MULTIPLE SEMIDIRECT PRODUCTS OF ASSOCIATIVE SYSTEMS

by RICHARD STEINER

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1. Introduction. Suppose that a group G is the semidirect product of a subgroup N and a normal subgroup M. Then the elements of G have unique expressions mn ($m \in M$, $n \in N$) and the commutator function

$$(n, m) \mapsto [n, m] = nmn^{-1}m^{-1}$$

maps $N \times M$ into M. In fact there is an action (by automorphisms) of N on M given by

$$(n, m) \mapsto [n, m]m : N \times M \to M.$$

Conversely, if one is given an action of a group N on a group M then one can construct a semidirect product.

These results have been generalised in [**1**, 1]. Let $A(1), \ldots, A(k)$ be the subsets of a finite set $\{1, \ldots, n\}$ listed in some arbitrary order. Suppose that $M_{A(1)}, \ldots, M_{A(k)}$ are groups and $[\ ,\ ] : M_{A(i)} \times M_{A(j)} \to M_{A(i) \cup A(j)}$ are functions. Then [**1**] gives necessary and sufficient conditions for there to be a group G with subgroups $M_{A(1)}, \ldots, M_{A(k)}$ such that

(i) the elements of G have unique expressions $a_1 \ldots a_k$ ($a_i \in M_{A(i)}$),

(ii) for each i and j, the function $[\ ,\ ] : M_{A(i)} \times M_{A(j)} \to M_{A(i) \cup A(j)}$ is a restriction of the commutator function in G.

There are topological applications [**1**, **2**].

Semidirect products have also been constructed for monoids and semigroups [**4**, **5**, **6**], and there is an analogous construction for categories, the split fibration (see [**3**, I.1] or Section 7 below). This paper describes how the multiple semidirect product construction works for monoids and categories. For topological purposes, the appropriate application is to fundamental groupoids, which are special cases of categories. The ideas do not work for semigroups, because one wants a generalized commutator $[a, b]$ to be invertible (with inverse $[b, a]$), and this makes no sense unless one has identities.

The theory for monoids is described in Sections 3 to 6, and the theory for categories in Sections 9 to 11. Section 2 contains conventions on semilattices, which are used to index the monoids and categories. In Section 7, we recall split fibrations of categories, and, in Section 8, we describe "liftings", which are essentially multiple-valued functors.

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2. Semilattices. In [**1**], the groups are indexed by the subsets of $\{1, \ldots, n\}$. In fact all one really needs for these subsets is the partial ordering by inclusion and the operation of union. So the family of subsets of $\{1, \ldots, n\}$ will be replaced by a *semilattice*, that is, a set with a partial ordering $\leq$ in which any two elements A and B have a least upper bound or *join* $A \vee B$. We now give some notation and recall some definitions.

Throughout this paper, Ω will be a semilattice. If two elements A and B of Ω are *incomparable* (neither $A \leq B$ nor $B \leq A$ is true) then we write $A \parallel B$. A *subsemilattice* of

Ω is a subset of Ω closed under join. The family of subsemilattices of Ω will be denoted $\mathcal{S}$; it is partially ordered by set inclusion $\subseteq$. If Φ and Ψ are subsemilattices of Ω then Φ is an *ideal* in Ψ, notation $\Phi \lhd \Psi$, if $\Phi \subseteq \Psi$ and Φ is "closed below" in Ψ: that is, $A \in \Phi$ whenever $A \in \Psi$ and $A \leq B$ for some $B \in \Phi$.

We shall use the subsemilattices of Ω as indices; for instance, we shall consider families of monoids M_Φ $(\Phi \in \mathcal{S})$. If Φ is a finite member of $\mathcal{S}$, say $\Phi = \{A(1), \ldots, A(k)\}$, then we shall write $M_{A(1),\ldots,A(k)}$ for M_Φ, and so on.

We note two facts.

Proposition 2.1. *If F is a finite subset of Ω then it generates a finite subsemilattice of Ω, consisting of the joins of subsets of F.*

Proposition 2.2. *Let Φ be a subsemilattice of Ω. Then Φ has a total ordering $\leq'$ extending the partial ordering $\leq$.*

Proof. Zorn's Lemma.

3. Families of monoids split over a semilattice. Suppose that a group G is a semidirect product of a subgroup N and a normal subgroup M. One can regard this as a system of groups $\{M_\Phi\}$ indexed by the subsemilattices of a two-element semilattice $\{U, V\}$, where $U < V$. Indeed the subsemilattices are $\{U, V\}$, $\{U\}$, $\{V\}$ and $\varnothing$, and we take

$$M_{U,V} = G, \qquad M_U = N, \qquad M_V = M, \qquad M_\varnothing = 1.$$

We then find that each M_Φ is set-theoretically the cartesian product of those M_A for which $A \in \Phi$. We note also that if $\Phi \subseteq \Psi$ then there is in general a projection homomorphism $M_\Psi \to M_\Phi$ precisely when $\Phi \lhd \Psi$ (to be explicit, this happens when $\Phi = \Psi$ or when $\Phi = \varnothing$ or when $\Phi = \{U\}$, $\Psi = \{U, V\}$, but not when $\Phi = \{V\}$, $\Psi = \{U, V\}$).

By generalising these ideas, we obtain the following definition of our object of study. Recall that $\mathcal{S}$ is the family of subsemilattices of the semilattice Ω.

Definition 3.1. A family of monoids M_Φ $(\Phi \in \mathcal{S})$ is *split over* Ω if the following hold.

(i) M_Φ is a submonoid of M_Ψ if $\Phi \subseteq \Psi$ in $\mathcal{S}$.

(ii) If $\Phi \in \mathcal{S}$ and $\leq'$ is a total ordering of Φ extending the partial ordering $\leq$ (see 2.2), then every element x of M_Φ has a unique factorisation

$$x = a_1 \ldots a_k$$

with $a_i \in M_{A(i)}$, $A(1) >' \ldots >' A(k)$ in Φ, $a_i \neq 1$.

(iii) If $\Phi \lhd \Psi$ in $\mathcal{S}$ then there is a homomorphism $f^\Psi_\Phi : M_\Psi \to M_\Phi$ given by: if $a \in M_A$, $A \in \Psi$ then

$$f^\Psi_\Phi a = \begin{cases} a & \text{if } A \in \Phi, \\ 1 & \text{if } A \in \Psi \setminus \Phi. \end{cases}$$

Remark 3.2. Let (M_Φ) be a family of monoids split over Ω such that the factors M_A $(A \in \Omega)$ are all groups. It is clear that all the monoids M_Φ are then groups. Also the uniqueness requirement in 3.1(ii) is then redundant. To see this, suppose that Φ is a subsemilattice of Ω with a total ordering $\leq'$ extending $\leq$, and suppose that

$$x = a_1 \dots a_k = b_1 \dots b_l \text{ in } M_\Phi$$

with $a_i \in M_{A(i)}$, $b_j \in M_{B(j)}$, $A(1) >' \dots >' A(k)$ in Φ, $B(1) >' \dots >' B(l)$ in Φ, $a_i \neq 1$, $b_j \neq 1$. For $C \in \Phi$, let $\Theta(C) = \{D \in \Phi : D \leq C\}$. Then $\Theta(C) \lhd \Phi$; so we can apply $f^{\Phi}_{\Theta(C)}$ to the two factorisations of x. This gives

$$\prod_{A(i) \leq C} a_i = \prod_{B(j) \leq C} b_j \text{ in } M_{\Theta(C)}.$$

Now $M_{\Theta(C)}$ is a group; so if

(i) $\{A(i) : A(i) < C\} = \{B(j) : B(j) < C\}$

and

(ii) $a_i = b_j$ for $A(i) = B(j) < C$

then

(i) $C \in \{A(1), \dots, A(k)\} \Leftrightarrow C \in \{B(1), \dots, B(l)\}$

and

(ii) $a_i = b_j$ if $A(i) = B(j) = C$.

