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It is known (see e.g. [**1**, 7.5]) that if $M[G]$ is a CCC extension of a model M, then every closed unbounded subset of ω_1 in the extension includes a closed unbounded set in the ground model. Therefore $⫯_{\mathscr{J}}$ is preserved by CCC extensions. In particular then, there are models of $\sim ⫯$ indeed of $MA + \sim CH$ in which $⫯_{\mathscr{J}}$ holds and hence $\Sigma(\mathscr{J})$ is not $\aleph_1$-Baire. Indeed we do not know the answer to the following:

Problem. Find a model for $\sim ⫯_{\mathscr{J}}$.

Note added in proof. After seeing the first version of this paper, Baumgartner constructed a model in which $\Sigma(\mathscr{J})$ is $\aleph_1$-Baire. Indeed, as he later noted, this conclusion follows from Shelah's "Proper Poset Axiom," which is a strengthening of Martin's Axiom plus $\sim CH$. It follows that whether or not $\Sigma(\mathscr{J})$ is $\aleph_1$-Baire is independent of Martin's Axiom plus $\sim CH$.

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