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GAUSSIAN BOUNDS FOR COMPLEX SUBELLIPTIC OPERATORS ON LIE GROUPS OF POLYNOMIAL GROWTH

A.F.M. ter Elst and Derek W. Robinson

We prove large time Gaussian bounds for the semigroup kernels associated with complex, second-order, subelliptic operators on Lie groups of polynomial growth.

1. Introduction

Let G be a connected Lie group of polynomial growth with Lie algebra $\mathfrak{g}$, $a_1, \ldots, a_{d'}$ an algebraic basis of $\mathfrak{g}$ and $A_1 = dL_G(a_1), \ldots, A_{d'} = dL_G(a_{d'})$ the corresponding representatives in the left regular representation L_G of G. Consider thé complex subelliptic operator

$$H = -\sum_{k,l=1}^{d'} c_{kl} A_k A_l \tag{1}$$

where the matrix $C = (c_{kl})$ of complex constant coefficients satisfies $2^{-1}(C + C^*) \geqslant \mu I > 0$. Then H generates a continuous semigroup S with a bounded integrable kernel K which satisfies local, that is, small t, Gaussian bounds [**7, 8**] in terms of the subelliptic modulus $|\cdot|'$ and the volume $V'(r)$ of the Haar measure of the ball $B_r = \{g \in G : |g|' < r\}$. In fact one can establish global Gaussian bounds.

Theorem 1.1. *There are $a, b > 0$ such that*

$$\left|K_t(g)\right| \leqslant a\, V'(t)^{-1/2} e^{-b(|g|')^2 t^{-1}} \tag{2}$$

for all $g \in G$ and $t > 0$.

The bounds (2) are well known for real symmetric operators (see, for example, [**19**, Theorem VIII.2.9], or [**16**, Chapter IV]) and are relatively easy to deduce with the aid of Nash–Sobolev inequalities and Davies perturbation theory. In the real case Alexopoulos

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[**1**, **2**] combined the bounds with ideas of homogenisation theory to analyze the asymptotic behaviour of the kernel and to prove boundedness of Riesz transforms.

The derivation of (2) for complex operators is more complicated and partly based on homogenisation theory. It uses a number of distinct ideas. First, one may assume the group G is simply connected since bounds in the general case follow by transference, as in [**11**, p. 201], from bounds for the simply connected covering group. Secondly, one may assume that the local dimension D' and the dimension at infinity D are equal because one can always arrange equality by tensoring with copies of the Heisenberg group, if $D' > D$, or the Euclidean motions group, if $D' < D$, as in the proof of [**10**, Theorem 3.1]. Thirdly, one may deduce local Gaussian bounds, that is, bounds for small t, from De Giorgi estimates, as in [**3, 9**]. Fourthly, and this is the contents of the next two sections, global bounds follow from local bounds if the De Giorgi estimates are uniform in a suitable scaling parameter, as in [**4**]. Homogenisation theory is used to control the scaling.

Throughout the rest of the paper we assume G is simply connected and $D' = D \geqslant 2$.

2. Structure theory

Let $\mathfrak{g} = (\mathfrak{g}, [\,\cdot\,,\,\cdot\,])$ be the Lie algebra of the connected, simply connected, Lie group G and $\mathfrak{q}$, $\mathfrak{n}$ and $\mathfrak{m}$ the radical, the nil-radical and a Levi-subalgebra of $\mathfrak{g}$. Then $\mathfrak{g}$ is the semidirect product $\mathfrak{m} \ltimes \mathfrak{q}$ where

$$\big[(m_1, q_1), (m_2, q_2)\big]_{\mathfrak{m}\ltimes\mathfrak{q}} = \big([m_1, m_2], [q_1, q_2] + [m_1, q_2] - [m_2, q_1]\big)$$

for all $m_1, m_2 \in \mathfrak{m}$ and $\mathfrak{q}_1, \mathfrak{q}_2$. Let Q and M be the connected subgroups of G which have Lie algebras $\mathfrak{q}$ and $\mathfrak{m}$. We assume that G has polynomial growth or, equivalently, M is compact and Q has polynomial growth. Then $\mathfrak{g}$ is of type R, that is, the operators $\mathrm{ad}a$ have purely imaginary eigenvalues for all $a \in \mathfrak{g}$ (see [**13**]).

Next for all $a \in \mathfrak{q}$ let $S(a)$ and $K(a)$ be the semisimple and nilpotent part of the Jordan decomposition of the derivation $\mathrm{ad}a$. Note that $S(\mathfrak{n}) = \{0\}$. It follows from [**1**], Sections 2 and 3, that there exists a subspace $\mathfrak{v} \subseteq \mathfrak{q}$ such that $\mathfrak{q} = \mathfrak{v} \oplus \mathfrak{n}$, $[\mathfrak{v}, \mathfrak{m}] = \{0\}$, $S(\mathfrak{v})\mathfrak{v} = \{0\}$ and $\big[S(\mathfrak{v}), S(\mathfrak{v})\big] = \{0\}$. Then the nilshadow of $\mathfrak{q}$ is defined as the nilpotent Lie algebra $\mathfrak{q}_N = (\mathfrak{q}, [\,\cdot\,,\,\cdot\,]_N)$ where

$$[a, b]_N = [a, b] - S(a_{\mathfrak{v}})b + S(b_{\mathfrak{v}})a$$

with $a_{\mathfrak{v}}, b_{\mathfrak{v}}$ the $\mathfrak{v}$-components of $a, b \in \mathfrak{q}$. The lower central series $\mathfrak{q}_{N;k}$ of $\mathfrak{q}_N$ is given by $\mathfrak{q}_{N;1} = \mathfrak{q}_N$ and $\mathfrak{q}_{N;k+1} = [\mathfrak{q}, \mathfrak{q}_{N;k}]$ for $k \geqslant 1$. Hence $\mathfrak{q}_{N;r+1} = \{0\}$ with r the nilpotency rank of $\mathfrak{q}_N$. Now one can choose vector subspaces $\mathfrak{h}_1, \ldots, \mathfrak{h}_r, \mathfrak{k}$ of $\mathfrak{q}$ and an inner product on $\mathfrak{g}$ with the following properties.

I $\mathfrak{q}_{N;k} = \mathfrak{h}_k \oplus \cdots \oplus \mathfrak{h}_r$ for all $k \in \{1, \ldots, r\}$, $\mathfrak{h}_1 = \mathfrak{v} \oplus \mathfrak{k}$ and $\mathfrak{n} = \mathfrak{k} \oplus \mathfrak{q}_{N;2}$.

II $S(\mathfrak{v})\mathfrak{h}_k \subseteq \mathfrak{h}_k$ and $[\mathfrak{m}, \mathfrak{h}_k] \subseteq \mathfrak{h}_k$ for all $k \in \{1, \ldots, r\}$.

III There exists a (real) inner product $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{g}$ such that $\mathrm{ad}a$ and $S(v)$ are skew-symmetric, for all $a \in \mathfrak{m}$ and $v \in \mathfrak{v}$, and the spaces $\mathfrak{m}, \mathfrak{v}, \mathfrak{k}, \mathfrak{h}_2, \ldots, \mathfrak{h}_r$ are mutually orthogonal.

The first two statements are contained in [**1**]. The third follows because $\mathfrak{g}$ is type R and $[\mathfrak{m}, \mathfrak{v}] = \{0\}$. First one chooses an arbitrary inner product $(\cdot\,, \cdot)$ on $\mathfrak{g}$ for which the subspaces $\mathfrak{m}, \mathfrak{v}, \mathfrak{k}, \mathfrak{h}_2, \ldots, \mathfrak{h}_r$ are mutually orthogonal. Secondly, one defines $\langle \cdot\,, \cdot \rangle$ by averaging. Explicitly,

$$\langle a, b \rangle = \lim_{R \to \infty} V(R)^{-1} \int_{\{x \in \mathbf{R}^{d_0} : |x| < R\}} dx \int_M dm\, (U(x)\mathrm{Ad}(m)a, U(x)\mathrm{Ad}(m)b)$$

where $V(R)$ is the volume (Lebesgue measure) of a ball of radius R in $\mathbf{R}^{d_0}$, dm is the normalised Haar measure on M and $U(x)$ is defined by

$$U(x) = e^{x_1 S(b_1) + \ldots + x_{d_0} S(b_{d_0})}$$

with $b_1, \ldots, b_{d_0}$ a basis of $\mathfrak{v}$. The average exists because M is compact and the $S(v)$ have purely imaginary eigenvalues, since $\mathfrak{g}$ is type R. Note that $U(x)$ commutes with $\mathrm{Ad}(m)$. It follows automatically that $\mathrm{Ad}(m)$ is unitary, with respect to $\langle \cdot\,, \cdot \rangle$, for all $m \in M$. Hence the operators $\mathrm{ad}a$ are skew-symmetric for all $a \in \mathfrak{m}$. Moreover, each $v \in \mathfrak{v}$ has a unique decomposition $v = x_1 b_1 + \ldots + x_{d_0} b_{d_0}$. Hence $S(v) = x_1 S(b_1) + \ldots + x_{d_0} S(b_{d_0})$ and it follows from the averaging that the $S(v)$ are also skew-symmetric. Since the subspaces $\mathfrak{m}, \mathfrak{v}, \mathfrak{k}, \mathfrak{h}_2, \ldots, \mathfrak{h}_r$ are invariant under $S(\mathfrak{v})$ and $\mathrm{ad}(\mathfrak{m})$ it follows that they remain orthogonal with respect to the averaged inner product.

