GENERALISED DIRICHLET SERIES AND HECKE'S FUNCTIONAL EQUATION †

by BRUCE C. BERNDT

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1. Introduction

The generalised zeta-function $\zeta(s, a)$ is defined by

$$\zeta(s, a) = \sum_{n=0}^{\infty} (a+n)^{-s},$$

where $a>0$ and $\text{Re}\, s>1$. Clearly, $\zeta(s, 1) = \zeta(s)$, where $\zeta(s)$ denotes the Riemann zeta-function. In this paper we consider a general class of Dirichlet series satisfying a functional equation similar to that of $\zeta(s)$. If $\phi(s)$ is such a series, we analogously define $\phi(s, a)$. We shall derive a representation for $\phi(s, a)$ which will be valid in the entire complex s-plane. From this representation we determine some simple properties of $\phi(s, a)$.

Throughout the sequel we let $s = \sigma+it$ and $z = x+iy$ with σ, t, x and y real. If c is real, we denote the integral $\int_{c-i\infty}^{c+i\infty}$ by $\int_{(c)}$. The summation sign Σ appearing with no indices will always mean $\sum_{n=1}^{\infty}$.

The following definition is essentially that of Chandrasekharan and Narasimhan (1).

Definition 1. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be two sequences of positive numbers tending to ∞, and $\{a(n)\}$ and $\{b(n)\}$ two sequences of complex numbers not identically zero. Consider the functions ϕ and ψ representable as Dirichlet series

$$\phi(s) = \Sigma a(n)\lambda_n^{-s}, \quad \psi(s) = \Sigma b(n)\mu_n^{-s}$$

with finite abscissae of absolute convergence σ_a and σ_a^*, respectively. If r is real, we say that ϕ and ψ satisfy the functional equation

$$\Gamma(s)\phi(s) = \Gamma(r-s)\psi(r-s)$$

if there exists in the s-plane a domain D which is the exterior of a bounded closed set S such that in D a holomorphic function $\chi(s)$ exists with these properties:

(i) $\chi(s) = \Gamma(s)\phi(s), \ (\sigma>\sigma_a),$

$\chi(s) = \Gamma(r-s)\psi(r-s), \ (\sigma<r-\sigma_a^*);$

† The main result of this paper appeared in the author's Ph.D. dissertation written under the direction of Professor J. R. Smart at the University of Wisconsin in 1966.

(ii) if c and η are chosen so that $c>\max(0, \sigma_a, \sigma_a^*)$, and S lies outside $R=\{s: r-c<\sigma<c, |t|>\eta\}$ but in $r-c<\sigma<c$, then, for some constant $\theta<1$,

$$\chi(s) = O(\exp[e^{\theta\pi|s|/(2c-r)}]), \tag{1.1}$$

uniformly in R as $|s|\to\infty$.

If $b(n)=\gamma a(n)$ with $\gamma=\pm 1$, $r>0$, $\lambda_n=\mu_n=2\pi n/\lambda$ with $\lambda>0$, and $(s-r)\phi(s)$ is entire, then $(2\pi/\lambda)^s\phi(s)$ is a Dirichlet series of signature (λ, r, γ) according to the definition of Hecke (**2**). Thus, $\zeta(2s)$ is of signature $(2, \frac{1}{2}, 1)$.

Definition 2. If ϕ satisfies definition 1, we define a generalised Dirichlet series $\phi(s, a)$ by

$$\phi(s, a) = \Sigma a(n)(a+\lambda_n)^{-s},$$

where $a>0$ and $\sigma>\sigma_a$.

Note that $\zeta(s, a)$ does not satisfy definition 2.

2. Preliminary results

We collect here some lemmas to be employed in the sequel.

Lemma 1. *We have*

$$\Gamma(s) = O(|t|^{\sigma-\frac{1}{2}}e^{-\pi|t|/2}),$$

uniformly for $-\infty<\sigma_1 \leqq \sigma \leqq \sigma_2<\infty$, *as* $|t|\to\infty$.

Lemma 2. *For* $x>0$ *and* $0<c<\sigma$,

$$\frac{1}{2\pi i}\int_{(c)} \Gamma(z)\Gamma(s-z)x^{-z}dz = \Gamma(s)/(1+x)^s.$$

This result is stated in (**4**), p. 192.