An inductive argument now shows that the two factorisations of x are the same, as required.

Remark 3.3. The condition in 3.1(ii) that $\leq'$ should extend $\leq$ could be replaced by the condition that it should extend $\geq$. This would give a different but isomorphic theory. For groups the results would coincide, by considering inverses. In fact, if one has a family of groups split over Ω then unique factorisation holds as in 3.1(ii) without restriction on the total ordering $\leq'$; see 5.1 below.

Remark 3.4. Suppose that (M_Φ) is a family of monoids split over Ω, that $\Phi = \{A(1), \dots, A(k)\}$ is a finite subsemilattice of Ω, and that Φ has a total ordering $\leq'$ extending $\leq$ given by $A(1) >' \dots >' A(k)$. Then unique factorisation amounts to saying that each element x of M_Φ has a unique expression $x = a_1 \dots a_k$ with $a_i \in M_{A(i)}$, where $a_i = 1$ is permitted.

4. Actions and commutators. In this section we shall show how to describe a family of monoids (M_Φ) split over Ω in terms of the factors M_A $(A \in \Omega)$.

Proposition 4.1. *Let (M_Φ) be a family of monoids split over Ω with factors M_A. If $A < B$ in Ω then there is a unique action*

$$(a, b) \mapsto {}^a b : M_A \times M_B \to M_B$$

of M_A on M_B such that

$$ab = {}^a b a \text{ in } M_{A,B}.$$

If $A \parallel B$ in Ω then there is a unique function

$$(a, b) \mapsto [a, b] : M_A \times M_B \mapsto M_{A \vee B}$$

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such that

$$ab = [a, b]ba \text{ in } M_{A,B,A\vee B}.$$

Proof. Take first the case $A < B$. Let $a \in M_A$, $b \in M_B$. By unique factorisation in $M_{A,B}$ (see 3.4), we have $ab = b'a'$ for unique $b' \in M_B$, $a' \in M_A$. Applying $f_A^{A,B}$ to this equation shows that $a = a'$ (see 3.1(iii)), and we define ${}^a b$ to be b'. This makes $(a, b) \mapsto {}^a b$ the unique function such that $ab = {}^a ba$. To see that we have an action of M_A on M_B, note that ${}^a 1a = a1 = 1a$ $(a \in M_A,\ 1 \in M_B)$, ${}^a(b'b)a = ab'b = {}^a b'ab = {}^a b'{}^a ba$ $(a \in M_A, b', b \in M_B)$, ${}^1 b1 = 1b = b1$ $(1 \in M_A, b \in M_B)$, ${}^{(a'a)}ba'a = a'ab = a'{}^a ba = {}^{a'}({}^a b)a'a$ $(a', a \in M_A, b \in M_B)$, and use unique factorisation to deduce ${}^a 1 = 1$, ${}^a(b'b) = {}^a b'{}^a b$, ${}^1 b = b$, ${}^{(a'a)}b = {}^{a'}({}^a b)$.

The case $A \parallel B$ is similar. Give $\{A, B, A \vee B\}$ the total ordering $A \vee B >' B >' A$ and let $a \in M_A$, $b \in M_B$. By unique factorisation in $M_{A,B,A\vee B}$, we have $ab = cb'a'$ for unique $c \in M_{A\vee B}$, $b' \in M_B$, $a' \in M_A$. By applying $f_A^{A,B,A\vee B}$ and $f_B^{A,B,A\vee B}$, we find that $a' = a$ and $b' = b$. We define $[a, b]$ to be c.

The action and commutator functions determine the entire family of monoids.

THEOREM 4.2. *Let (M_Φ) be a family of monoids split over Ω with factors M_A. Then for each $\Phi \in \mathcal{S}$ the monoid M_Φ can be obtained from the free product $\mathop{*}\limits_{A\in\Phi} M_A$ by applying the relations*

$$ab = {}^a ba \qquad (a \in M_A, b \in M_B, A < B \text{ in } \Phi),$$

$$ab = [a, b]ba \qquad (a \in M_A, b \in M_B, A \parallel B \text{ in } \Phi).$$

Proof. Let $\leq'$ be a total ordering of Φ extending the partial ordering $\leq$ and let $(a_1, \dots, a_k)$ be any word with $a_i \in M_{A(i)}$, $A(i) \in \Phi$. It suffices to show that $(a_1, \dots, a_k)$ can be transformed by use of the relations to a *reduced* word, i.e. one of the form $(b_1, \dots, b_l)$ with $b_j \in M_{B(j)}$, $B(1) >' \dots >' B(l)$, $b_j \neq 1$. For two words representing the same element of M_Φ must then be transformable to the same reduced word, by unique factorisation.

Now, if a word is not reduced then one can apply at least one of the following operations:

(i) delete a if $a = 1$,
(ii) replace a, a' by aa' if $a, a' \in M_A$,
(iii) replace a, b by ${}^a b, a$ if $a \in M_A$, $b \in M_B$, $A < B$,
(iv) replace a, b by $[a, b], b, a$ if $a \in M_A$, $b \in M_B$, $A \parallel B$, $A <' B$.

It suffices to show that repeated application of these operations must reach a reduced word in finitely many steps.

Note that, when we start with the particular word $(a_1, \dots, a_k)$, there are only finitely many members of Φ that can appear, namely those that lie in the subsemilattice Ψ generated by $A(1), \dots, A(k)$ (see 2.1). Say that the elements of Ψ are $B(1), \dots, B(l)$ with $B(1) >' \dots >' B(l)$. Given a word $(c_1, \dots, c_m)$ with the c_r in the $M_{B(i)}$, we assign to it the $(l+1)$-tuple of non-negative integers $(\mu, \lambda_1, \dots, \lambda_l)$ for which μ is the number of c_r which are equal to 1 and for which λ_j is the number of pairs c_r, c_s with $r < s$, $c_r \in M_{B(i)}$, $c_s \in M_{B(j)}$, $B(i) \leq' B(j)$. to complete the proof, one uses the following result.

LEMMA 4.3. *If one of the operations* (i)–(iv) *is applied to a word with* $(l+1)$–*tuple* $(\mu, \lambda_1, \ldots, \lambda_l)$ *then the result is a word with* $(l+1)$-*tuple* $(\mu', \lambda_1', \ldots, \lambda_l')$ *such that* $\lambda_l' = \lambda_l, \ldots, \lambda_{j+1}' = \lambda_{j+1}, \lambda_j' \leq \lambda_j$ *for some* j *or such that* $\lambda_l' = \lambda_l, \ldots, \lambda_1' = \lambda_1, \mu' < \mu$.

Indeed 4.3 implies by induction on l that the reduction process must terminate with the $(l+1)$-tuple $(0, 0, \ldots, 0)$, since the $(l+1)$-tuples consist of non-negative integers. So the process terminates with a reduced word as required.

Proof of 4.3. For operations (i)–(iii) this is easy, For (iv), suppose that one replaces b_i, b_j by $[b_i, b_j]$, b_j, b_i, where $b_i \in M_{B(i)}$, $b_j \in M_{B(j)}$, $[b_i, b_j] \in M_{B(u)}$, $B(i) \parallel B(j)$, $B(i) <' B(j)$, $B(u) = B(i) \vee B(j)$. The reversal of b_i and b_j makes $\lambda_j' < \lambda_j$. If the insertion of $[b_i, b_j] \in M_{B(u)}$ makes λ_v' different from λ_v then one has $B(v) \geq' B(u) = B(i) \vee B(j) >' B(j)$; hence $v < j$. It follows that $\lambda_l' = \lambda_l, \ldots, \lambda_{j+1}' = \lambda_{j+1}, \lambda_j' < \lambda_j$, as required.