For all $u > 0$ let $\gamma_u \colon \mathfrak{g} \to \mathfrak{g}$ be the linear map such that $\gamma_u(b_i) = u^{w_i} b_i$ for all $i \in \{-d_m, \ldots, d\}$, where $w_i = 0$ if $b_i \in \mathfrak{m}$ and $w_i = k$ if $b_i \in \mathfrak{h}_k$. Next define $[\cdot\,, \cdot]_u \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by $[a, b]_u = \gamma_u^{-1}([\gamma_u(a), \gamma_u(b)])$. Then $\mathfrak{g}_u = (\mathfrak{g}, [\cdot\,, \cdot]_u)$ is a Lie algebra and $\gamma_u \colon \mathfrak{g}_u \to \mathfrak{g}$ a Lie algebra isomorphism. Define similarly the nilpotent Lie algebra $\mathfrak{q}_{Nu} = (\mathfrak{q}, [\cdot\,, \cdot]_{Nu})$ with $[a, b]_{Nu} = \gamma_u^{-1}([\gamma_u(a), \gamma_u(b)]_N)$. Then $\mathfrak{q}_{Nu}$ is the nilshadow of $\mathfrak{g}_u$. If $a *_{Nu} b$ denotes the Campbell–Baker–Hausdorff formula in a and b with respect to $[\cdot\,, \cdot]_{Nu}$ on $\mathfrak{q}_{Nu}$ then $Q_{Nu} = (\mathfrak{q}, *_{Nu})$ is the connected simply connected nilpotent Lie group with Lie algebra $\mathfrak{q}_{Nu}$. Set $G_{Nu} = M \times Q_{Nu}$. We denote by $*_{Nu}$ the multiplication on G_{Nu} and by $g^{(-1)_{Nu}}$ the inverse of g. Define $\tau_u \colon \mathfrak{g}_{Nu} \to \mathcal{L}(\mathfrak{g}_{Nu})$ by $\tau_u(a)b = (\mathrm{ad}a_{\mathfrak{m}} + S(\gamma_u(a_{\mathfrak{v}}))b_{\mathfrak{q}}$, where $a_{\mathfrak{m}}$ and $a_{\mathfrak{v}}$ are the components of a in $\mathfrak{m}$ and $\mathfrak{v}$ and $b_{\mathfrak{q}}$ is the component of b in $\mathfrak{q}$. If $\overline{T_u} \colon \mathfrak{g}_{Nu} \to \mathrm{Aut}(\mathfrak{g}_{Nu})$ is the homomorphism such that $\overline{T_u}(\exp_{G_{Nu}} a) = e^{\tau_u(a)}$ and $T_u \colon G_{Nu} \to \mathrm{Aut}(G_{Nu})$ is the Lie group homomorphism such that

$$T(\exp_{G_{Nu}} a) \exp_{G_{Nu}} b = \exp_{G_{Nu}} (\overline{T_u}(\exp_{G_{Nu}} a) b)$$

for all $a, b \in \mathfrak{g}_{Nu}$ then $(g, h) \mapsto g \;{}_{T_u}\!* \; h = (T_u(h^{(-1)_{Nu}})g) *_{Nu} h$ defines a Lie group multiplication on the set G_{Nu} of which the Lie algebra is isomorphic to $\mathfrak{g}_u$ (see [**18**,

p. 229]). Here $\exp_{G_{Nu}}$ denotes the usual exponential map on G_{Nu}. We set $G_u = (G_{Nu}, {}_{T_u}*)$ and $T = T_1$. Then with $u = 1$ the Lie group G is isomorphic to $(G_{Nu}, {}_T*)$ and from now on we identify G with $(G_{N1}, {}_T*)$. We also delete the u in a symbol if $u = 1$. As a consequence

$$(3) \qquad \big(dL_{G_u}(a)\varphi\big)(g) = \Big(dL_{G_{Nu}}\big(\overline{T_u}(g^{(-1)_{Nu}})a\big)\varphi\Big)(g)$$

for all $a \in \mathfrak{g}$, $g \in G_u$ and $\varphi \in C_c^\infty(G_u)$. But it follows from Statement 2 that $\overline{T_u}$ is a unitary representation of G_u on $\mathfrak{g}$ equipped with the inner product $\langle \cdot\,, \cdot \rangle$. Fix an orthonormal basis $b_{-d_m}, \ldots, b_d$ of $\mathfrak{g}$ passing through $\mathfrak{m}, \mathfrak{v}, \mathfrak{k}, \mathfrak{h}_2, \ldots, \mathfrak{h}_r$ with $b_{-d_m}, \ldots, b_0$ a basis of $\mathfrak{m}$, $b_1, \ldots, b_{d_0}$ a basis of $\mathfrak{v}$ and $b_{d_0+1}, \ldots, b_d$ a basis of $\mathfrak{n}$. If $a_1, \ldots, a_{d'}$ is the algebraic basis of $\mathfrak{g}$ then $u\gamma_u^{-1}(a_1), \ldots, u\gamma_u^{-1}(a_{d'})$ is an algebraic basis for $\mathfrak{g}_u$. Now set $A_k^{[u]} = dL_{G_u}\big(u\gamma_u^{-1}(a_k)\big)$ for all $k \in \{1, \ldots, d'\}$. Then

$$\begin{aligned}(A_k^{[u]}\varphi)(g) &= u\Big(dL_{G_{Nu}}\big(\overline{T_u}(g^{-1})\gamma_u^{-1}(a_k)\big)\varphi\Big)(g) \\ &= \sum_{j=-d_m}^{d} u^{1-w_j}\Big\langle \overline{T}\big(\Gamma_u(g)\big)b_j, a_k\Big\rangle\, \big(\widetilde{B}_j^{(u)}\varphi\big)(g)\end{aligned}$$

where $\widetilde{B}_j^{(u)} = dL_{G_{Nu}}(b_j)$ and $\Gamma_u \colon G_u \to G$ is the lifting of the isomorphism γ_u.

Next, define the subelliptic operator $H_{[u]}$ on G_u by

$$H_{[u]} = -\sum_{k,l=1}^{d'} c_{kl}\, A_k^{[u]}\, A_l^{[u]} = -\sum_{i,j=-d_m}^{d} \widetilde{B}_i^{(u)}\, \widetilde{c}_{ij}^{[u]}\, \widetilde{B}_j^{(u)}$$

where the $\widetilde{c}_{ij}^{[u]}$ are multiplication operators, $\widetilde{c}_{ij}^{[u]}(g) = u^{2-w_i-w_j}\widetilde{c}_{ij}^{(u)}$ with

$$\widetilde{c}_{ij}(g) = \sum_{k,l=1}^{d'} \big\langle \overline{T}(g)b_i, a_k\big\rangle\, c_{kl}\, \big\langle \overline{T}(g)b_j, a_l\big\rangle$$

and for any function $\psi \colon G \to \mathbf{C}$ we write $\psi^{(u)} = \psi \circ \Gamma_u$. One calculates that

$$\sum_{i=-d_m}^{d} (\widetilde{B}_j\widetilde{c}_{ij})(g) = \sum_{k,l=1}^{d'} c_{kl}\, \big\langle \overline{T}(g)b_j, \tau(a_k)a_l\big\rangle\,.$$

In particular $\sum_{i=-d_m}^{d} \widetilde{B}_j\widetilde{c}_{ij} = 0$ if $j \leqslant d_0$. Therefore the equations

$$H\chi_j = -\sum_{i=-d_m}^{d} \widetilde{B}_j\widetilde{c}_{ij}$$

can be solved for the correctors χ_j. If $j \leqslant d_0$ then $\chi_j = 0$ and if $j > d_0$ then $\chi_j(g) = \big\langle \overline{T}(g)b_j, x\big\rangle$ with $x \in \mathfrak{n}$ any solution of $H_\tau x = c_\tau$, where $H_\tau = -\sum_{k,l=1}^{d'} c_{kl}\tau(a_k)\tau(a_l)$ and $c_\tau = -\sum_{k,l=1}^{d'} c_{kl}\tau(a_k)a_l$.