Lemma 3. *Let f be holomorphic in a strip S given by $a<\sigma<b$, $|t|>\eta>0$, and continuous on the boundary. If for some constant $\theta<1$,*

$$f(s) = O(\exp[e^{\theta\pi|s|/(b-a)}]),$$

uniformly in S, $f(a+it)=o(1)$ and $f(b+it)=o(1)$ as $|t|\to\infty$, then $f(\sigma+it)=o(1)$ uniformly in S as $|t|\to\infty$.

This is a version of the Phragmén-Lindelöf theorem ((**3**), p. 109).

Lemma 4. *Let $K_\nu(x)$ denote the usual modified Bessel function. Then,*

$$2K_\nu(x) = \frac{1}{2\pi i}\int_{(c)} (x/2)^{\nu-2s}\Gamma(s-\nu)\Gamma(s)ds, \quad c>\max(0, \operatorname{Re}\nu); \tag{2.1}$$

$$K_\nu(x) = 0(x^{-\frac{1}{2}}e^{-x}), \text{ as } x\to\infty; \tag{2.2}$$

$$K_\nu(x) = K_{-\nu}(x); \tag{2.3}$$

$$K_{\frac{1}{2}}(x) = (\pi/2x)^{\frac{1}{2}}e^{-x}. \tag{2.4}$$

Result (2.1) is given in (**4**), p. 197; (2.2), (2.3) and (2.4) are found in (**5**), pp. 202, 79 and 80, respectively. We note from (2.1) that $K_\nu(x)$ is an entire function of ν.

3. Main results

We now establish the representation theorem for $\phi(s, a)$.

Theorem. *Let $\phi(s, a)$ denote a generalised Dirichlet series and let*

$$R(s, a) = \frac{1}{2\pi i}\int_C \Gamma(s-z)\chi(z)a^z dz,$$

where C is a curve, or curves, chosen so that C encircles all of S and does not contain s, if possible, for a given fixed value of s. Then,

$$\Gamma(s)a^s\phi(s, a) = 2a^{(r+s)/2}\Sigma b(n)\mu_n^{(s-r)/2}K_{r-s}(2\sqrt{a\mu_n}) + R(s, a), \tag{3.1}$$

for those values of s such that the series on the right-hand side converges uniformly and which are not contained in C for a suitable choice of C. In particular, if $2\sqrt{a\mu_n} \geqq (1+\varepsilon)\log n$, $\varepsilon > 0$, for $n \geqq N$, and the singularities of $\chi(s)$ are isolated, (3.1) is valid for all values of s, except these isolated singularities.

Proof. Let c be given as in definition 1 and consider s as fixed with $\sigma > c$. Then,

$$\frac{1}{2\pi i}\int_{(c)} \Gamma(s-z)\Gamma(z)\phi(z)a^z dz = \Sigma a(n)\frac{1}{2\pi i}\int_{(c)} \Gamma(s-z)\Gamma(z)(\lambda_n/a)^{-z}dz$$

$$= \Gamma(s)a^s\phi(s, a), \tag{3.2}$$

upon an application of lemma 2. The change in order of summation and integration is justified by absolute convergence, since by lemma 1,

$$\Gamma(s-z)\Gamma(z) = O(|y|^{\sigma-1}e^{-\pi|y|}),$$

as $|y| \to \infty$.

We now move the line of integration to $r-c+it$, $-\infty < t < \infty$, by integrating along the boundary of a rectangle with vertices $c \pm iT$ and $r-c \pm iT$ and then letting $T \to \infty$. By (1.1), lemma 1 and lemma 3, the integrals along the horizontal sides tend to 0 as $T \to \infty$. Hence,

$$\frac{1}{2\pi i}\int_{(c)} \Gamma(s-z)\chi(z)a^z dz = I(s, a) + R(s, a), \tag{3.3}$$

where

$$I(s, a) = \frac{1}{2\pi i}\int_{(r-c)} \Gamma(s-z)\chi(z)a^z dz.$$

Replacing z by $r-z$, using the functional equation, and interchanging the order of summation and integration by absolute convergence, we find

$$I(s, a) = a^r\Sigma b(n)\frac{1}{2\pi i}\int_{(c)} \Gamma(z-\{r-s\})\Gamma(z)(a\mu_n)^{-z}dz$$

$$= 2a^{(r+s)/2}\Sigma b(n)\mu_n^{(s-r)/2}K_{r-s}(2\sqrt{a\mu_n}), \tag{3.4}$$

upon an application of (2.1), provided $c > \max(0, r-c)$. Combining (3.2), (3.3) and (3.4), we have established (3.1). By analytic continuation (3.1) is valid for those values of s such that (3.4) converges uniformly and which are not contained in C. However, by (2.2)

$$I(s, a) = O(\Sigma\,|\,b(n)|\,\mu_n^{(\sigma-r-\frac{1}{2})/2} e^{-2\sqrt{a\mu_n}}). \tag{3.5}$$

