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ON BERNDT'S METHOD IN ARITHMETICAL FUNCTIONS AND CONTOUR INTEGRATION

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Introduction. If f is a suitable meromorphic function then, by a classical technique in the calculus of residues, one can evaluate in closed form series of the form

$$(1.1)\qquad \sum_{n=-\infty}^{\infty} a(n)f(n) \quad \text{or} \quad \sum_{n=-\infty}^{\infty} (-1)^n f(n).$$

Recently Bruce C. Berndt ([1]) considered using contour integration for the evaluation of series of the form

$$(1.2)\qquad \sum_{n=-\infty}^{\infty} f(n) \quad \text{or} \quad \sum_{n=-\infty}^{\infty} (-1)^n a(n) f(n).$$

where f belongs to a suitable class of rational functions and $a(n)$ is an arithmetical function. He developed a new method to transform series of type (1.2) above (and of slightly more general types) into series generally involving a different arithmetical function. As he claims his method is applicable to arithmetical functions $a(n)$ which have the representation

$$a(n) = \sum_{d/n} g(d)h(d, n)$$

where g and h are arithmetical functions and for each fixed d, $h(d, z)$ is a polynomial in z. In fact he illustrated his method (cf. [1], theorems 1, 2, 3 and 4) by considering four well known arithmetical functions.

In §2 of this paper we refine Berndt's method so as to be applicable to series of type (1.2) (and more general types also) where the arithmetical function $a(n)$ and the rational function f are arbitrary subject to the only requirement that $\sum f(n)$ and $\sum a(n)f(n)$ are absolutely convergent. In §3, in addition to deducing theorems 1 through 4 of [1] as special cases of our theorem in §2, we further illustrate it by choosing particular rational functions while allowing $a(n)$ to be arbitrary. In §4 we specialize $a(n)$ to evaluate some series, involving the Möbius function $\mu(n)$, the Liouville's function $\lambda(n)$, the Jordan totient function $J(n)$ and the well known Ramanujan's trigonometric sum $c_n(m)$, in closed form.

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2. Main result. In the sequel we use the following notation:

Let $a(n)$ be an arithmetical function. For each positive integer n we define $a(-n)$ to be $a(n)$. We denote by $a^*(n)$ the unique arithmetical function satisfying $a(n)=\sum_{d\mid n} a^*(d)$ or equivalently $a^*(n)=\sum_{d\mid n} \mu(d)a(n/d)$ where μ is the Möbius function. For any set X of positive integers we write $a_X(n)$ for $\sum_{d\in X, d\mid n} a^*(d)$.

For complex z and positive integral d we write

$$S(d, z)=\sum_{j=0}^{d-1} \exp(2\pi izj/d) \quad \text{and} \quad T(d, z)=\sum_{j=-[(d-1)/2]}^{[d/2]} \exp(2\pi izj/d)$$

Clearly for integral n

$$S(d, n)=T(d, n)=\begin{cases} d & \text{if } d \mid n \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

If f is meromorphic in the complex plane we write $P(=P(f))$ to denote the set consisting of the origin and all the poles of f (when f is rational P is clearly finite). Also $R(f(z): z=\omega)$ stands for the residue of f at ω. We write

$$s(n)=s_f(n)=\sum_{\omega\in P} R\left(\frac{\pi e^{-\pi iz}S(n, z)f(z)}{\sin \pi z}: z=\omega\right)$$

and

$$t(n)=t_f(n)=\sum_{\omega\in P} R\left(\frac{\pi T(n, z)f(z)}{\sin \pi z}: z=\omega\right).$$

We now prove the following

THEOREM. *Let X be a set of positive integers, $a(n)$ an arithmetical function and f a rational function such that $\sum f(n)$ and $\sum a(n)f(n)$ are absolutely convergent. Then*

$$\sum_{\substack{n=-\infty \\ n\notin P}}^{\infty} a_X(n)f(n)=-\sum_{n\in X} \frac{a^*(n)s(n)}{n} \tag{2.2}$$

and

$$\sum_{\substack{n=-\infty \\ n\notin P}}^{\infty} (-1)^n a_X(n)f(n)=-\sum_{n\in X} \frac{a^*(n)t(n)}{n}. \tag{2.3}$$

We require the following

LEMMA. *Let $a'(n)$ denote $\sum_{d\mid n} |a^*(d)|$. Then, under the hypotheses of the theorem,* $\sum a'(n)\,|f(n)|$ converges.