Using the parametrisation $v = x_1 b_1 + \ldots + x_{d_0} b_{d_0}$ the coefficients $\widetilde{c}_{ij}$ and the correctors are functions over $M \times \mathbf{R}^{d_0}$ of the form

$$\widetilde{\psi}(m,x) = \sum_{\lambda} \psi_\lambda(m) e^{i\lambda.x}$$

where the sum is over a finite subset of $\mathbf{R}^{d_0}$ and the $\psi_\lambda \in C^\infty(M)$. The mean value of functions of this type can be defined by

$$\mathcal{M}(\widetilde{\psi}) = \lim_{R\to\infty} V(R)^{-1} \int_{\{x\in\mathbf{R}^{d_0}:|x|<R\}} dx \int_M dm\, \widetilde{\psi}(m,x) = \int_M dm\, \psi_0(m)\,.$$

Before we can define the homogenisation $\widehat{H}$ of H we have to introduce one more Lie group. Define $[\,\cdot\,,\,\cdot\,]_H \colon \mathfrak{q} \times \mathfrak{q} \to \mathfrak{q}$ by $[a,b]_H = \lim_{u\to\infty} [a,b]_{Nu}$. Then $\mathfrak{q}_H = (\mathfrak{q}, [\,\cdot\,,\,\cdot\,]_H)$ is a homogeneous (nilpotent) Lie algebra (see [**14**].) If $a *_H b$ denotes the Campbell–Baker–Hausdorff formula in a and b with respect to $[\,\cdot\,,\,\cdot\,]_H$ on $\mathfrak{q}_H$ then $Q_H = (\mathfrak{q}, *_H)$ is the connected simply connected homogeneous (nilpotent) Lie group with Lie algebra $\mathfrak{q}_H$. Then set $G_H = M \times Q_H$. The homogenisation $\widehat{H}$ of H is defined, in analogy with standard homogenisation theory [**5**] [**20**], as the operator with constant coefficients $\widehat{c}_{ij}$ on Q_H given by

$$\widehat{H} = -\sum_{i=1}^{d_1} \widehat{c}_{ij}\, B_i^{(H)} B_j^{(H)}$$

where d_1 is the dimension of $\mathfrak{h}_1$, $B_i^{(H)} = dL_{Q_H}(b_i)$ and

$$\widehat{c}_{ij} = \mathcal{M}\Big(\widetilde{c}_{ij} - \sum_{k=-d_m}^{d} \widetilde{c}_{ik} \widetilde{B}_k \chi_j\Big)\,.$$

Then $\widehat{H}$ is subelliptic on Q_H for the following reason. First, $b_1, \ldots, b_{d_1}$ is an algebraic basis for $\mathfrak{q}_N$, and hence $\mathfrak{q}_H$, by [**12**, Lemma 3.5]. Secondly, one can reexpress the $\widehat{c}_{ij}$ as

$$\widehat{c}_{ij} = \sum_{k,l=-d_m}^{d} \mathcal{M}\big(\overline{\widetilde{B}_k(\xi_i - \chi_i)}\, \widetilde{c}_{kl}\, \widetilde{B}_l(\xi_j - \chi_j)\big)\,,$$

where $\xi_i \colon G_N \to \mathbf{R}$ is defined by $\xi_i(m, \exp_{Q_N} a) = -\langle a, b_i\rangle$ for all $i \in \{1, \ldots, d\}$. Next, if $\tau_1, \ldots, \tau_{d_1} \in \mathbf{C}$ and $\varphi_\tau = \sum_{i=1}^{d_1} \tau_i(\xi_i - \chi_i)$ then

$$\operatorname{Re} \sum_{i,j=1}^{d_1} \overline{\tau_i}\, \widehat{c}_{ij}\, \tau_j = \operatorname{Re} \sum_{k,l=1}^{d'} \mathcal{M}\big(\overline{(A_k\varphi_\tau)}\, c_{kl}\, (A_l \varphi_\tau)\big) \geqslant \mu \sum_{k=1}^{d'} \mathcal{M}\big(|A_k\varphi_\tau|^2\big)\,.$$

Hence $A_k \varphi_\tau = 0$ for all $k \in \{1, \ldots, d'\}$. Since $a_1, \ldots, a_{d'}$ is an algebraic basis for $\mathfrak{g}$ this implies that $dL_G(a)\varphi_\tau = 0$ for all $a \in \mathfrak{g}$ and φ_τ must be constant. Therefore $\sum_{i=1}^{d_1} \tau_i \xi_i$ is bounded which implies $\tau_1 = \ldots = \tau_{d_1} = 0$. Hence $\widehat{C} = (\widehat{c}_{ij})$ is strictly positive-definite, that is, $\widehat{H}$ is a subelliptic operator on Q_H.

It also follows by a calculation analogous to [**5**, pp. 27–28], that $\widehat{C^*} = \widehat{C}^*$.

The important point is that the homogenisation is obtained from H by a scaling limit. This is established by an elaboration of the arguments of [**5**, **20**].

We choose and fix a Lebesgue measure on the vector space $\mathfrak{q}$. Then we fix the Haar measure on Q_{Nu} and Q_H such that $\int_{Q_{Nu}} \varphi = \int_{\mathfrak{q}} \varphi \circ \exp_{Q_{Nu}}$ and $\int_{Q_H} \psi = \int_{\mathfrak{q}} \psi \circ \exp_{Q_H}$ for all $\varphi \in C_c(Q_{Nu})$ and $\psi \in C_c(Q_H)$. Then the Haar measure on G_{Nu} or G_H is the product measure of the normalised Haar measure on the compact group M and the Haar measure on Q_{Nu} or Q_H, respectively. Finally since $|\det \overline{T_u}(g)| = 1$ for all g it follows that we can choose the Haar measure on G_u such that $\int_{G_u} \varphi = \int_{G_{Nu}} \varphi$ for all $\varphi \in C_c(G_u)$. Note that this fixes the Haar measure on $G = G_1$.

If $K_t^{[u]}$ is the semigroup kernel associated with the scaled subelliptic operator $H_{[u]}$ then

$$K_t(g) = u^{-D} K^{[u]}_{u^{-2}t}\big(\Gamma_u^{-1}(g)\big) \tag{4}$$

for all $g \in G$ and $u, t > 0$.

Let $|\cdot|'_u$ be the modulus on G_u with respect to the algebraic basis $u\gamma_u^{-1}(a_1), \ldots, u\gamma_u^{-1}(a_{d'})$ and let $B'^{(u)}(r)$ be the corresponding balls. Then $B'(ru) = \Gamma_u\big(B'^{(u)}(r)\big)$ and $|B'^{(u)}(r)| = u^{-D}|B'(ur)|$ for all $u, r > 0$. But there exists a $\widetilde{c} > 0$ such that $\widetilde{c}^{-1} r^D \leqslant |B'(r)| \leqslant \widetilde{c}\, r^D$ for all $r > 0$, since $D = D'$ by assumption. Hence

$$\widetilde{c}^{-1} r^D \leqslant |B'^{(u)}(r)| \leqslant \widetilde{c}\, r^D \tag{5}$$

for all $r > 0$ uniformly for $u > 0$.

Note that $G_u = M \times \mathfrak{q}$ as manifold for all $u > 0$. If Ω is an open subset of $M \times \mathfrak{q}$ introduce the Sobolev subspace $H'^{(u)}_{2;1}(\Omega) = \bigcap_{k=1}^{d'} D(A_k^{[u]})$ of $L_2(\Omega)$ with the usual Sobolev norm. Then $\varphi_u \in H'^{(u)}_{2;1}(\Omega)$ is defined to satisfy $H^{[u]}\varphi_u = 0$ weakly on Ω if $\sum_{k,l=1}^{d'} c_{kl}\,(A_k^{[u]}\psi, A_l^{[u]}\varphi_u) = 0$ for all $\psi \in C_c^\infty(\Omega)$. Similarly, if Ω' is an open subset of $\mathfrak{q}$ then $H'^{(H)}_{2;1}(\Omega') = \bigcap_{i=1}^{d_1} D(B_i^{(H)}) \subseteq L_2(\Omega')$ and $\varphi \in H'^{(H)}_{2;1}(\Omega')$ satisfies $\widehat{H}\varphi = 0$ weakly on Ω' if $\sum_{k,l=1}^{d_1} \widehat{c}_{kl}(B_k^{(H)}\psi, B_l^{(H)}\varphi) = 0$ for all $\psi \in C_c^\infty(\Omega')$.

If $\Omega = M \times \Omega'$ and $\varphi \in L_p(\Omega)$ define $\varphi^\flat \in L_p(\Omega')$ by $\varphi^\flat(q) = \int_M dm\, \varphi(m, q)$. Thus $\mathbb{1} \otimes \varphi^\flat = P\varphi$ with $P = \int_M dm$.

Proposition 2.1. *Let $u_n \geqslant 1$ be a sequence tending to infinity, Ω' an open subset of $\mathfrak{q}$ and $\Omega = M \times \Omega'$. Assume $\varphi_n \in H'^{(u_n)}_{2;1}(\Omega)$ satisfy $H_{[u_n]}\varphi_n = 0$ weakly on Ω,*

$$\sup_{n \in \mathbb{N}} \sum_{k=1}^{d'} \|A_k^{[u_n]}\varphi_n\|^2_{L_2(\Omega)} < \infty\,, \tag{6}$$

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and $\varphi_n \to \varphi_\infty$ *weakly in* $L_2(\Omega)$.

Then $P\varphi_\infty = \varphi_\infty$, $\varphi_\infty^\flat \in H_{2;1}^{\prime(H)}(\Omega')$ *and* $\widehat{H}\varphi_\infty^\flat = 0$ *weakly on* Ω'.

The first step in the proof is a strong convergence result.

Lemma 2.2. *Suppose* φ_n *satisfy* (6) *and* $\varphi_n \to \varphi_\infty$ *weakly in* $L_2(\Omega)$. *Then* $\varphi_\infty = P\varphi_\infty$ *and* $\varphi_n \to \varphi_\infty$ *strongly in* $L_2(M \times \Omega'')$ *for any open subset* Ω'' *of* $\mathfrak{q}$ *such that* $\overline{\Omega''} \subset \Omega'$.