5. The case of groups. In this section we justify Remark 3.3; that is, we show that if (M_Φ) is a family of groups split over Ω then unique factorisation holds in each M_Φ for any total ordering of Φ whatsoever.

PROPOSITION 5.1. *Let* (M_Φ) *be a family of groups split over* Ω. *Let* Φ *be a subsemilattice of* Ω *and let* $\leq'$ *be any total ordering of* Φ. *Then each element* x *of* M_Φ *has a unique expression*

$$x = a_1 \ldots a_k \ (a_i \in M_{A(i)}, A(i) \in \Phi, A(1) >' \ldots >' A(k), a_i \neq 1).$$

Proof. Uniqueness is as in 3.2. For existence one uses the method of 4.2, but one uses operation (iii) $(a, b \to {}^a b, a)$ only when $a \in M_A$, $b \in M_B$ with $A < B$, $A <' B$, and one also uses the operation:

$$\text{replace } a, b \text{ by } b, {}^{b^{-1}}a \text{ if } a \in M_A, b \in M_B, B < A, A <' B.$$

6. Conditions for monoids to be factors in a split family. We now come to a deeper question: given monoids M_A $(A \in \Omega)$, actions of M_A on M_B $(A < B$ in $\Omega)$, and functions $[\,,\,]: M_A \times M_B \to M_{A \vee B}$ $(A \parallel B$ in $\Omega)$, what conditions must they satisfy in order to come from a family of monoids split over Ω? The answer is given by the following theorem.

THEOREM 6.1. *A family of monoids split over* Ω *is equivalent to the following*:
(a) *monoids* M_A $(A \in \Omega)$,
(b) *actions* $(a, b) \mapsto {}^a b$ *of* M_A *on* M_B *for* $A < B$ *in* Ω,
(c) *functions* $(a, b) \mapsto [a, b]: M_A \times M_B \to M_{A \vee B}$ $(A \parallel B$ *in* $\Omega)$,
satisfying the identities in Table 1.

REMARK 6.2. Thus the identities in Table 1 can be regarded as a basic set of commutator identities written without inverses (provided that one rewrites the first identity as $[a, b][b, a] = 1$) and subject to "homogeneity" conditions requiring each identity to lie entirely in some factor $M_{A \vee B}$ ((1)–(3)) or $M_{A \vee B \vee C}$ ((4)–(13)). The second identity in (3) is redundant: it can be deduced from the identity in (1) and the first identity in (3). Similarly the identity in (2) could be replaced by $[a, b] = 1$ if $a = 1$. I do not think there are any other redundancies.

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TABLE 1. IDENTITIES IN A FAMILY OF MONOIDS OR CATEGORIES SPLIT OVER A SEMILATTICE

	Condition	Identity
(1)	$A \parallel B$	$[a, b]$ is invertible and $[a, b]^{-1} = [b, a]$
(2)	$A \parallel B$	$[a, b] = 1$ if $a = 1$ or $b = 1$
(3)	$A \parallel B$	$[a'a, b] = {}^{a'}[a, b][a', b]$ $[a, b'b] = [a, b']^{b'}[a, b]$
(4)	$A < B < C$	${}^{({}^{a}b)}({}^{a}c) = {}^{a}({}^{b}c)$
(5)	$A < B, A < C$	$[{}^{a}b, {}^{a}c] = {}^{a}[b, c]$
(6)	$A \vee B = C$	$[a, b]^{b}({}^{a}c) = {}^{a}({}^{b}c)[a, b]$
(7)	$A \vee B < C$	${}^{[a,b]}({}^{b}({}^{a}c)) = {}^{a}({}^{b}c)$
(8)	$A < B, A \vee C = B \vee C$	${}^{({}^{a}b)}[a, c][{}^{a}b, c] = {}^{a}[b, c][a, c]$
(9)	$A < B$	$[{}^{a}b, [a, c]]^{[a,c]}[{}^{a}b, c] = {}^{a}[b, c]$
(10)	$A \vee B = A \vee C = B \vee C$	$[a, b]^{b}[a, c][b, c] = {}^{a}[b, c][a, c]^{c}[a, b]$
(11)	$A \vee C = B \vee C$	${}^{[a,b]}({}^{b}[a, c])^{[a,b]}[b, c] = {}^{a}[b, c][a, c][c, [a, b]]$
(12)	$A < B \vee C$	${}^{[a,b]}[b, [a, c]][[a, b], [a, c]]^{[a,c]}({}^{[a,b]}[b, c]) = {}^{a}[b, c]^{[a,c]}[c, [a, b]]$
(13)	$A, B, C, A \vee B$ $A \vee C, B \vee C,$ $A \vee B \vee C$ distinct	${}^{[a,b]}[b, [a, c]][[a, b], [a, c]]^{[a,c]}[[a, b], [b, c]][[a, c], [b, c]]$ $= [a, [b, c]]^{[b,c]}({}^{[a,c]}[c, [a, b]])$

Note.—In this table A, B, C are elements of a semilattice Ω and $a, a' \in M_A$, $b, b' \in M_B$, $c \in M_C$, where (M_Φ) is a family of monoids or categories split over Ω. In (4)–(12), there are no relations in the subsemilattice generated by A, B, C except those that can be deduced from the condition given.

Proof of 6.1. We shall show how to get the data of 6.1 from a family of monoids split over Ω and vice versa; it will be obvious that the constructions are mutually inverse.

Obtaining the data of 6.1. Suppose that (M_Φ) is a family of monoids split over Ω with factors M_A. Then, by Section 4, one has actions and functions [,]. To see that they satisfy the identities of Table 1, one uses unique factorisation. For identity (1), note that $[a, b][b, a]ab = [a, b]ba = 1ab$, so $[a, b][b, a] = 1$ (see Remark 3.4), and similarly $[b, a][a, b] = 1$. For (2), note that $[a, b]ba = ab = 1ba$ if $a = 1$ or *if* $b = 1$. For the first identity in (3), note that $[a'a, b]ba'a = a'ab = a'[a, b]ba = {}^{a'}[a, b]a'ba = {}^{a'}[a, b][a', b]ba'a$; the second identity in (3) is similar. For identities (4)–(13), one applies the reduction process of the proof of 4.2 to the word (a, b, c) for the various configurations of A, B, C. This requires a total ordering $\leq'$ of the subsemilattice generated by A, B, C extending the partial ordering $\leq$. In each case, we suppose $A <' B <' C$. In most cases this suffices to specify $\leq'$; in the remaining cases $\leq'$ is specified by the following additional conditions:

(11) $C <' A \vee B$,

(12) $C <' A \vee B <' A \vee C$.

(13) $C <' A \vee B <' A \vee C <' B \vee C$.

In each case, to get the left side one first moves a past b and c and then moves b to the right hand end, while to get the right side one first moves b past c and then moves a to the right hand end. In each case, the reduced words are the same on the two sides, except for the first factors (that is, the factors in $M_{A \vee B \vee C}$), and equating these two factors gives the identity required.

For instance, for the left side of identity (4) the reduction is

$$(a, b, c) \to ({}^a b, a, c) \to ({}^a b, {}^a c, a) \to ({}^{({}^a b)a} c, {}^a b, a)$$

and for the right side of (4) it is

$$(a, b, c) \to (a, {}^b c, b) \to ({}^a({}^b c), a, b) \to ({}^a({}^b c), {}^a b, a).$$

In some cases, one has a choice of steps. To get the identities in Table 1, I have taken the leftmost reduction where there is a choice. It is possible for alternative reductions to end up with different first factors (I have found by exhaustive computation that the remaining factors are always the same, however the reduction is done). In such cases, one can see that the results are equivalent to those in the table by using the properties of actions and earlier identities in the table. For instance, on the left side of (11), one might get ${}^{[a,b]}({}^b[a, c][b, c])$, which is the same as what is given since ${}^{[a,b]}(\ )$ is a homomorphism.