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Proof. The hypotheses imply that there is an integer $p \geq 2$ such that $z^p f(z)$ tends to a nonzero limit as $z \to \infty$. Thus it is enough to consider $\sum_{n=1}^{\infty} a'(n)n^{-p}$. We have

$$a'(n) = \sum_{rs=n} |a^*(r)| \leq \sum_{rs=n} \sum_{d\delta=r} |a(d)| = \sum_{d|n} |a(d)|\tau(n/d)$$

where $\tau(n)$ denotes the number of positive divisors of n. Since $\sum_{n=1}^{\infty} |a(n)|\, n^{-p}$ and $\sum_{n=1}^{\infty} \tau(n)n^{-p}$ are convergent series of non-negative terms, their Dirichlet product series, namely

$$\sum_{n=1}^{\infty} \left(\sum_{d|n} |a(d)|\, \tau(n/d)\right) n^{-p},$$

converges. The lemma now follows.

Proof of the theorem. For each positive integer m, we put

$$A(m, z) = \sum_{d=1}^{m} a^*(d)S(d, z)\, d^{-1}$$

and

$$B(m, z) = \sum_{d=1}^{m} a^*(d)T(d, z)\, d^{-1}.$$

If N is a positive integer, let C_N denote the positively oriented square with centre at the origin and sides of length $(2N+1)$ parellel to the real and imaginary axes. For values of N large enough to ensure that P is contained in the interior of C_N we consider the integrals

$$I(m, N) = \frac{1}{2\pi i} \int_{C_N} \frac{\pi e^{-\pi i z} A(m, z) f(z)}{\sin \pi z}\, dz$$

and

$$J(m, N) = \frac{1}{2\pi i} \int_{C_N} \frac{\pi B(m, z) f(z)}{\sin \pi z}\, dz.$$

By Cauchy's residue theorem, each of the above integrals equals the sum of the residues of the integrand at its poles in the interior of C_N. If n is an integer, not in P, then

$$R\left(\frac{\pi e^{-\pi i z} A(m, z) f(z)}{\sin \pi z} : z = n\right) = A(m, n) f(n)$$
$$= f(n) \sum_{\substack{d=1 \\ d|n}}^{m} a^*(d)$$

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in virtue of (2.1). Hence

$$I(m, N)=\sum_{\substack{n=-N\\ n\notin P}}^{N} A(m, n)f(n)+\sum_{\omega\in P} R\left(\frac{\pi e^{-\pi iz}A(m, z)f(z)}{\sin \pi z} : z=w\right)$$

(2.4) $$=\sum_{\substack{n=-N\\ n\notin P}}^{N} A(m, n)f(n)+\sum_{d=1}^{m} \frac{a^*(d)s(d)}{d}.$$

If $z=x+iy$,

$$\left|\frac{e^{-\pi iz}A(m, z)}{\sin \pi z}\right|\leq\frac{2e^{\pi y}\sum_{d=1}^{m}\left|\frac{a^*(d)}{d}\right|\sum_{j=0}^{d-1'} e^{-2\pi yj/d}}{|e^{\pi y}-e^{-\pi y}|}$$

which is bounded as $y\to\pm\infty$. Thus there exists a constant $M=M(m)$, independent of N, such that for all $z\in C_N$,

$$\left|\frac{e^{-\pi iz}A(m, z)}{\sin \pi z}\right|\leq M.$$

Since the convergence of $\sum|f(n)|$ implies the existence of a $k>0$ and of an integer $c\geq 2$ such that $|f(z)|\leq k|z|^{-c}$ for all large $|z|$, we have

$$|I(m, N)|\leq 2Mk(2N+1)\left(N+\frac{1}{2}\right)^{-c}.$$

Thus $I(m, N)\to 0$ as $N\to\infty$ and from (2.4) we have

$$\sum_{\substack{n=-\infty\\ n\notin P}}^{\infty} A(m, n)f(n)=-\sum_{d=1}^{m} \frac{a^*(d)s(d)}{d}.$$

Taking limits as $m\to\infty$ on both sides we obtain

(2.5) $$\sum_{\substack{n=-\infty\\ n\notin P}}^{\infty} a(n)f(n)=-\sum_{n=1}^{\infty} \frac{a^*(n)s(n)}{n},$$

since on the left side the above lemma allows us to take the limit on m inside the summation sign. Replacing $a^*(n)$ by its product with the characteristic function of X one obtains (2.2). The proof of (2.3) follows along the same lines. In particular we obtain

(2.6) $$\sum_{\substack{n=-\infty\\ n\notin p}}^{\infty} (-1)^n a(n)f(n)=-\sum_{n=1}^{\infty} \frac{a^*(n)t(n)}{n}.$$

3. Applications. As mentioned in the introduction we deduce in this section, theorems 1 through 4 of [1] as special cases of the above theorem and record some more results illustrating that theorem.