Proof: Let $\psi^{(H)} \in C_c^\infty(\Omega')$ and set $\psi = \mathbb{1} \otimes \psi^{(H)}$. Then $\lim_{u\to\infty} \widetilde{B}_i^{(u)}\psi = \widetilde{B}_i^{(H)}\psi$ uniformly on $M \times \mathfrak{q}$ for all $i \in \{1, \ldots, d\}$, where $\widetilde{B}_i^{(H)} = dL_{G_H}(b_i)$, since

$$\begin{aligned} \lim_{u\to\infty} (\widetilde{B}_i^{(u)}\psi)(m,q) &= \lim_{u\to\infty} \frac{d}{dt}\psi^{(H)}(-tb_i *_{N u} q)\Big|_0 \\ &= \frac{d}{dt}\psi^{(H)}(-tb_i *_H q)\Big|_0 \\ &= (\widetilde{B}_i^{(H)}\psi)(m,q)\,. \end{aligned} \tag{7}$$

It follows from (3) that

$$\sup_{u\geqslant 1}\sup_{g\in\Omega}\left|(A_k^{[u]}\psi)(g)\right| \leqslant \sup_{u\geqslant 1}\sup_{g\in\Omega}\sum_{j=1}^{d} u^{1-w_j}\left|\langle \overline{T_u}(g)b_j, a_i\rangle\right|\left|(\widetilde{B}_j^{(u)}\psi)(g)\right| < \infty$$

for all $k \in \{1, \ldots, d'\}$. Next $\|\varphi_n\|_{L_2(\Omega)}$ is bounded uniformly in n and $\widetilde{\varphi}_n = \varphi_n\psi \in H_{2;1}^{\prime(u_n)}(G_{u_n})$ for all $n \in \mathbf{N}$. Moreover,

$$C = \sup_{n\in\mathbf{N}} \left(\sum_{k=1}^{d'} \|A_k^{[u_n]}\widetilde{\varphi}_n\|_{L_2(G_{u_n})}^2\right)^{1/2} < \infty\,. \tag{8}$$

Then

$$\begin{aligned} \left\|(I - L_{G_H}(m))\widetilde{\varphi}_n\right\|_{L_2(G_H)} &= \left\|(I - L_{G_{u_n}}(m))\widetilde{\varphi}_n\right\|_{L_2(G_{u_n})} \\ &\leqslant |m|'_{u_n}\left(\sum_{k=1}^{d'} \|A_k^{[u_n]}\widetilde{\varphi}_n\|_{L_2(G_{u_n})}^2\right)^{1/2} \\ &\leqslant C\,|m|'_{u_n} = C\,u_n^{-1}\left|\Gamma_{u_n}(m)\right|' = C\,u_n^{-1}\,|m|' \end{aligned}$$

for all $m \in M$ and $n \in \mathbf{N}$. Therefore $\big(I - L_{G_H}(m)\big)\widetilde{\varphi}_n$ converges strongly to zero in $L_2(\Omega)$ and so $\varphi_\infty\psi = L_{G_H}(m)(\varphi_\infty\psi)$ for all $m \in M$. This implies that $\varphi_\infty\psi = P\,\varphi_\infty\,\psi$. Hence $\varphi_\infty = P\varphi_\infty$.

Next we argue that the set $\{\widetilde{\varphi}_n : n \in \mathbf{N}\}$ is relatively compact in $L_2(G_H)$ by checking the Fréchet–Kolmogorov conditions of [**16**, Appendix D.1.3]. Then the second statement of the lemma follows by choosing $\psi^{(H)}$ such that $\psi^{(H)}(q) = e$ for all $q \in \Omega''$.

The set $\{\widetilde{\varphi}_n : n \in \mathbf{N}\}$ is uniformly bounded in $L_2(G_H)$ and the $\widetilde{\varphi}_n$ have a support in a common compact set. Hence it suffices to prove that

$$\lim_{g \to e} \sup_n \Big\| \big(I - L_{G_H}(g)\big)\widetilde{\varphi}_n \Big\|_{L_2(G_H)} = 0 \tag{9}$$

and a similar statement with respect to right translations.

First, $b_1, \ldots, b_{d_1}$ is an algebraic basis for $\mathfrak{q}$, by [**12**, Lemma 3.4]. Hence $b_{-d_m}, \ldots, b_{d_1}$ is an algebraic basis for $\mathfrak{g}$. Let $|\cdot|_u'^{(b)}$ be the modulus on G_u with respect to the scaled algebraic basis $u\gamma_u^{-1}(b_{-d_m}), \ldots, u\gamma_u^{-1}(b_{d_1})$ and s the rank of the algebraic basis $a_1, \ldots, a_{d'}$ in $\mathfrak{g}$. Then there is a $c > 0$ such that

$$|g|_u' \leqslant u^{-1} c \left(u^{1/s} \big(|g|_u'^{(b)}\big)^{1/s} + u\, |g|_u'^{(b)} \right) \leqslant 2c \left(|g|_u'^{(b)}\right)^{1/s}$$

for all $u \geqslant 1$ and $g \in G_u$ with $|g|_u'^{(b)} \leqslant 1$, where the first inequality follows from [**15**, Proposition 1.1], for $u = 1$, and then by scaling. Secondly, if $L'_{2;1}(G_u)$ is the Sobolev space defined with respect to the algebraic basis $u\gamma_u^{-1}(a_1), \ldots, u\gamma_u^{-1}(a_{d'})$ then one has $\Big\| \big(I - L_{G_u}(g)\big)\varphi \Big\|_{L_2(G_u)} \leqslant d' \, \|\varphi\|_{L'_{2;1}(G_u)} \, |g|_u'$. Combining these estimates one deduces $L'_{2;1}(G_u)$ is continuously embedded in $\mathcal{L}'^{(b)}_{2;1/s}(G_u)$, where $\mathcal{L}'^{(b)}_{2;\gamma}(G_u)$ is the Lipschitz space defined with respect to left translations on $L_2(G_u)$ and the modulus $|\cdot|_u'^{(b)}$. Moreover, the embedding is continuous uniformly for all $u \geqslant 1$.

Next by standard reasoning (see, for example, the proof of [**6**, Theorem 3.2]) one establishes a chain of uniform embeddings of a similar nature. First, one proves that $\mathcal{L}'^{(b)}_{2;\gamma}(G_u)$ is continuously embedded in the Lipschitz space $L^{S,u}_{2;\gamma/2}$ associated with the semigroup $S^{[u]}$ generated by the sublaplacian $-\sum_{i=-d_m}^{d_1} (B_i^{[u]})^2$ where $B_i^{[u]} = dL_{G_u}\big(u\gamma_u^{-1}(b_i)\big)$. The proof uses both the upper and lower Gaussian bounds on the kernel of $S^{[u]}$ which follow from (4) and [**19**, Theorem VIII.2.9], or [**16**, Chapter IV], since the sublaplacian is a real symmetric, subelliptic, operator. Secondly, one argues that $L^{S,u}_{2;\gamma/2}$ is continuously embedded in the real interpolation space $\big(L_2(G_u), L'^{(b)}_{2;1}(G_u)\big)_{\gamma,\infty;\mathrm{K}}$ defined by the K-method of Peetre, where $L'^{(b)}_{2;1}(G_u)$ is the Sobolev space defined with respect to the algebraic basis $u\gamma_u^{-1}(b_{-d_m}), \ldots, u\gamma_u^{-1}(b_{d_1})$. Thirdly, one establishes that $\big(L_2(G_u), L'^{(b)}_{2;1}(G_u)\big)_{\gamma,\infty;\mathrm{K}}$ is continuously embedded in the interpolation space $\big(L_2(G_{Nu}), L'^{(b)}_{2;1}(G_{Nu})\big)_{\gamma,\infty;\mathrm{K}}$, where $L'^{(b)}_{2;1}(G_{Nu})$ is defined with respect to the algebraic basis $u\gamma_u^{-1}(b_{-d_m}), \ldots, u\gamma_u^{-1}(b_{d_1})$. This embedding, and its uniformity for $u \geqslant 1$, follow by another use of (3). Fourthly, the interpolation space $\big(L_2(G_{Nu}), L'^{(b)}_{2;1}(G_{Nu})\big)_{\gamma,\infty;\mathrm{K}}$ is continuously embedded in the interpolation space $\big(L_2(G_{Nu}), L'^{(b)}_{2;r}(G_{Nu})\big)_{\gamma/r,\infty;\mathrm{K}}$, uniformly for all $u > 0$. This follows for $u = 1$ from [**6**, Theorem 2.1.II], and for general u by application of the scaling Γ_u after replacing the norms on $L'^{(b)}_{2;1}(G_{Nu})$ and $L'^{(b)}_{2;r}(G_{Nu})$ by their seminorms. Note that these seminorms satisfy a scaling relation. Fifthly, the space $\big(L_2(G_{Nu}), L'^{(b)}_{2;r}(G_{Nu})\big)_{\gamma/r,\infty;\mathrm{K}}$ is continuously embedded in the interpolation space $\left(L_2(G_{Nu}), L'_{2;r}(G_{Nu})\right)_{\gamma/r,\infty;\mathrm{K}}$, uniformly for all $u \geqslant 1$, where $L'_{2;r}(G_{Nu})$ is defined with respect to the algebraic basis $b_{-d_m}, \dots, b_{d_1}$. Sixthly, fix $\widetilde{\psi}^{(H)} \in C_c^\infty(\mathfrak{q})$ such that $\widetilde{\psi}^{(H)}(q) = 1$ for all $q \in \operatorname{supp} \psi^{(H)}$ and set $\widetilde{\psi} = \mathbb{1} \otimes \widetilde{\psi}^{(H)}$. Define the multiplication operator $E\colon L_2(G_H) \to L_2(G_H)$ by $E\varphi = \widetilde{\psi}\,\varphi$. Then E maps the space $\left(L_2(G_{Nu}), L'_{2;r}(G_{Nu})\right)_{\gamma/r,\infty;\mathrm{K}}$ continuously into the space $\left(L_2(G_H), L_{2;1}(G_H)\right)_{\gamma/r,\infty;\mathrm{K}}$, uniformly for all $u \geqslant 1$, where the Sobolev space $L_{2;1}(G_H)$ is defined with respect to the basis $b_{-d_m}, \dots, b_d$. The proof requires some work that we next describe. For all $q = \sum_{i=1}^{d} \xi_i\, b_i \in \mathfrak{q}$ and $\beta = (i_1, \dots, i_n) \in J(\mathfrak{q})$ define $q^\beta = \xi_{i_1} \cdot \dots \cdot \xi_{i_n}$. For all $i \in \{1, \dots, d\}$ let D_i denote the partial derivative $(D_i\varphi)(m,q) = \frac{d}{dt}\varphi(m, q + tb_i)|_{t=0}$. Then it follows from the Campbell–Baker–Hausdorff formula that there are $c_{ij\beta} \in \mathbf{R}$ such that