Constructing a split family from the data of 6.1. Suppose conversely that one is given monoids M_A $(A \in \Omega)$, together with actions and functions $[\ ,\ ]$ satisfying the identities of Table 1. Then we construct a family (M_Φ) of monoids split over Ω as follows: for Φ a subsemilattice of Ω, the monoid M_Φ is the quotient of the free product of the M_A $(A \in \Phi)$ got by applying the relations

$$ab = {}^a ba \qquad (a \in M_A,\ b \in M_B,\ A < B \text{ in } \Phi),$$
$$ab = [a, b]ba \qquad (a \in M_A,\ b \in M_B,\ A \parallel B \text{ in } \Phi).$$

It is easy to check that one has homomorphisms $f^\Psi_\Phi : M_\Psi \to M_\Phi$ for $\Phi \lhd \Psi$ in $\mathscr{S}$ as required by 3.1(iii). To complete the proof, it suffices to show that one has unique factorisation as required by 3.1(ii) (this will imply that M_Φ is a submonoid of M_Ψ for $\Phi \subseteq \Psi$ in $\mathscr{S}$ as required by 3.1(i)).

Because of 2.1, if $\Phi \in \mathscr{S}$ then M_Φ is the colimit of the $M_{\Phi'}$ with Φ' a finite subsemilattice of Φ; so it suffices to prove unique factorisation for M_Φ when Φ is finite. We use induction on the size of Φ. The result is trivial when Φ has size 0 or 1. For the inductive step, let Φ be a finite subsemilattice of Ω of size at least 2 with a total ordering $\leq'$ extending the partial ordering $\leq$. Let A be the first element of Φ in the total ordering $\leq'$ and let Ψ be $\Phi \backslash \{A\}$. Then Ψ is a subsemilattice of Ω; so the inductive hypothesis applies to M_Ψ. We complete the proof by showing that M_Φ is a semidirect product of M_Ψ and M_A.

Indeed M_Φ is got from the free product of M_Ψ and M_A by applying the relations

$$\begin{aligned} ab &= {}^a ba \qquad (a \in M_A,\, b \in M_B,\, B \in \Psi,\, A < B),\\ ab &= [a, b]ba \qquad (a \in M_A,\, b \in M_B,\, B \in \Psi,\, A \parallel B),\\ ba &= [b, a]ab \qquad (a \in M_A,\, b \in M_B,\, B \in \Psi,\, A \parallel B). \end{aligned}$$

Because of identity (1) in Table 1, the third family of relations is redundant. So, to see that M_Φ is a semidirect product of M_Ψ and M_A, it suffices to show that there is an action $(a, x) \mapsto a_* x$ of M_A on M_Ψ given by

$$\begin{aligned} a_* b &= {}^a b \qquad (a \in M_A,\, b \in M_B,\, B \in \Psi,\, A < B),\\ a_* b &= [a, b]b \qquad (a \in M_A,\, b \in M_B,\, B \in \Psi,\, A \parallel B). \end{aligned}$$

To do this we need to verify that

(i) for $a \in M_A$ and $B \in \Psi$, $a_* : M_B \to M_\Psi$ is a homomorphism,

(ii) for $a \in M_A$ the $a_* : M_B \to M_\Psi$ are compatible with the relations of M_Ψ, i.e. for $b \in M_B$, $c \in M_C$, $B, C \in \Psi$ we have

$$(a_* b)(a_* c) = \begin{cases} a_*({}^b c)(a_* b) & (B < C),\\ a_*[b, c](a_* c)(a_* b) & (B \parallel C), \end{cases}$$

(iii) the resultant function $a \mapsto a_* : M_A \to \operatorname{End} M_\Psi$ is a homomorphism.

Proof of (i). If $A < B$ then $a_* : M_B \to M_\Psi$ is a homomorphism because ${}^a(\ )$ is an endomorphism of M_B. If $A \parallel B$ then $a_* : M_B \to M_\Psi$ is a homomorphism by identities (2) and (3) in Table 1: in M_Ψ we have

$$a_* 1 = [a, 1]1 = 1,$$

$$a_*(b'b) = [a, b'b]b'b = [a, b']^{b'}[a, b]b'b = [a, b']b'[a, b]b = (a_* b')(a_* b)$$

for $b', b \in M_B$ (here ${}^{b'}[a, b]b' = b'[a, b]$ by the defining relations of M_Ψ).

Proof of (iii). Given (ii), it is clearly enough to show that $1_* b = b$, $(a'a)_* b = a'_* a_* b$ for $1, a', a \in M_A$, $b \in M_B$, $B \in \Psi$. This can be done as in the proof of (i).

Proof of (ii). Suppose first that A, B, C are in one of the configurations of identities (4)–(13) in Table 1. Then reducing $(a_* b)(a_* c)$ and $a_*({}^b c)(a_* b)$ or $a_*[b, c](a_* c)(a_* b)$ by the method of the proof of 4.2, using the relations of M_Ψ, will lead (assuming that one makes the leftmost reduction where one has a choice) to expressions of the form $lx_1 \ldots x_s$ and $rx_1 \ldots x_s$, where l and r are the left and right sides of the appropriate identity in Table 1. So the required equation follows from the identity in Table 1.

There are eight other possible configurations of A, B, C; that is, there are eight other semilattices generated by three distinct elements A, B, C for which neither $A > B$ nor

$B > C$ is true. These semilattices have defining relations given by

(14) $B < C, A \vee B = A \vee C$,

(15) $B < C$,

(16) $A \vee B = A \vee C$,

(17) $B < A \vee C$,

(18) $A < C, A \vee B = B \vee C$,

(19) $A < C$,

(20) $A \vee B = B \vee C$,

(21) $C < A \vee B$.

These configurations can be got from those in (4)–(13) of Table 1 by permuting A, B, C. Indeed (14)–(17) are got from (8), (9), (11), (12) by substituting B, C, A for A, B, C and (18)–(21) are got from (8), (9), (11), (17) by interchanging B and C.

Consider first (18)–(21), assuming that we already have the required result in (8), (9), (11), (17). The assumption tells us that $(a_*c)(a_*b) = a_*[c, b](a_*b)(a_*c)$. Using identity (1) gives $(a_*b)(a_*c) = a_*[b, c](a_*c)(a_*b)$, as required.

Next consider (14)–(17). From (8), (9), (11) or (12), we get

$$(b_*c)[b, a]a = b_*[c, a][b, a]a(b_*c) \text{ in } M_\Theta,$$

where $\Theta = \{A, C, A \vee B, A \vee C, B \vee C, A \vee B \vee C\}$,

$$b_*c = \begin{cases} {}^bc & (B < C) \\ [b, c]c & (B \parallel C), \end{cases}$$

and similarly for $b_*[c, a]$ (note that in all these cases $A \parallel B$ and $A \parallel C$; so it is correct to write $[b, a]a$ rather than ${}^ba$, and so on). Hence

$$(b_*c)[b, a]a = b_*[c, a][b, a]a_*(b_*c)a \text{ in } M_\Theta$$

by the relations of M_Θ. Now the inductive hypothesis can be applied to M_Θ because $B \notin \Theta$, so that Θ is smaller than Φ. We deduce that

$$(b_*c)[b, a] = b_*[c, a][b, a]a_*(b_*c)$$

in M_Θ by unique factorisation. Now $M_{\Theta\setminus\{A\}} \subseteq M_\Theta$ by the inductive hypothesis of unique factorisation; so this equation holds in $M_{\Theta\setminus\{A\}}$, hence also in M_Ψ since $\Theta\setminus\{A\} \subseteq \Psi$. We now work on M_Ψ. Using identity (1) of Table 1, we get

$$[a, b]b_*[a, c](b_*c) = a_*(b_*c)[a, b];$$

hence

$$[a, b]b_*[a, c](b_*c)b = a_*(b_*c)[a, b]b;$$

hence

$$[a, b]b[a, c]c = a_*(b_*c)[a, b]b$$

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by the relations of M_Ψ, and this is

$$(a_*b)(a_*c) = a_*(b_*c)(a_*b)$$

as required. This completes the proof.