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For integers $a \geq 0$, $q \geq 1$ and $(q, a) = 1$, put $A(q, a) = \{mq + a \mid m \geq 0, m$ integral$\}$. Let $\sigma_\nu(n)$, $\varphi_{r,s,t}(n)$, $\chi(n)$, $r(n)$ and $\Lambda(n)$ be the arithmetical functions defined by

$$\sigma_\nu(n) = \sum_{d\mid n} d^\nu, \; n^{-t}\varphi_{r,s,t}(n) = \sum_{d\mid n} \mu_r^s(d)\, d^{-t},$$

$\chi(n) = 0$ or $(-1)^{(n-1)/2}$ according as n is even or odd,

$r(n) = 4\sum_{d\mid n} \chi(d)$ (the number of representations of n as a sum of two integral squares) and

$\Lambda(n) = \log p$ or 0 according as n is a power of the prime p or not (known as von Mangoldt's function). Here r, s, t are positive integers with $s \leq 2$ and $\mu_r(n) = \mu(n^{1/r})$ or 0 according as $n^{1/r}$ is an integer or not (known as Klee's generalization of the Möbius function).

Writing $X = A(q, a)$ and taking $a(n) = \sigma_\nu(n)$, $r(n)$ and $\log n$ in turn in the theorem of §2, we obtain theorems 1, 3 and 4 of [1] (noting that $\log n = \sum_{d\mid n} \Lambda(d)$). Further, taking $a(n) = n^{-t}\varphi_{r,s,t}(n)$ and replacing $f(z)$ with $z^t f(z)$ in the theorem of §2, we obtain theorem 2 of [1]. It may be noted that the condition that $f(z) = 0(|z|^{-c})$ as $|z| \to \infty$ together with the inequality imposed on c in each of the theorems 1 through 4 of [1] is equivalent to the requirement of absolute convergence of the corresponding series $\sum f(n)$ and $\sum a(n)f(n)$.

Now let α, β, γ, δ be complex numbers such that $\alpha \neq ni$, $\beta \neq n$; $\gamma \neq n$, $\omega^{\pm 1}n$ and $\delta \neq P^{\pm 1}n$ for all integers n where $\omega = \exp(2\pi i/3)$ and $P = \exp(\pi i/4)$.

Taking $f(z) = 1/(z^2 + \alpha^2)$ and adding appropriate residues at 0, $i\alpha$ and $-i\alpha$ we obtain

$$\begin{aligned} s(n) &= \sum_{j=0}^{n-1} \left\{ \frac{1}{\alpha^2} + \frac{\pi e^{\pi\alpha(1-2j/n)} + \pi e^{-\pi\alpha(1-2j/n)}}{2i\alpha \sin \pi i\alpha} \right\} \\ &= -\frac{n}{\alpha^2}\left(\frac{\pi\alpha}{n}\coth\frac{\pi\alpha}{n} - 1\right) \end{aligned}$$

and

$$\begin{aligned} t(n) &= \sum_{j=-[(n-1)/2]}^{[n/2]} \left\{ \frac{1}{\alpha^2} + \frac{\pi e^{-2\pi\alpha j/n} + \pi e^{2\pi\alpha j/n}}{2i\alpha \sin \pi i\alpha} \right\} \\ &= -\frac{n}{\alpha^2}\left\{\frac{\pi\alpha}{n}\operatorname{cosech}\frac{\pi\alpha}{n}\cosh\left(\frac{1+(-1)^n}{2}\frac{\pi\alpha}{n}\right) - 1\right\} \end{aligned}$$

by a straight forward calculation. Hence by (2.5) and (2.6) we obtain

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^2 + \alpha^2} = \frac{1}{2\alpha^2} \sum_{n=1}^{\infty} a^*(n)\left(\frac{\pi\alpha}{n}\coth\frac{\pi\alpha}{n} - 1\right) \tag{3.1}$$

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and

$$(3.1')\qquad \sum_{n=1}^{\infty}\frac{(-1)^n a(n)}{n^2+\alpha^2}=\frac{1}{2\alpha^2}\sum_{n=1}^{\infty}a^*(n)\left(\frac{\pi\alpha}{n}ch_n(\pi\alpha)-1\right)$$

where $ch_n(\theta)=\coth\theta/n$ or $\operatorname{cosech}\theta/n$ according as n is even or odd.