$$(10) \qquad (B_i^{(H)}\varphi)(m,q) = -(D_i\varphi)(m,q) + \sum_{w_j - w_i = \|\beta\| > 0} c_{ij\beta}\, q^\beta\, (D_j\varphi)(m,q)$$

for all $\varphi \in C_c^\infty(G_H)$ and $(m,q) \in G_H$. Similarly,

$$(\widetilde{B}_i\varphi)(m,q) = -(D_i\varphi)(m,q) + \sum_{w_j - w_i \geqslant \|\beta\| > 0} c'_{ij\beta}\, q^\beta\, (D_j\varphi)(m,q)$$

for all $i \in \{1, \dots, d\}$. The transition matrix from $\widetilde{B}_i$ to D_i is triangular, with -1 on the diagonal. Then it can be inverted and there are $c''_{ij\beta} \in \mathbf{R}$ such that

$$(D_i\varphi)(m,q) = -(\widetilde{B}_i\varphi)(m,q) + \sum_{w_j - w_i \geqslant \|\beta\| > 0} c''_{ij\beta}\, q^\beta\, (\widetilde{B}_j\varphi)(m,q)\ .$$

Since $\mathfrak{h}_1$ generates $\mathfrak{q}_N$ for all $i \in \{1, \dots, d\}$, $\alpha \in J(d_1)$ and $\beta \in J(d)$ there are $c_{i\alpha\beta} \in \mathbf{R}$ such that

$$(D_i\varphi)(m,q) = \sum_{r \geqslant |\alpha| \geqslant w_i + \|\beta\|} c_{i\alpha\beta}\, q^\beta\, (\widetilde{B}_{[\alpha]}\varphi)(m,q)\ ,$$

where $\widetilde{B}_{[\alpha]}^{(u)} = \left[\widetilde{B}_{i_1}^{(u)}, \left[\dots\left[\widetilde{B}_{i_{n-1}}^{(u)}, \widetilde{B}_{i_n}^{(u)}\right]\dots\right]\right]$ for all $u > 0$ and $\alpha = (i_1, \dots, i_n) \in J(d_1)$. Then by scaling

$$(11) \qquad (D_i\varphi)(m,q) = \sum_{r \geqslant |\alpha| \geqslant w_i + \|\beta\|} c_{i\alpha\beta}\, u^{-(|\alpha| - w_i - \|\beta\|)}\, q^\beta\, (\widetilde{B}_{[\alpha]}^{(u)}\varphi)(m,q)\ .$$

Hence E is continuous from $L'_{2;r}(G_{Nu})$ into $L_{2;1}(G_H)$, uniformly for all $u \geqslant 1$, by (10) and (11). Then by interpolation the claim follows. Seventhly, by [6] the interpolation space $\left(L_2(G_H), L_{2;1}(G_H)\right)_{\gamma/r,\infty;\mathrm{K}}$ is continuously embedded in the Lipschitz space $\mathcal{L}_{2;\gamma/r}(G_H)$ on G_H defined with respect to the basis $b_{-d_m}, \dots, b_d$. Since all the foregoing embeddings are uniform for $u \geqslant 1$ there exists a $c > 0$ such that

$$(12) \qquad \left\|\left(I - L_{G_H}(g)\right)E\varphi\right\|_{L_2(G_H)} \leqslant c\,|g|_{G_N}^{1/(rs)} \left(\|\varphi\|_{L_2(G_u)} + \sum_{k=1}^{d'} \|A_k^{[u]}\varphi\|_{L_2(G_u)}\right)$$

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uniform for all $u \geqslant 1$, $\varphi \in L'^{(u)}_{2;1}(G_u)$ and $g \in G_N$ with $|g|_{G_N} \leqslant 1$ where $|\cdot|_{G_H}$ is the modulus on G_H with respect to the basis $b_{-d_m}, \ldots, b_d$. Therefore (9) is proved using (8) and (12). A similar conclusion follows for right translations since $dR_{G_H}(a) = \sum_{j=-d_m}^{d} c_j \, dL_{G_H}(b_j)$ with $c_j(g) = \langle b_j, \mathrm{Ad}_{G_H}(g^{-1})a \rangle$ and the c_j are uniformly bounded on $M \times \Omega''$. □

PROOF OF PROPOSITION 2.1: First, $\varphi_\infty = P\varphi_\infty$ by Lemma 2.2. The remainder of the proof is similar to the derivation of the analogous result in the $\mathbf{R}^d$-homogenisation theory (see, for example, [**5**, pp. 24–28]). Care is needed, however, since the operators are subelliptic and their form domains $H'^{(u)}_{2;1}(\Omega)$ vary with the scaling parameter u.

Introduce

$$\eta_{n;i} = \sum_{k,l=1}^{d'} u_n^{1-w_i} \langle \overline{T_{u_n}}(\cdot) b_i, a_k \rangle \, c_{kl} \, A_l^{[u_n]} \varphi_n$$

for all $i \in \{1, \ldots, d\}$. Then $\|\eta_{n;i}\|_{L_2(\Omega)}$ is bounded uniformly in n by (6). Hence one may assume the $\eta_{n;i}$, or a subsequence, converge weakly to a limit η_i in $L_2(\Omega)$. Clearly $\eta_i = 0$ if $i > d_1$. But if $\chi' \in C_c^\infty(\Omega')$ with $\operatorname{supp} \chi' \subset \Omega'$ and $\chi = \mathbb{1} \otimes \chi'$ then by (8) one has

$$\sum_{i=1}^{d_1} (\widetilde{B}_i^{(H)} \chi, \eta_i) = \lim_{n \to \infty} \sum_{i=1}^{d_1} (\widetilde{B}_i^{(u_n)} \chi, \eta_{n;i}) = 0 \tag{13}$$

since $H_{[u_n]} \varphi_n = 0$ weakly on Ω and $\widetilde{B}_i^{(u_n)} \chi = 0$ if $i \leqslant 0$, where $\widetilde{B}_i^{(H)} = dL_{G_H}(b_i)$.

Now set $\xi_{n;i} = \xi_i - u_n^{-1} \chi_i^{\dagger(u_n)}$ for $i \in \{1, \ldots, d_1\}$ where ξ_i is again given by $\xi_i(m, a) = -\langle a, b_i \rangle$ and the $\chi_j^\dagger$ are the correctors of the adjoint H^*. It follows that $H^*_{[u_n]} \xi_{n;i} = 0$ as a distribution on Ω by the corrector equation for H^*. Therefore by a density argument

$$(\chi H^*_{[u_n]} \xi_{n;i}, \varphi_n) = 0 \, .$$

But since $H_{[u_n]} \varphi_n = 0$ weakly one then has

$$\big([H^*_{[u_n]}, \chi] \xi_{n;i}, \varphi_n \big) = 0 \, . \tag{14}$$

Now the commutator $[H^*_{[u_n]}, \chi]$ is linear in the $A_k^{[u]}$,

$$[H^*_{[u]}, \chi] = - \sum_{k,l=1}^{d'} \overline{c_{lk}} \left(A_k^{[u]} \, (A_l^{[u]} \chi) + (A_k^{[u]} \chi) \, A_l^{[u]} \right) ,$$

and one may use this to evaluate the limit of (14) as $n \to \infty$. The calculation is very similar to the standard argument of [**5**, pp. 24–27]. The term with the $A_k^{[u]}$ acting on φ_n can be rewritten in terms of $\eta_{n;i}$ and it converges to $\sum_{j=1}^{d_1} \big((\widetilde{B}_j^{(H)} \chi) \xi_i, \eta_j \big)$. The term with $A_l^{[u]}$ acting on $\xi_{n;i}$ is more complicated. It gives a contribution

$$- \sum_{j=1}^{d} \sum_{k=-d_m}^{d_1} u_n^{1-w_j} \Big((\widetilde{B}_j^{(u_n)} \chi) \, \overline{\widetilde{c}_{kj}^{(u_n)}} \, (\delta_{ik} - (\widetilde{B}_k \chi_i^\dagger)^{(u_n)}), \varphi_n \Big) \, .$$