7. Split fibrations of categories. We begin with some notation and conventions on categories. Generally categories will be small. We write $\mathrm{Ob}\, G$ and $\mathrm{Mor}\, G$ for the objects and morphisms of a category G, and if x is a morphism then we write Sx and Tx for its source and target.

If $f: G \to N$ is a functor then the *kernel* of f is the subcategory $\mathrm{Ker} f$ of G given by

$$\mathrm{Ob}\,\mathrm{Ker} f = \mathrm{Ob}\, G,$$

$$\mathrm{Mor}\,\mathrm{Ker} f = \{m \in \mathrm{Mor}\, G : fm = 1\}.$$

Suppose that we have a functor from some category N to sets given by sets $a(X)$ $(X \in \mathrm{Ob}\, N)$ and functions $n_*: a(Sn) \to a(Tn)$ $(n \in \mathrm{Mor}\, N)$. Suppose also that the sets $a(X)$ $(X \in \mathrm{Ob}\, N)$ are disjoint and have union A. Then we write $N \ltimes A$ for the category with

$$\mathrm{Ob}(N \ltimes A) = A,$$

$$\mathrm{Mor}(N \ltimes A) = \{(n, Y) : n \in \mathrm{Mor}\, N,\ Y \in a(Sn)\},$$

$$S(n, Y) = Y, \qquad T(n, Y) = n_*Y \quad \text{for} \quad (n, Y) \in \mathrm{Mor}(N \ltimes A),$$

$$1_Y = (1_X, Y) \quad \text{for} \quad X \in \mathrm{Ob}\, N, \qquad Y \in a(X),$$

$$(n', n_*Y)(n, Y) = (n'n, Y) \quad \text{for} \quad n', n \in \mathrm{Mor}\, N, \qquad Y \in a(Sn), \qquad Tn = Sn'.$$

We now recall the definition of a split fibration from [**3**, 1.1] (but recast it and change the order of composition).

DEFINITION 7.1. A *split fibration of categories* consists of two categories G and N, a functor $f: G \to N$, and functions $n_*: f^{-1}(Sn) \to f^{-1}(Tn)$ $(n \in \mathrm{Mor}\, N)$ yielding a functor from N to sets and so a category $N \ltimes \mathrm{Ob}\, G$ with $\mathrm{Ob}(N \ltimes \mathrm{Ob}\, G) = \mathrm{Ob}\, G$, such that

(i) $N \ltimes \mathrm{Ob}\, G$ is a subcategory of G,

(ii) $f(n, Y) = n$ for $(n, Y) \in \mathrm{Mor}(N \ltimes \mathrm{Ob}\, G)$,

(iii) each morphism x of G has a unique factorisation $m(n, Y)$ with $m \in \mathrm{Mor}\,\mathrm{Ker} f$, $(n, Y) \in \mathrm{Mor}(N \ltimes \mathrm{Ob}\, G)$.

This generalises the semidirect product construction for monoids, i.e. categories with one object: in 7.1, if G and N are monoids then $N \ltimes \mathrm{Ob}\, G \cong N$ and G is the semidirect product of N and $\mathrm{Ker} f$.

The following result is well-known.

PROPOSITION 7.2. *A split fibration is essentially equivalent to a functor with target the category of categories and functors.*

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Proof. Given a split fibration $f:G\to N$ as in 7.1, one gets a functor F from N to categories by: if X is an object of N then

$$\operatorname{Ob} F(X)=f^{-1}(X),$$
$$\operatorname{Mor} F(X)=\{m\in \operatorname{Mor} G: fm=1_X\},$$

and if n is a morphism in N then the induced functor from $F(Sn)$ to $F(Tn)$ is given by n_* on objects and by $m\mapsto {}^n m$ on morphisms, where ${}^n m(n, Sm)$ is the unique factorisation of $(n, Tm)m$.

Conversely, let $F:N\to$ categories be a functor such that the $\operatorname{Ob} F(X)$ ($X\in \operatorname{Ob} N$) are disjoint (this restriction accounts for the "essentially" in the statement of the proposition). Let A be the union of the $\operatorname{Ob} F(X)$. By restricting to the objects of the $F(X)$, we get a functor from N to sets, hence a category $N\ltimes A$ with objects A. The disjoint union category $\coprod_{X\in \operatorname{Ob} N} F(X)$ also has objects A. Let G be the category with objects A and with morphisms generated by those of $\coprod F(X)$ and $N\ltimes A$, subject to the relations

$$(n, Tm)m=F(n)(m)(n, Sm)\qquad (n\in \operatorname{Mor} N,\ m\in \operatorname{Mor} F(Sn)).$$

Then there is a functor $f:G\to N$ given by

$$\begin{aligned} f(Y)&=X &&(X\in \operatorname{Ob} N,\ Y\in \operatorname{Ob} F(N)),\\ f(m)&=1_X &&(X\in \operatorname{Ob} N,\ m\in \operatorname{Mor} F(X)),\\ f(n, Y)&=n &&(n\in \operatorname{Mor} N,\ Y\in \operatorname{Ob} F(Sn)),\end{aligned}$$

and it gives a split fibration.

8. Liftings. In the last section, we have considered functors on categories of the form $N\ltimes A$. One can regard $N\ltimes A$ as a "covering" of N; so functors on $N\ltimes A$ can be regarded as "multiple-valued functors" on N. We introduce some terminology to make this more convenient.

DEFINITION 8.1. Let N and G be categories. A *lifting* $(\alpha, f):N\to G$ consists of the following:

(a) a function $f:\operatorname{Ob} G\to \operatorname{Ob} N$,

(b) a functor from N to sets which sends $X\in \operatorname{Ob} N$ to the set $f^{-1}(X)$ and sends $n\in \operatorname{Mor} N$ to a function $n_*:f^{-1}(Sn)\to f^{-1}(Tn)$, and so yields a category $N\ltimes \operatorname{Ob} G$,

(c) a functor from $N\ltimes \operatorname{Ob} G$ to G which is the identity on objects.

Thus a split fibration consists of a functor and a lifting satisfying suitable conditions.

In practice, instead of saying that (α, f) is a lifting from N to G, we will usually say that $\alpha:N\to G$ is a lifting for f. Also the functor from $N\ltimes \operatorname{Ob} G$ to G in 8.1(c) is usually an inclusion: so it will be suppressed from the notation.

We record two obvious facts.

REMARK 8.2. Let N and G be categories and $f:\operatorname{Ob} G\to \operatorname{Ob} N$ a bijection. Then a lifting for f is just a functor from N to G which is f^{-1} on objects.

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PROPOSITION 8.3. *Let N, P, Q be categories,*

$$\mathrm{Ob}\,N \xleftarrow{f} \mathrm{Ob}\,P \xleftarrow{g} \mathrm{Ob}\,Q$$

be functions, and let $\alpha: N \to P$, $\beta: P \to Q$ be liftings for f and g. Then there is a composite lifting $\beta\alpha: N \to Q$ for fg given by

$$n_*Z = (n, gZ)_*Z \text{ in } \mathrm{Ob}\,Q, \qquad (n, Z) = ((n, gZ), Z) \text{ in } \mathrm{Mor}\,Q$$

for $n \in \mathrm{Mor}\,N$, $Z \in g^{-1}f^{-1}(Sn)$. With this composition, liftings form the morphisms of a category whose objects are categories.