It is easy to see that the series of (3.5) converges uniformly if

$$2\sqrt{a\mu_n} \geqq (1+\varepsilon)\log n,\ \varepsilon > 0, \text{ for } n \geqq N.$$

In particular, we have

Corollary 1. *Let $f(s)$ denote a Dirichlet series of signature (λ, r, γ), and let ρ denote the residue of $f(s)$ at $s = r$. Then, for all s,*

$$f(s, a) = a^{-s} f(0) + \Gamma(r)\Gamma(s-r)a^{r-s}\rho/\Gamma(s) + \frac{2\gamma}{\Gamma(s)}\left(\frac{2\pi}{\lambda}\right)^s \Sigma a(n)\left(\frac{a}{n}\right)^{(r-s)/2} K_{r-s}(4\pi\sqrt{an}/\lambda).$$

Proof. The result is immediate on noting that $\chi(s)$ has at most simple poles at $s = 0$ and $s = r$ and on replacing a by $2\pi a/\lambda$ in (3.1).

In the following corollaries we assume $f(s)$ is a Dirichlet series of signature (λ, r, γ).

Corollary 2. *If $f(s)$ is entire, $f(s, a)$ has simple zeros at $s = 0, -1, -2, \ldots.$*

Corollary 2 is clear, since, from the functional equation, $f(0) = 0$ if f is entire.

Corollary 3. *If $f(s)$ is not entire and r is not an integer, then*

$$f(s, a) = a^{-s} f(0),\ s = 0, -1, -2, \ldots.$$

Corollary 4. *Suppose $f(s)$ is not entire. If r is not an integer, $f(s, a)$ has simple poles at $s = r, r-1, r-2, \ldots.$ If r is an integer, $f(s, a)$ has simple poles at $s = 1, 2, \ldots, r$.*

Corollary 5. *$f(s, a)$ is entire if and only if $f(s)$ is entire.*

4. Examples

Let $f(s) = \zeta(2s)$. Since $\zeta(0) = -\frac{1}{2}$ and $\rho = \frac{1}{2}$, we have, by corollary 1 on replacing n by n^2,

$$f(s, a) = -a^{-s}/2 + \pi^{\frac{1}{2}}\Gamma(s-\tfrac{1}{2})a^{\frac{1}{2}-s}/2\Gamma(s) + \frac{2\pi^s}{\Gamma(s)}\Sigma\left(\frac{a}{n^2}\right)^{(\frac{1}{2}-s)/2} K_{\frac{1}{2}-s}(2\pi n\sqrt{a}).$$

If $s = 1$, we find by using (2.3) and (2.4) that

$$\Sigma \frac{1}{n^2+a} = -\frac{1}{2a} + \frac{\pi}{a^{\frac{1}{2}}}\left(\frac{1}{2} + \frac{e^{-2\pi\sqrt{a}}}{1-e^{-2\pi\sqrt{a}}}\right).$$

This result is, of course, known. Also $f(-n, a) = -a^n/2$, where n is a non-negative integer, and $f(s, a)$ has poles at $s = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \ldots.$

Next, consider $f(s) = \Sigma\tau(n)n^{-s}$, where $\tau(n)$ denotes Ramanujan's arithmetical function. It is well known that $f(s)$ is entire and of signature (1, 12, 1). By corollary 1,

$$f(s, a) = \frac{2(2\pi)^s}{\Gamma(s)} \Sigma\tau(n)\left(\frac{a}{n}\right)^{(12-s)/2} K_{12-s}(4\pi\sqrt{an}).$$

$f(s, a)$ is entire and has simple zeros at $s = 0, -1, -2, \ldots$.

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University of Glasgow
Glasgow, W.2
and
University of Illinois
Urbana, Illinois