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Since the inner product is uniformly bounded in n all terms with $w_j > 1$ vanish as $n \to \infty$. Also φ_n converges strongly to φ_∞ on the support of χ, by Lemma 2.2. In addition

$$\lim_{u\to\infty}\big((\psi\circ\Gamma_u)\,(\widetilde{B}_j^{(u)}\chi),\varphi_\infty\big) = \big((\widetilde{B}_j^{(H)}\chi),\varphi_\infty\big)\,\overline{\mathcal{M}(\psi)}$$

since $(I-P)(\widetilde{B}_j^{(u)}\chi) = 0 = (I-P)\varphi_\infty$. Therefore one deduces from (14) and $\widehat{C^*} = \widehat{C}^*$ that

$$\sum_{j=1}^{d_1}\big((\widetilde{B}_j^{(H)}\chi)\xi_i,\eta_j\big) - \sum_{k=1}^{d_1}\widehat{c}_{ik}(\widetilde{B}_k^{(H)}\chi,\varphi_\infty) = 0\,.$$

But $\sum_{j=1}^{d_1}(\widetilde{B}_j^{(H)}(\chi\xi_i),\eta_j) = 0$ by (13), with χ replaced by $\chi\xi_i$, and $\widetilde{B}_j^{(H)}\xi_i = \delta_{ij}$. Therefore

$$(\chi,\eta_j) = -\sum_{k=1}^{d_1}\widehat{c}_{jk}(\widetilde{B}_k^{(H)}\chi,\varphi_\infty)\,. \tag{15}$$

Since $\widehat{C}$ is strictly positive-definite equations (15) can be solved to give

$$(\widetilde{B}_k^{(H)}\chi,\varphi_\infty) = -\sum_{j=1}^{d_1}(\widehat{C}^{-1})_{kj}(\chi,\eta_j)$$

for all $\chi = \mathbb{1}\otimes\chi'$ with $\chi' \in C_c^\infty(\Omega')$. One immediately concludes that $\varphi_\infty^\flat \in H_{2;1}'^{(H)}(\Omega')$. Then (13) and (15) imply that $\big((I\otimes\widehat{H})^*\chi,\varphi_\infty\big) = \sum_{j=1}^{d_1}(\widetilde{B}_j^{(H)}\chi,\eta_j) = 0$. Hence $\widehat{H}\varphi_\infty^\flat = 0$ weakly and the proof of Proposition 2.1 is complete. $\square$

3. Gaussian bounds

In order to establish the Gaussian bounds (1) on K_t it suffices to prove that $K_t^{[u]}$ satisfies Gaussian bounds for all $t \in \langle 0,1]$ uniformly for $u \geqslant 1$ by taking $u = t^{1/2}$ in (4). Local bounds on $K_t^{[u]}$ can be deduced either by a parametrix argument [8] or, following Auscher [3], by De Giorgi estimates [9]. But the first method is ill suited to the deduction of uniformity. One can, however, obtain uniform local estimates by adaptation of the homogenisation arguments of Avellaneda and Lin [4]. These require uniform De Giorgi estimates together with the uniform growth properties (5) and a uniform Poincaré inequality.

Let $\|\varphi\|_{2,u,r}$ denote the norm of $\varphi \in L_2\big(B'^{(u)}(r)\big)$ and for $\varphi \in H_{2;1}'^{(u)}\big(B'^{(u)}(r)\big)$ set

$$\|\nabla_u'\varphi\|_{2,u,r} = \Big(\sum_{k=1}^{d'}\|A_k^{[u]}\varphi\|_{2,u,r}^2\Big)^{1/2}\,. \tag{16}$$

Further let $\langle\varphi\rangle_{u,r}$ denote the average of $\varphi \in L_{1,\mathrm{loc}}(M\times\mathfrak{q})$ over $\big(B'^{(u)}(r)\big)$. Then the required Poincaré inequality states that there is a $c_N > 0$ such that

$$\|\varphi - \langle\varphi\rangle_{u,r}\|_{2,u,r}^2 \leqslant c_N\, r^2\|\nabla_u'\varphi\|_{2,u,r}^2 \tag{17}$$

uniformly for all $u > 0$, $r > 0$ and $\varphi \in H'^{(u)}_{2;1}\big(B'^{(u)}(2r)\big)$. This follows from [**17**, p. 1], which establishes that there exists a $c_N > 0$ such that

$$\|\varphi - \langle\varphi\rangle_{1,r}\|^2_{2,1,r} \leqslant c_N\, r^2 \|\nabla'_1\varphi\|^2_{2,1,r} \tag{18}$$

uniformly for all $r > 0$ and $\varphi \in C_b^\infty(G)$. Then (18) is valid for all $r > 0$ and $\varphi \in H'_{2;1}\big(B'(2r)\big)$ by approximation and (17) follows by scaling.

The key De Giorgi estimates are the following.

Proposition 3.1. *For all $\nu \in \langle 0,1\rangle$ there exists a $c_{DG} > 0$ such that for all $u > 0$, $R \in \langle 0,1]$ and $\varphi \in H'^{(u)}_{2;1}\big(B'^{(u)}(R)\big)$ satisfying $H_{[u]}\varphi = 0$ weakly on $B'^{(u)}(R)$ one has*

$$\|\nabla'_u\varphi\|^2_{2,u,r} \leqslant c_{DG}\, (r/R)^{D-2+2\nu} \|\nabla'_u\varphi\|^2_{2,u,R} \tag{19}$$

for all $0 < r \leqslant R$.

Proof: The De Giorgi estimates (19) are valid for each $u \geqslant 1$ by [**9**, Proposition 3.3]. The problem is to prove uniformity. This requires several lemmas.

Let $\|\varphi\|_{2,H,r}$ denote the norm of $\varphi \in L_2\big(B'^{(H)}(r)\big)$ and if $\varphi \in H'^{(H)}_{2;1}\big(B'^{(H)}(r)\big)$ define $\|\nabla'^{(H)}\varphi\|_{2,H,r}$ similarly to (16), where $B'^{(H)}(r)$ is the ball on Q_H with respect to the algebraic basis $b_1, \ldots, b_{d_1}$. Then for $\varphi \in L_{1,\mathrm{loc}}(Q_H)$ let $\langle\varphi\rangle_{H,r}$ denote the average over $B'^{(H)}(r)$. We begin with two Caccioppoli inequalities.

Lemma 3.2. *There exist $c_1 \geqslant 1$ and $\sigma \in \langle 0,1\rangle$ such that*

I
$$\|\nabla'_u\varphi\|^2_{2,u,\sigma r} \leqslant c_1\, r^{-2} \big\|\varphi - \langle\varphi\rangle_{u,r}\big\|^2_{2,u,r}$$

uniformly for all $u > 0$, $r \in \langle 0,1]$ and $\varphi \in H'^{(u)}_{2;1}\big(B'^{(u)}(r)\big)$ satisfying $H_{[u]}\varphi = 0$ weakly on $B'^{(u)}(r)$, and,

II
$$\|\nabla'^{(H)}\varphi\|^2_{2,H,\sigma r} \leqslant c_1\, r^{-2} \big\|\varphi - \langle\varphi\rangle_{H,r}\big\|^2_{2,H,r}$$

for all $r \in \langle 0,1]$ and $\varphi \in H'^{(H)}_{2;1}\big(B'^{(H)}(r)\big)$ satisfying $\widehat{H}\varphi = 0$ weakly on $B'^{(H)}(r)$.

Proof: Statement 3.2 has been proved in [**9**, Lemma 2.7], using cut-off functions. The proof of Statement 3.2 is more delicate, since it requires the constant to be uniform in u.

By [**12**, p. 30], there exist $c, \sigma > 0$ and for all $R > 0$ a function $\eta_R \in C_c^\infty\big(B'(R)\big)$ such that $\eta_R(g) = 1$ for all $g \in B'(\sigma R)$, $0 \leqslant \eta_R \leqslant 1$ and $\|A_k\eta_R\|_\infty \leqslant c\,R^{-1}$ uniformly for all $R > 0$ and $k \in \{1, \ldots, d'\}$. For $u > 0$ and $R > 0$ define $\eta^{(u)}_R = \eta_{Ru} \circ \Gamma_u$. If $u > 0$ then $\eta^{(u)}_R \in C_c^\infty\big(B'^{(u)}(R)\big)$, $\eta^{(u)}_R(g) = 1$ for all $g \in B'^{(u)}(\sigma R)$, $0 \leqslant \eta^{(u)}_R \leqslant 1$ and $\|A^{[u]}_k\eta^{(u)}_R\|_\infty \leqslant c\,R^{-1}$ uniformly for all $R > 0$ and $k \in \{1, \ldots, d'\}$, by scaling. But then it follows from the proof of [**9**, Lemma 2.7] that Statement 3.2 is valid. ∎

The $B'^{(u)}(r)$ have the following uniform geom..... property. Let $B^{(M)}(\rho)$ be the ball of radius ρ on M with respect to the vector space basis $b_{-d_m}, \ldots, b_0$.