9. Families of categories split over a semilattice. A family (M_Φ) of categories split over Ω should be like a family of monoids together with additional information about the objects. As in 6.1, the categories M_Φ (Φ a subsemilattice of Ω) should be determined by the factors M_A ($A \in \Omega$). Consider the case of a split fibration $f: G \to N$ (see Section 7). As for a semidirect product (see the beginning of Section 3), one can regard this as a family of categories (M_Φ) indexed by the subsemilattices of $\{U, V\}$, where $U < V$. One takes

$$M_{U,V} = G, \qquad M_U = N, \qquad M_V = \mathrm{Ker}\,f, \qquad M_\varnothing = \mathbf{1},$$

where $\mathbf{1}$ denotes the trivial category on one object. Now $\mathrm{Ob}\,M_V = \mathrm{Ob}\,G$; so the functor $f: G \to N$ gives a function $\mathrm{Ob}\,M_V \to \mathrm{Ob}\,M_U$, which can be interpreted as a contravariant functor from $(\{U, V\}, \leq)$ to sets. One then observes that the $\mathrm{Ob}\,M_\Phi$ (Φ a subsemilattice of $\{U, V\}$) can be regarded as (inverse) limits:

$$\mathrm{Ob}\,M_\Phi = \lim_{A \in \Phi} \mathrm{Ob}\,M_A.$$

Also, if Φ and Ψ are subsemilattices of $\{U, V\}$ with $\Phi \subseteq \Psi$ then there is a canonical projection $q: \mathrm{Ob}\,M_\Psi \to \mathrm{Ob}\,M_\Phi$ and there is a lifting $\alpha: M_\Phi \to M_\Psi$ for q (compare 3.1(i)); if further $\Phi \lhd \Psi$ then there is a functor $M_\Psi \to M_\Phi$ given by q on objects (compare 3.1(iii)).

It is now obvious how to generalise the definition (3.1) of a split family of monoids. The inclusions of 3.1(i) are replaced by liftings. Recall that $\mathcal{S}$ is the family of subsemilattices of Ω.

DEFINITION 9.1. A family of categories M_Φ ($\Phi \in \mathcal{S}$) is *split over* Ω if the following hold.

(i) There are functions $p_A^B: \mathrm{Ob}\,M_B \to \mathrm{Ob}\,M_A$ for $A \leq B$ in Ω forming a contravariant functor from $(\Omega, \leq)$ to sets such that

$$\mathrm{Ob}\,M_\Phi = \lim_{A \in \Phi} \mathrm{Ob}\,M_A$$

for $\Phi \in \mathcal{S}$; we write $q_\Phi^\Psi: \mathrm{Ob}\,M_\Psi \to \mathrm{Ob}\,M_\Phi$ for the canonical projection if $\Phi \subseteq \Psi$ in $\mathcal{S}$.

(ii) There are liftings $\alpha_\Psi^\Phi: M_\Phi \to M_\Psi$ for q_Φ^Ψ for $\Phi \subseteq \Psi$ in $\mathcal{S}$ forming a covariant functor from $(\mathcal{S}, \subseteq)$ to categories and liftings (see 8.3).

(iii) If $\Phi \in \mathcal{S}$ and $\leq'$ is a total ordering of Φ extending the partial ordering $\leq$ (see 2.2) then every morphism x in M_Φ has a unique factorisation

$$x = (a_1, Y_1) \ldots (a_k, Y_k)$$

with $a_i \in \mathrm{Mor}\,M_{A(i)}$, $A(1) >' \ldots >' A(k)$ in Φ, $Y_i \in \mathrm{Ob}\,M_\Phi$, $a_i \neq 1$.

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(iv) If $\Phi \lhd \Psi$ in $\mathscr{S}$ then there is a functor $f_\Phi^\Psi : M_\Psi \to M_\Phi$ which is q_Φ^Ψ on objects and which is given on morphisms by: if $A \in \Psi$, $a \in \operatorname{Mor} M_A$, $Z \in \operatorname{Ob} M_\Psi$ and $q_A^\Psi Z = Sa$ then

$$f_\Phi^\Psi(a, Z) = \begin{cases} (a, q_\Phi^\Psi Z) & \text{if } A \in \Phi, \\ 1 & \text{if } A \in \Psi \backslash \Phi. \end{cases}$$

REMARK 9.2. In the notation (a, Y) for a morphism y, the letter Y indicates the source of y. This is often determined by the context or irrelevant; so we shall sometimes abbreviate (a, Y) to a.

REMARK 9.3. If (M_Φ) is a family of categories split over Ω and Φ is a subsemilattice of Ω with a maximal element C (this happens, for instance, if Φ is finite and non-empty) then we shall identify $\operatorname{Ob} M_\Phi$ with $\operatorname{Ob} M_C$, using 9.1(i).

10. Properties of split families of categories. The properties described in this section are the analogues of those in Sections 3–5, together with additional information about objects. Throughout the section, (M_Φ) is a family of categories split over Ω.

The analogues of 3.2–3.4 are simple.

REMARK 10.1 Suppose that the factors M_A ($A \in \Omega$) are all groupoids. Then, as in 3.2, the M_Φ are groupoids for all $\Phi \in \mathscr{S}$, and the uniqueness in 9.1(iii) is redundant. Also, as in 5.1, one gets unique factorisation for any total orderings of the subsemilattices Φ whatsoever; see 10.7 below.

REMARK 10.2. Just as in Remark 3.4, if (M_Φ) is a family of categories split over Ω and $\Phi = \{A(1), \dots, A(k)\}$ is a finite subsemilattice of Ω with a total ordering $\leq'$ extending $\leq$ given by $A(1) >' \dots >' A(k)$ then the morphisms of M_Φ have unique factorisations

$$(a_1, Y_1) \dots (a_k, Y_k)$$

($a_i \in \operatorname{Mor} M_{A(i)}$, $Y_i \in \operatorname{Ob} M_\Phi$), where the a_i are permitted to be identity morphisms.

As in Section 4, we get actions when we have $A < B$ in Ω. Before considering the actions, we give a basic fact about the behaviour of objects in this situation.

PROPOSITION 10.3. *If $A < B$ in Ω and $b \in \operatorname{Mor} M_B$ then $p_A^B Sb = p_A^B Tb$.*

Proof. There is a morphism (b, Sb) in $M_{A,B}$ (see 9.3) and $\{A\} \lhd \{A, B\}$; so there is a morphism $f_A^{A,B}(b, Sb)$ in M_A by 9.1(iv). By 9.1(iv), this morphism has source $p_A^B Sb$ and target $p_A^B Tb$, and it is an identity. Hence the result.

We now consider the actions. Suppose that $A < B$ in Ω. It is easy to check that $f_A^{A,B} : M_{A,B} \to M_A$ is a split fibration with lifting $\alpha_{A,B}^A : M_A \to M_{A,B}$ (see 7.1). By 7.2, this gives a functor F_B^A from M_A to categories and functors. Let a be a morphism in M_A; then we shall write ${}^a(\ )$ for $F_B^A(a)$. Explicitly, this works out as follows. For every object Y in M_B such that $p_A^B Y = Sa$, there is an object ${}^aY$ in M_B such that $p_A^B({}^aY) = Ta$. And, for every morphism b in M_B such that $p_A^B Sb = p_A^B Tb = Sa$ (note that $p_A^B Sb = p_A^B Tb$ in any case, by 10.3), there is a unique morphism ${}^ab : {}^aSb \to {}^aTb$ in M_B such that

$$(a, Tb)(b, Sb) = ({}^ab, {}^aSb)(a, Sb) : Sb \to {}^aTb \text{ in } M_{A,B}$$

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(where we identify $\operatorname{Ob} M_{A,B}$ and $\operatorname{Ob} M_B$—see 9.3). Following 9.2, we may also write this equation as $ab = {}^a ba$. The functoriality of ${}^a(\ )$ and of F_B^A can be expressed by the familiar equations

$$ {}^a 1 = 1, \quad {}^a(b'b) = {}^a b' {}^a b, \quad {}^1 Y = Y, \quad {}^1 b = b, \quad {}^{(a'a)}Y = {}^{a'}({}^a Y), \quad {}^{(a'a)}b = {}^{a'}({}^a b). $$

We can use the actions to describe the behaviour of the liftings $\alpha_\Psi^\Phi : M_\Phi \to M_\Psi$ for $\Phi \subseteq \Psi$ in $\mathcal{S}$. Recall, from 9.1(i), that $\operatorname{Ob} M_\Psi = \lim_{B \in \Psi} \operatorname{Ob} M_B$, which is a subset of $\prod_{B \in \Psi} \operatorname{Ob} M_B$; we write Z_B for the B-component of Z, where $Z \in \operatorname{Ob} M_\Psi$ and $B \in \Psi$.