Taking

$$f(z)=\frac{1}{(z+\beta)^2},\quad \frac{1}{z^3+\gamma^3}\quad\text{and}\quad\frac{1}{z^4+\delta^4}$$

in turn and proceeding as above, we obtain

$$(3.2)\qquad \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{a(n)}{(n+\beta)^2}=\frac{1}{\beta^2}\sum_{n=1}^{\infty}a^*(n)\left\{\left(\frac{\pi\beta}{n}\operatorname{cosec}\frac{\pi\beta}{n}\right)^2-1\right\},$$

$$(3.2')\qquad \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^n a(n)}{(n+\beta)^2}=\frac{1}{\beta^2}\sum_{n=1}^{\infty}a^*(n)\left\{\left(\frac{\pi\beta}{n}\operatorname{cosec}\frac{\pi\beta}{n}\right)^2\cos\left(\frac{1-(-1)^n}{2}\frac{\pi\beta}{n}\right)-1\right\},$$

$$(3.3)\qquad \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{a(n)}{n^3+\gamma^3}=\frac{1}{3\gamma^3}\sum_{n=1}^{\infty}a^*(n)\left\{\frac{\pi\gamma}{n}\cot\frac{\pi\gamma}{n}+\frac{\pi\gamma\omega}{n}\cot\frac{\pi\gamma\omega}{n}+\frac{\pi\gamma\bar{\omega}}{n}\cot\frac{\pi\gamma\bar{\omega}}{n}-3\right\},$$

$$(3.3')\qquad \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^n a(n)}{n^3+\gamma^3}=\frac{1}{3\gamma^3}\sum_{n=1}^{\infty}a^*(n)\left\{\frac{\pi\gamma}{n}c_n(\pi\gamma)+\frac{\pi\gamma\omega}{n}c_n(\pi\gamma\omega)+\frac{\pi\gamma\bar{\omega}}{n}c_n(\pi\gamma\bar{\omega})-3\right\}$$

where $c_n(\theta)=\cot\theta/n$ or $\operatorname{cosec}\theta/n$ according as n is even or odd,

$$(3.4)\qquad \sum_{n=1}^{\infty}\frac{a(n)}{n^4+\delta^4}=\frac{1}{4\delta^4}\sum_{n=1}^{\infty}a^*(n)\left\{\frac{\pi\delta\rho}{n}\cot\frac{\pi\delta\rho}{n}+\frac{\pi\delta\bar{\rho}}{n}\cot\frac{\pi\delta\bar{\rho}}{n}-2\right\}$$

and

$$(3.4')\qquad \sum_{n=1}^{\infty}\frac{(-1)^n a(n)}{n^4+\delta^4}=\frac{1}{4\delta^4}\sum_{n=1}^{\infty}a^*(n)\left\{\frac{\pi\delta\rho}{n}c_n(\pi\delta\rho)+\frac{\pi\delta\bar{\rho}}{n}c_n(\pi\delta\bar{\rho})-2\right\}.$$

4. Evaluation of some series. In this section, by suitably specializing $a(n)$, we deduce interesting formulae from the theorem of §2 as well as from the identities listed in §3.

If $a^*(n)$ is the characteristic function of $\{x\}$, x a positive integer, we see that $a(n)$ turns out to be the characteristic function of the set of all non-zero

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multiples of x so that, by (2.5) and (2.6), we have

$$\sum_{n\neq 0, n\equiv 0 (\text{mod } x)} f(n) = -s(x)/x \tag{4.1}$$

and

$$\sum_{n\neq 0, n\equiv 0 (\text{mod } x)} (-1)^n f(n) = -t(x)/x \tag{4.1'}$$

If $a^*(n)$ is an arithmetical function vanishing outside a finite set then $a^*(n)$ is a (complex) linear combination of characteristic functions of singletons so that, in this case, one easily obtains formulae analogous to (4.1) and (4.1′).

If $a(n)$ is the characteristic function of a set X of positive integers, the corresponding function $a^*(n)$, usually called the inversion function of the set X, is denoted by $\mu_X(n)$. Formulae (3.1) through (3.4′) may be specialized for this characteristic function. For example, one can write, in virtue of (3.1),

$$\sum_{\pm n \in X} \frac{1}{n^2+\alpha^2} = \frac{1}{\alpha^2} \sum_{n=1}^{\infty} \mu_X(n)\left(\frac{\pi\alpha}{n}\coth\frac{\pi\alpha}{n} - 1\right). \tag{4.2}$$

Taking $X = \{1\}$ we see that $\mu_X(n) = \mu(n)$ so that (4.2) yields

$$\sum_{n=1}^{\infty} \mu(n)\left(\frac{\pi\alpha}{n}\coth\frac{\pi\alpha}{n} - 1\right) = \frac{2\alpha^2}{\alpha^2+1}. \tag{4.3}$$