LEMMA 3.3. *There exist $c \in \langle 0,1]$ and $c' \geqslant 1$ such that $B^{(M)}(cru) \times B'^{(H)}(cr) \subset B'^{(u)}(r) \subset B^{(M)}(c'ru) \times B'^{(H)}(c'r)$ uniformly for all $r, u > 0$ with $ru \geqslant 1$.*

PROOF: Since the $\overline{T}(g)$ with $g \in G_N$ are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ the balls $B(r)$ and $B_{G_N}(r)$ of radius r on G and G_N with respect to the vector space basis $b_{-d_m}, \ldots, b_d$ are equal, by (3). Hence by [**19**, Proposition III.4.2], there exists a $c_1 > 0$ such that $B'(c_1^{-1}r) \subset B_{G_N}(r) \subset B'(c_1 r)$ for all $r \geqslant 1$. Next for all $r > 0$ define $R_r = \left\{\sum_{i=1}^{d} \xi_i b_i : |\xi_i| \leqslant r^{w_i} \text{ for all } i \in \{1, \ldots, d\}\right\}$. Then it follows from the proofs of [**19**, Proposition IV.5.6 and IV.5.7], that there exists a $c_2 > 0$ such that $\exp_{Q_N} R_{c_2^{-1}r} \subset B_{Q_N}(r) \subset \exp_{Q_N} R_{c_2 r}$ for all $r \geqslant 1$, where $B_{Q_N}(r)$ is the ball on Q_N with respect to the basis $b_1, \ldots, b_d$. But obviously there exists a $c_3 > 0$ such that $\exp_{Q_H} R_{c_3^{-1}r} \subset B'^{(H)}(r) \subset \exp_{Q_H} R_{c_3 r}$ for all $r > 0$. Hence the inclusions of the lemma are valid for $u = 1$. But then the general case follows by scaling. $\square$

Let $c_N \geqslant 1$ be the constant in (17). We may assume that

$$\left\|\psi - \langle\psi\rangle_{H,r}\right\|_{2,H,r}^2 \leqslant c_N\, r^2 \|\nabla'^{(H)}\psi\|_{2,H,r}^2 \tag{20}$$

uniformly for all $r > 0$ and $\psi \in H_{2;1}'^{(H)}\big(B'^{(H)}(2r)\big)$. Further, let c, c' be as in Lemma 3.3 and c_1 and σ as in Lemma 3.2.

LEMMA 3.4. *For all $\nu_0 \in \langle 0,1\rangle$ there is an $r_0 \in \big\langle 0, c\sigma(c')^{-1}\big\rangle$ such that*

$$\left\|\psi - \langle\psi\rangle_{H,c'\sigma^{-1}r_0}\right\|_{2,H,c'\sigma^{-1}r_0}^2 \leqslant c_1^{-1}\sigma^{-2}\, r_0^{D+2\nu_0}\left\|\psi - \langle\psi\rangle_{H,c^{-1}}\right\|_{2,H,c}^2$$

uniformly for all $\psi \in H_{2;1}'^{(H)}\big(B'^{(H)}(2c)\big)$ satisfying $\widehat{H}\psi = 0$ weakly on $B'^{(H)}(c)$.

PROOF: Let $\nu_1 \in \langle \nu_0, 1\rangle$. By [**9**, Proposition 3.3], there exists a $c_{DG} > 0$ such that for all $R \in \langle 0,1]$ and $\psi \in H_{2;1}'^{(H)}\big(B'^{(H)}(R)\big)$ which satisfy $\widehat{H}\psi = 0$ weakly on $B'^{(H)}(R)$ one has

$$\|\nabla'^{(H)}\psi\|_{2,H,r}^2 \leqslant c_{DG}\,(r/R)^{D-2+2\nu_1}\|\nabla'^{(H)}\psi\|_{2,H,R}^2$$

for all $0 < r \leqslant R$. Hence in combination with (20) and Lemma 3.2.3.2 it follows that

$$\begin{aligned}\left\|\psi - \langle\psi\rangle_{H,r}\right\|_{2,H,r}^2 &\leqslant c_{DG}\, c_N\, r^2 (r/R)^{D-2+2\nu_1}\|\nabla'^{(H)}\psi\|_{2,H,\sigma R}^2\\ &\leqslant c_1\, c_{DG}\, c_N\, \sigma^{-D+2-2\nu_1}(r/R)^{D+2\nu_1}\|\psi - \langle\psi\rangle_{H,R}\|_{2,H,R}^2\end{aligned}$$

whenever $0 < r \leqslant (\sigma \wedge 2^{-1})R \leqslant R \leqslant 1$ and $\psi \in H_{2;1}'^{(H)}\big(B'^{(H)}(R)\big)$ satisfying $\widehat{H}\psi = 0$ weakly on $B'^{(H)}(R)$. But

$$c_1 c_{DG} c_N \sigma^{-D+2-2\nu_1}(r/R)^{D+2\nu_1} = \Big(c_1 c_{DG} c_N (\sigma R)^{-D+2-2\nu_1} r^{2(\nu_1-\nu_0)}\Big) r^{D+2\nu_0}.$$

So take $R = c$ and r small enough. $\square$

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Proposition 3.5. *For all $\nu \in \langle 0,1\rangle$ there exist $r_0 \in \langle 0,1\rangle$ and $u_0 \geqslant 1$ such that for all $u \geqslant u_0$ and $\varphi \in H'^{(u)}_{2;1}\big(B'^{(u)}(2)\big)$ satisfying $H_{[u]}\varphi = 0$ weakly on $B'^{(u)}(1)$ one has*

$$\|\nabla'_u\varphi\|^2_{2,u,r_0} \leqslant r_0^{D-2+2\nu}\|\nabla'_u\varphi\|^2_{2,u,1} .$$

Proof: Let $\nu_0 \in \langle \nu, 1\rangle$. Let r_0 be as in Lemma 3.4. Suppose there is no such u_0. Then for all $n \in \mathbf{N}$ there exist $u_n \geqslant \sigma r_0^{-1} \vee \rho \vee n$ and $\varphi_n \in H'^{(u_n)}_{2;1}\big(B'^{(u_n)}(2)\big)$ such that $H_{[u_n]}\varphi_n = 0$ weakly on $B'^{(u_n)}(1)$ and

$$\|\nabla'_{u_n}\varphi_n\|^2_{2,u_n,r_0} > r_0^{D-2+2\nu}\|\nabla'_{u_n}\varphi_n\|^2_{2,u_n,1} ,$$

where $\rho > 0$ is such that $B^{(M)}(c\rho) = M$. We may assume that $\langle\varphi_n\rangle_{u_n,1} = 0$ and $\|\nabla'_{u_n}\varphi_n\|_{2,u_n,1} = 1$ for all $n \in \mathbf{N}$. Then $\|\varphi_n\|_{2,u_n,1} \leqslant c_N$ for all $n \in \mathbf{N}$ by the Poincaré inequality (17). But $M \times B'^{(H)}(c) \subset B'^{(u_n)}(1)$ for all $n \in \mathbf{N}$ since $u_n \geqslant \rho$, by Lemma 3.3. Applying Proposition 2.1 to the set $M \times B'^{(H)}(c)$ and the restrictions of the functions φ_n to the set $M \times B'^{(H)}(c)$ it follows that there exists a subsequence of $\varphi_1, \varphi_2, \ldots$, also denoted by $\varphi_1, \varphi_2, \ldots$, such that φ_n converges weakly on $M \times B'^{(H)}(c)$ to a φ satisfying $\varphi = P\varphi$ and $\widehat{H}\psi = 0$ weakly on $B'^{(H)}(c)$, where $\psi = \varphi^\flat$. Moreover, since $c'\sigma^{-1}r_0 < c$ one may assume by Lemma 2.2 that φ_n converges to $\mathbb{1}\otimes\psi$ strongly in $L_2(M\times B'^{(H)}\big(c'\sigma^{-1}r_0)\big)$. Then

$$\begin{aligned}
r_0^{D-2+2\nu} &= \liminf_{n\to\infty} r_0^{D-2+2\nu}\|\nabla'_{u_n}\varphi_n\|^2_{2,u_n,1}\\
&\leqslant \liminf_{n\to\infty} \|\nabla'_{u_n}\varphi_n\|^2_{2,u_n,r_0}\\
&\leqslant \liminf_{n\to\infty} c_1\,\sigma^2\, r_0^{-2}\big\|\varphi_n - \langle\varphi_n\rangle_{u_n,\sigma^{-1}r_0}\big\|^2_{2,u_n,\sigma^{-1}r_0}
\end{aligned}$$

by the Caccioppoli inequality of Lemma 3.2.

Next note that $\nu \mapsto \int_\Omega |\varphi - \nu|^2$ has its minimum for $\nu = \langle\varphi\rangle_\Omega$, the average of φ over Ω. Moreover, $B'^{(u)}(\sigma^{-1}r_0) \subset B^{(M)}(c'\sigma^{-1}r_0u) \times B'^{(H)}(c'\sigma^{-1}r_0) = M \times B'^{(H)}(c'\sigma^{-1}r_0)$ by Lemma 3.3 whenever $\sigma^{-1}r_0u \geqslant 1$ and $c'\sigma^{-1}r_0u \geqslant c\rho$. Therefore with $r_1 = c'\sigma^{-1}r_0$ one has

$$\begin{aligned}
\liminf_{n\to\infty}\big\|\varphi_n - \langle\varphi_n\rangle_{u_n,\sigma^{-1}r_0}\big\|^2_{2,u_n,\sigma^{-1}r_0} &\leqslant \liminf_{n\to\infty}\int_{M\times B'^{(H)}(c'\sigma^{-1}r_0)}\big|\varphi_n - \langle\psi\rangle_{H,r_1}\big|^2\\
&= \int_{M\times B'^{(H)}(c'\sigma^{-1}r_0)}\big|\mathbb{1}\otimes\psi - \langle\psi\rangle_{H,r_1}\big|^2\\
&= \big\|\psi - \langle\psi\rangle_{H,r_1}\big\|^2_{2,H,r_1} .
\end{aligned}$$