Proposition 10.4. *Let $\Phi \subseteq \Psi$ in $\mathcal{S}$, let (a, Y) ($a \in \operatorname{Mor} M_A$, $A \in \Phi$) be a generating morphism of M_Φ, let $Z \in \operatorname{Ob} M_\Psi$ be such that $q_\Phi^\Psi Z = Y$, and let $B \in \Psi$. Then $(a, Y)_* Z = a_* Z$ and*

$$ (a_* Z)_B = \begin{cases} Ta & \text{if } B = A, \\ {}^a(Z_B) & \text{if } B > A, \\ Z_B & \text{otherwise.} \end{cases} $$

Proof. By 9.1(ii), $\alpha_\Psi^\Phi \alpha_\Phi^A = \alpha_\Psi^A$, so $(a, Y)_* Z = a_* Z$ by 8.3 Again, by 9.1(ii), α_Ψ^A is a lifting for q_A^Ψ; so $q_A^\Psi a_* Z = Ta$; that is $(a_* Z)_A = Ta$. Similarly, if $B > A$ then $\alpha_\Psi^A = \alpha_\Psi^{A,B} \alpha_{A,B}^A$; so $a_* Z = (a, Z_B)_* Z$, whence

$$ q_{A,B}^\Psi a_* Z = T(a, Z_B) = {}^a(Z_B); $$

that is, $(a_* Z)_B = {}^a(Z_B)$ (we are identifying $\operatorname{Ob} M_{A,B}$ and $\operatorname{Ob} M_B$). Finally, suppose that $B \geq A$ is false. Let $\Theta = \{C \in \Psi : C \leq B\}$. Then $\Theta \lhd \Psi$ and $A \notin \Theta$; so $f_\Theta^\Psi(a, Z) = 1$ by 9.1(iv) and, with further use of 9.1(iv),

$$ q_\Theta^\Psi a_* Z = f_\Theta^\Psi T(a, Z) = T f_\Theta^\Psi(a, Z) = S f_\Theta^\Psi(a, Z) = f_\Theta^\Psi S(a, Z) = q_\Theta^\Psi Z. $$

On identifying $\operatorname{Ob} M_\Theta$ and $\operatorname{Ob} M_B$, this gives $(a_* Z)_B = Z_B$.

Corollary 10.5. *Let $A < C$ and $B \leq C$ in Ω, and let $a \in \operatorname{Mor} M_A$ and $Z \in \operatorname{Ob} M_C$ be such that $p_A^C Z = Sa$. Then*

$$ p_B^C({}^a Z) = \begin{cases} Ta & \text{if } B = A, \\ {}^a(p_B^C Z) & \text{if } B > A, \\ p_B^C Z & \text{otherwise.} \end{cases} $$

Proof. In 10.4, let Φ be $\{A\}$ and Ψ the subsemilattice of Ω generated by A, B, C. Then $p_B^C[(a_* Z)_C] = (a_* Z)_B$ by the description of $\operatorname{Ob} M_\Psi$ as a limit (9.1(i)), and the result follows on substituting the values from 10.4 for $(a_* Z)_C$ and $(a_* Z)_B$.

We also get commutators. Now let $A \parallel B$ in Ω. Let $a \in \operatorname{Mor} M_A$, $b \in \operatorname{Mor} M_B$, $Z \in \operatorname{Mor} M_{A \vee B}$ be such that $p_A^{A \vee B} Z = Sa$, $p_B^{A \vee B} Z = Sb$. Equate $\operatorname{Ob} M_{A,B,A \vee B}$ with $\operatorname{Ob} M_{A \vee B}$. In $M_{A,B,A \vee B}$, we then have morphisms

$$ (b, Z) : Z \to {}^b Z, $$
$$ (a, {}^b Z) : {}^b Z \to {}^a({}^b Z) $$

(note that $p_A^{A\vee B}({}^bZ) = p_A^{A\vee B}Z = Sa$ by 10.5), and the composite $(a, {}^bZ)(b, Z)$ has a unique factorisation of the form

$$(c, Z_2)(b', Z_1)(a', Z) \quad (a' \in \operatorname{Mor} M_A,\ b' \in \operatorname{Mor} M_B,\ c \in \operatorname{Mor} M_{A\vee B})$$

with $Tc = {}^a({}^bZ)$. Applying $f_A^{A,B,A\vee B}$ and $f_B^{A,B,A\vee B}$ shows that $a' = a$, $b' = b$, whence $Z_1 = {}^aZ$, $Z_2 = {}^b({}^aZ)$. We write

$$[a, b]_Z : {}^b({}^aZ) \to {}^a({}^bZ)$$

in $M_{A\vee B}$ for c; it is then the unique morphism such that

$$(a, {}^bZ)(b, Z) = ([a, b]_Z, {}^b({}^aZ))(b, {}^aZ)(a, Z) : Z \to {}^a({}^bZ)$$

in $M_{A,B,A\vee B}$. When possible, we shall abbreviate $[a, b]_Z$ to $[a, b]$, and this equation will be written $ab = [a, b]ba$.

As in Section 4, the family of categories (M_Φ) split over Ω can be reconstructed from the factors M_A ($A \in \Omega$), the functions $p_A^B : \operatorname{Ob} M_B \to \operatorname{Ob} M_A$, the actions and the commutators. Let Φ be a subsemilattice of Ω. The objects of M_Φ are determined from the p_A^B by 9.1(i), and the morphisms of M_Φ are generated by the (a, Y) ($a \in \operatorname{Mor} M_A$, $A \in \Phi$, $Y \in (q_A^\Phi)^{-1}(Sa)$) by 9.1(iii). The source of (a, Y) is Y and the target is a_*Y, which is determined by the actions as in 10.4. The liftings α_Ψ^Φ are determined from the actions on the generating morphisms of M_Φ by 10.4, and the functors f_Φ^Ψ are determined by 9.1(iv). Finally, in analogy to 4.2, one can show that M_Φ is generated by the $M_A \ltimes \operatorname{Ob} M_\Phi$ ($A \in \Phi$) subject to the relations $ab = {}^aba$ and $ab = [a, b]ba$. We state this formally as follows.

Proposition 10.6. *Let Φ be a subsemilattice of Ω. Then M_Φ is generated by the $M_A \ltimes \operatorname{Ob} M_\Phi$ ($A \in \Phi$) subject to the relations*

(i) $(a, b_*Y)(b, Y) = ({}^ab, a_*Y)(a, Y)$ *for* $A < B$ *in* Φ, $a \in \operatorname{Mor} M_A$, $b \in \operatorname{Mor} M_B$, $Y \in (q_B^\Phi)^{-1}(Sb)$, $p_A^B Sb = Sa$,

(ii) $(a, b_*Y)(b, Y) = ([a, b]_Z, b_*a_*Y)(b, a_*Y)(a, Y)$ *for* $A \parallel B$ *in* Φ, $a \in \operatorname{Mor} M_A$, $b \in \operatorname{Mor} M_B$, $Z \in \operatorname{Ob} M_{A\vee B}$, $Y \in (q_{A\vee B}^\Phi)^{-1}(Z)$, $p_A^{A\vee B}Z = Sa$, $p_B^{A\vee B}Z = Sb$.