If X is the set of all positive integers that are prime to a fixed positive integer m, one can easily verify that $\mu_X(n) = \mu(n)$ or 0 according as $n \mid m$ or not. Hence by (4.2) one has

$$\sum_{\substack{n=1\\(n,m)=1}}^{\infty} \frac{1}{n^2+\alpha^2} = \frac{1}{2\alpha^2} \sum_{n \mid m} \mu(n)\left(\frac{\pi\alpha}{n}\coth\frac{\pi\alpha}{n} - 1\right). \tag{4.4}$$

Taking X to the set of all integral squares we see that $\mu_X(n) = \lambda(n) = (-1)^{\Omega(n)}$ (λ is known as the Liouville's function) where $\Omega(n)$ is the number of prime factors of n, counting repetitions. Now replacing α by δ^2 in (4.2) we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda(n)\left(\frac{\pi\delta^2}{n}\coth\frac{\pi\delta^2}{n} - 1\right) &= 2\delta^4 \sum_{n=1}^{\infty} \frac{1}{n^4+\delta^4} \\ &= \tfrac{1}{2}(\pi\delta\rho \cot \pi\delta\rho + \pi\delta\bar{\rho} \cot \pi\delta\bar{\rho}) - 1.\end{aligned} \tag{4.5}$$

Here we made use of (3.4) with $a(n) = 1$ for all n (so that $a^*(n) = 1$ or 0 according as $n = 1$ or $n > 1$).

Taking $a^*(n) = J(n) = \varphi_{1,1,2}(n)$ (see §3 above), known as Jordan's totient

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function of order 2, we note that $a(n)=n^2$. Now (3.4) yields

$$(4.6)\qquad \sum_{n=1}^{\infty} J(n)\left(\frac{\pi\delta\rho}{n}\cot\frac{\pi\delta\rho}{n}+\frac{\pi\delta\bar{\rho}}{n}\cot\frac{\pi\delta\bar{\rho}}{n}-2\right)$$

$$=4\delta^4\sum_{n=1}^{\infty}\frac{n^2}{n^4+\delta^4}=\pi\sqrt{2}\,\delta^3\frac{\sinh\pi\delta\sqrt{2}-\sin\pi\delta\sqrt{2}}{\cosh\pi\delta\sqrt{2}-\cos\pi\delta\sqrt{2}}.$$

The last step is obtained by writing $n^2/(n^4+\delta^4)=\frac{1}{2}(1/(n^2+\rho^2\delta^2)+1/(n^2+\bar{\rho}^2\delta^2))$ and applying (3.1) separately with $a(n)=1$ for all n.

Concerning the well known Ramanujan's trigonometric sum $c_n(m)$, defined to be the sum of the mth powers of the primitive nth roots of unity, it is known that $\sum_{d/n} c_d(m)=n$ or 0 according as $n\mid m$ or not (cf. [2], §1.5). Hence, taking $a^*(n)$ to be $c_n(m)$ where m is a fixed positive integer in (3.1), we obtain

$$(4.7)\qquad \sum_{n=1}^{\infty} c_n(m)\left(\frac{\pi\alpha}{n}\coth\frac{\pi\alpha}{n}-1\right)=2\alpha^2\sum_{n\mid m}\frac{n}{n^2+\alpha^2}$$

Incidentally we note, since $c_n(1)=\mu(n)$, that (4.3) is a special case of (4.7).

Dividing both sides of (4.7) by α^2 and letting $\alpha\to 0$, one obtains

$$(4.8)\qquad \sum_{n=1}^{\infty}\frac{c_n(m)}{n^2}=\frac{6}{\pi^2}\sigma_{-1}(m).$$

Working similarly with results analogous to (4.7) that may be obtained from (3.1′), (3.4) and (3.4′) respectively, we get

$$(4.8')\qquad \sum_{n=1}^{\infty}\frac{(-1)^n c_n(m)}{n^2}=\frac{2}{\pi^2}(4E_{-1}(m)-\sigma_{-1}(m)),$$

$$(4.9)\qquad \sum_{n=1}^{\infty}\frac{c_n(m)}{n^4}=\frac{90}{\pi^4}\sigma_{-3}(m)$$

and

$$(4.9')\qquad \sum_{n=1}^{\infty}\frac{(-1)^n c_n(m)}{n^4}=\frac{6}{\pi^4}(16E_{-3}(m)-\sigma_{-3}(m))$$

where $E_\nu(m)$ stands for $\sum_{n\mid m}(-1)^n n^\nu$. Where as (4.8) and (4.9) are due to S. Ramanujan (cf. [2], §1.5), (4.8′) and (4.9′) are believed to be new.

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