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It follows from Lemma 3.4 and again the normalisation $|M| = 1$ that

$$\begin{aligned} r_0^{D-2+2\nu} \leqslant c_1 \sigma^2 r_0^{-2} \|\psi - \langle\psi\rangle_{H,r_1}\|_{2,H,r_1}^2 &\leqslant r_0^{D-2+2\nu_0} \|\psi\|_{2,H,c}^2 \\ &= r_0^{D-2+2\nu_0} \int_{M\times B'^{(H)}(c)} |\varphi|^2 \\ &\leqslant r_0^{D-2+2\nu_0} \liminf_{n\to\infty} \int_{M\times B'^{(H)}(c)} |\varphi_n|^2 \\ &\leqslant r_0^{D-2+2\nu_0} \liminf_{n\to\infty} \|\varphi_n\|_{2,u_n,1}^2 = r_0^{D-2+2\nu_0} \end{aligned}$$

Then one has a contradiction since $\nu < \nu_0$, $D \geqslant 2$ and $r_0 < 1$. □

These local estimates extend to global estimates by various applications of scaling. If $0 < r \leqslant R$, $s, u > 0$, $\varphi \in H_{2;1}'^{(u)}(B'^{(u)}(R))$ satisfying $H_{[u]}\varphi = 0$ weakly on $B'^{(u)}(r)$ and $\psi = \varphi \circ \Gamma_s$ then $\psi \in H_{2;1}'^{(us)}(B'^{(us)}(Rs^{-1}))$ and $H_{[us]}\psi = 0$ weakly on $B'^{(us)}(rs^{-1})$. Moreover,

$$\|\nabla'_{us}\psi\|_{2,us,\rho}^2 = s^{2-D}\, \|\nabla'_u\varphi\|_{2,u,\rho s}^2$$

for all $\rho \in \langle 0, Rs^{-1}]$.

Lemma 3.6. *For all $\nu \in \langle 0,1\rangle$ there exist $r_0 \in \langle 0,1\rangle$ and $u_0 \geqslant 1$ such that for all $r \in \langle 0, r_0]$, $u \geqslant r^{-1}u_0$ and $\varphi \in H_{2;1}'^{(u)}(B'^{(u)}(2))$ satisfying $H_{[u]}\varphi = 0$ weakly on $B'^{(u)}(1)$ one has*

$$\|\nabla'_u\varphi\|_{2,u,r}^2 \leqslant r_0^{-D+2-2\nu} r^{D-2+2\nu} \|\nabla'_u\varphi\|_{2,u,1}^2 \,.$$

Proof: Let $r_0 \in \langle 0,1\rangle$ and $u_0 \geqslant 1$ be as in Proposition 3.5. The proof is by induction. By Proposition 3.5 one has

$$\|\nabla'_u\varphi\|_{2,u,r_0}^2 \leqslant r_0^{D-2+2\nu} \|\nabla'_u\varphi\|_{2,u,1}^2$$

for all $u \geqslant u_0$ and $\varphi \in H_{2;1}'^{(u)}(B'^{(u)}(2))$ satisfying $H_{[u]}\varphi = 0$ weakly on $B'^{(u)}(1)$. Let $k \in \mathbf{N}$ and suppose that

$$\|\nabla'_u\varphi\|_{2,u,r_0^k}^2 \leqslant (r_0^k)^{D-2+2\nu} \|\nabla'_u\varphi\|_{2,u,1}^2 \,. \tag{21}$$

for all $u \geqslant r_0^{-k}u_0$ and $\varphi \in H_{2;1}'^{(u)}(B'^{(u)}(2))$ satisfying $H_{[u]}\varphi = 0$ weakly on $B'^{(u)}(1)$. Let $u \geqslant r_0^{-(k+1)}u_0$ and $\varphi \in H_{2;1}'^{(u)}(B'^{(u)}(2))$ satisfying $H_{[u]}\varphi = 0$ weakly on $B'^{(u)}(1)$. Set $s = r_0^k$ and $\psi = \varphi \circ \Gamma_s$. Then Proposition 3.5 implies that

$$\|\nabla'_{us}\psi\|_{2,us,r_0}^2 \leqslant r_0^{D-2+2\nu} \|\nabla'_{us}\psi\|_{2,us,1}^2 \,.$$

Hence

$$\|\nabla'_u\varphi\|_{2,u,r_0^{k+1}}^2 \leqslant r_0^{D-2+2\nu} \|\nabla'_u\varphi\|_{2,u,r_0^k}^2 \leqslant r_0^{(k+1)(D-2+2\nu)} \|\nabla'_u\varphi\|_{2,u,1}^2$$

where the induction hypothesis (21) is used in the last step.

This proves the lemma if $r \in r_0^{\mathbf{N}}$. The general case is an easy consequence. □

LEMMA 3.7. *For all $\nu \in \langle 0,1\rangle$ there exist $r_0 \in \langle 0,1\rangle$ and $c > 0$ such that for all $u > 0$ and $\varphi \in H'^{(u)}_{2;1}(B'^{(u)}(2))$ satisfying $H_{[u]}\varphi = 0$ weakly on $B'^{(u)}(1)$ one has*

$$\|\nabla'_u\varphi\|^2_{2,u,r} \leqslant c\, r^{D-2+2\nu}\|\nabla'_u\varphi\|^2_{2,u,1} \tag{22}$$

for all $r \in \langle 0, r_0]$.

PROOF: Let r_0, u_0 be as in Lemma 3.6. Let $u > 0$, $r \in \langle 0, r_0]$ and $\varphi \in H'^{(u)}_{2;1}(B'^{(u)}(2))$ satisfying $H_{[u]}\varphi = 0$ weakly on $B'^{(u)}(1)$. If $u \geqslant r^{-1}u_0$ then (22) is valid with $c = r_0^{-D+2-2\nu}$ by Lemma 3.6. So we may assume that $u \leqslant r^{-1}u_0$. Let c_{DG} be the De Giorgi constant for the operator $H_{[u_0]}$ associated to the order ν (see [**9**, Proposition 3.3]). Set $\psi = \varphi \circ \Gamma_{u^{-1}u_0}$. Then $H_{[u_0]}\psi = 0$ weakly on $B'^{(u_0)}(uu_0^{-1})$ and

$$\|\nabla'_{u_0}\psi\|^2_{2,u_0,\rho} \leqslant c_{DG}\,(\rho/R)^{D-2+2\nu}\|\nabla'_{u_0}\psi\|^2_{2,u_0,R}$$

for all $0 < \rho \leqslant R \leqslant (uu_0^{-1} \wedge 1)$. Therefore

$$\|\nabla'_u\varphi\|^2_{2,u,u^{-1}u_0\rho} \leqslant c_{DG}\,(\rho/R)^{D-2+2\nu}\|\nabla'_u\varphi\|^2_{2,u,u^{-1}u_0R}$$

for all $0 < \rho \leqslant R \leqslant (uu_0^{-1} \wedge 1)$.

Now if $u \leqslant u_0$ choose $\rho = uu_0^{-1}r$ and $R = uu_0^{-1}$. Then

$$\|\nabla'_u\varphi\|^2_{2,u,r} \leqslant c_{DG}\, r^{D-2+2\nu}\|\nabla'_u\varphi\|^2_{2,u,1}$$

as desired.

But if $u > u_0$ and since $u \leqslant r^{-1}u_0$ one can choose $\rho = uu_0^{-1}r$ and $R = 1$. Then

$$\|\nabla'_u\varphi\|^2_{2,u,r} \leqslant c_{DG}\,(uu_0^{-1}r)^{D-2+2\nu}\|\nabla'_u\varphi\|^2_{2,u,u^{-1}u_0}\,.$$

If, however, $u^{-1}u_0 \leqslant r_0$ then, by Lemma 3.6, one has

$$\|\nabla'_u\varphi\|^2_{2,u,u^{-1}u_0} \leqslant r_0^{-D+2-2\nu}(u^{-1}u_0)^{D-2+2\nu}\|\nabla'_u\varphi\|^2_{2,u,1}\,. \tag{23}$$

Alternatively, if $u^{-1}u_0 \geqslant r_0$ then (23) is valid since $u^{-1}u_0 \leqslant 1$. Hence

$$\begin{aligned}\|\nabla'_u\varphi\|^2_{2,u,r} &\leqslant c_{DG}\, r_0^{-D+2-2\nu}(uu_0^{-1}r)^{D-2+2\nu}(u^{-1}u_0)^{D-2+2\nu}\|\nabla'_u\varphi\|^2_{2,u,1}\\ &= c_{DG}\, r_0^{-D+2-2\nu}\, r^{D-2+2\nu}\|\nabla'_u\varphi\|^2_{2,u,1}\end{aligned}$$

as required. ∎

PROOF OF PROPOSITION 3.1: The De Giorgi estimates of the proposition follow from those of Lemma 3.7 by scaling. ∎

Although the estimates (5), (17) and (19) have been expressed in terms of balls centred at the identity the same bounds are true for balls centred at an arbitrary $g \in G$ by right invariance of the differential operators and unimodularity of G. Therefore one obtains the desired Gaussian bounds on $K^{[u]}_t$ for $t \in \langle 0,1]$, uniformly for $u \geqslant 1$, by [**9**, Theorem 4.1], since all the estimates are uniform in u. Then the bounds of Theorem 1.1 follow from (4).

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Centre for Mathematics and its Applications
Australian National University
Canberra, ACT 0200
Australia