Remark 10.7. As in Section 5, one can now prove that if each factor M_A ($A \in \Omega$) is a groupoid then one gets unique factorisation in the M_Φ for any total orderings whatsoever.

11. Conditions for categories to be factors in a split family. We conclude with the analogue of Theorem 6.1.

Theorem 11.1. *A family of categories split over Ω is equivalent to the following*:

(a) *categories* M_A ($A \in \Omega$),

(b) *functions* $p_A^B : \operatorname{Ob} M_B \to \operatorname{Ob} M_A$ ($A \le B$) *yielding a contravariant functor from* ($\Omega, \le$) *to sets such that if* $A < B$ *then* $p_A^B Sb = p_A^B Tb$ *for* $b \in \operatorname{Mor} M_B$,

(c) *functors* F_B^A *from* M_A *to categories and functors for* $A < B$ *in* Ω *such that if* $X \in \operatorname{Ob} M_A$ *then* $F_B^A(X)$ *is the full subcategory of* M_B *on the objects in* $(p_A^B)^{-1}(X)$ (*we write* ${}^aY$ *for* $F_B^A(a)(Y)$ *and* ${}^ab$ *for* $F_B^A(a)(b)$, *where* $a \in \operatorname{Mor} M_A$, $Y \in \operatorname{Ob} M_B$, $b \in \operatorname{Mor} M_B$ *and* $p_A^B Y = Sa$, $p_A^B Sb = Sa$),

(d) *morphisms* $[a, b]_Z = [a, b] : {}^b({}^aZ) \to {}^a({}^bZ)$ *in* $M_{A \vee B}$ *for* $A \parallel B$ *in* Ω, $a \in \operatorname{Mor} M_A$, $b \in \operatorname{Mor} M_B$, $Z \in \operatorname{Ob} M_{A \vee B}$ *with* $p_A^{A \vee B} Z = Sa$, $p_B^{A \vee B} Z = Sb$, *such that*

(i) *if* A, B, $C \in \Omega$, $a \in \operatorname{Mor} M_A$ *and* $Z \in \operatorname{Ob} M_C$ *are such that* $A < C$, $B \leq C$ *and* $p_A^C Z = Sa$, *then*

$$p_B^C({}^aZ) = \begin{cases} Ta & \text{if } B = A, \\ {}^a(p_B^C Z) & \text{if } B > A, \\ p_B^C Z & \text{otherwise}, \end{cases}$$

(ii) *the identities of Table* 1 *hold wherever they make sense.*

Proof. Essentially the same as for 6.1. It is straightforward to check that if (M_Φ) is a family of categories split over Ω then the factors M_A $(A \in \Omega)$ have the structure described; see 10.3 and 10.5 in particular.

Conversely, given the factors M_A $(A \in \Omega)$ with the structure described, one can construct a family (M_Φ) of categories split over Ω as follows.

One takes $\operatorname{Ob} M_\Phi$ $(\Phi \in \mathcal{S})$ to be $\lim_{A \in \Phi} \operatorname{Ob} M_A$, so as to satisfy 9.1(i). Write $q_\Phi^\Psi : \operatorname{Ob} M_\Psi \to \operatorname{Ob} M_\Phi$ for the canonical projection, where $\Phi \subseteq \Psi$ are subsemilattices of Ω.

Given $\Phi \in \mathcal{S}$, $A \in \Phi$, $a \in \operatorname{Mor} M_A$ and $Y \in \operatorname{Ob} M_\Phi$ such that $q_A^\Phi Y = Sa$, one defines an object $a_* Y$ in $\operatorname{Ob} M_\Phi$ by the formulae of 10.4; it follows from 11.1(b) and 11.1(ii) that this gives a well-defined function

$$a_* : (q_A^\Phi)^{-1}(Sa) \to (q_A^\Phi)^{-1}(Ta)$$

and that these functions give a functor from M_A to sets. This gives categories $M_A \ltimes \operatorname{Ob} M_\Phi$ $(A \in \Phi)$, and M_Φ is to be generated by them, subject to the relations of 10.6.

In order for this to make sense, we must check that the sources and targets of the morphisms involved match up. For instance, in relation 10.6(i) we require

$$q_A^\Phi b_* Y = Sa, \qquad q_B^\Phi a_* Y = S({}^ab), \qquad (a_* b_* Y)_C = (({}^ab)_* a_* Y)_C \quad \text{for} \quad C \in \Phi.$$

Most of these verifications are easy. There are two cases which may not be obvious:

(a) in relation 10.6(i),

$$(a_* b_* Y)_C = (({}^ab)_* a_* Y)_C \quad \text{for} \quad C \in \Phi \quad \text{with} \quad B < C,$$

(b) in relation 10.6(ii),

$$(a_* b_* Y)_C = (([a, b]_Z)_* b_* a_* Y)_C \quad \text{for} \quad C \in \Phi \quad \text{with} \quad A \vee B < C.$$

The verifications use identities (4) and (7) respectively from Table 1, and run as follows: for (a),

$$(a_* b_* Y)_C = {}^a({}^b(Y_C)) = T({}^a({}^b(1_{Y_C}))) = T({}^{({}^ab)}({}^a(1_{Y_C}))) = {}^{({}^ab)}({}^aY_C) = (({}^ab)_* a_* Y)_C,$$

and for (b),

$$(a_* b_* Y)_C = {}^a({}^bY_C) = T({}^a({}^b(1_{Y_C}))) = T({}^{[a,b]}({}^b({}^a(1_{Y_C})))) = (([a, b]_Z)_* b_* a_* Y)_C.$$

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It is now straightforward to check that the M_Φ can be made into a family of categories split over Ω. The liftings $\alpha^\Phi_\Psi: M_\Phi \to M_\Psi$ for the q^Ψ_Φ ($\Phi \subseteq \Psi$ subsemilattices of Ω) are given on generators as follows. Let $y=(a, Y)$ ($a \in \operatorname{Mor} M_A$, $A \in \Phi$, $Y \in (q^\Phi_A)^{-1}(Sa)$) be a generating morphism of M_Φ. Then $y_*: (q^\Psi_\Phi)^{-1}(Sy) \to (q^\Psi_\Phi)^{-1}(Ty)$ is the restriction of $a_*: (q^\Psi_A)^{-1}(Sa) \to (q^\Psi_A)^{-1}(Ta)$ and

$$(y, Z) = (a, Z): Z \to y_* Z \quad \text{in} \quad M_\Psi$$

for $Z \in (q^\Psi_\Phi)^{-1}(Sy)$. It is straightforward to check that the α^Φ_Ψ give a well-defined functor from $(\mathcal{S}, \subseteq)$ to categories and liftings as required by 9.1(ii). It is also straightforward to check that for $\Phi \lhd \Psi$ in $\mathcal{S}$ one has functors $f^\Psi_\Phi: M_\Psi \to M_\Phi$ satisfying the conditions of 9.1(iv). Finally, one verifies unique factorisation (9.1(iii)) much as in the proof of 6.1: one shows inductively that if Φ is a finite non-empty subsemilattice of Ω with a total ordering $\leq'$ extending $\leq$ and A is the minimal element of Φ in the total ordering, then $f^\Phi_A: M_\Phi \to M_A$ is a split fibration with kernel $M_{\Phi \setminus \{A\}}$. Indeed, we already have the lifting $\alpha^A_\Phi: M_A \to M_\Phi$, and we verify the required unique factorisation (7.1(iii)) as in the proof of 6.1.

This completes the proof.

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Department of Mathematics
University of Glasgow
University Gardens
Glasgow G12 8QW