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PRE-TITS SYSTEMS

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ABSTRACT. A Tits system gives rise to the corresponding pre-Tits system which is the free product of the standard parabolic subgroups of rank 1 amalgamating B. Some properties of this system are investigated.

According to a theorem of Tits, as given for instance in Serre [2] p. 92, the group of every Tits system is the sum of N and the standard parabolic subgroups of rank 1 amalgamated along their intersections. Tits [4] has established a converse of this result. We establish a different type of converse.

Given a Tits system we construct the corresponding *pre-Tits system* which is the free product of the standard parabolic subgroups of rank 1 amalgamating B. We investigate some of its properties. In a subsequent paper we shall use these results to derive some information concerning the set of all relations of certain 2-generator subgroups of GL_3.

We shall assume the reader is familiar with the elementary theory of Tits systems and free products with amalgamation as given in Serre [2]. We shall use his notation except that we shall deviate in one respect from the usual notation for a Tits system (G, B, N, S) in that we take S to be a *subset* of G.

Finally, we note that although it is possible to derive Theorem 2 of this paper directly from the initially mentioned Theorem of Tits—in fact of course they are equivalent —we shall for the sake of clarity proceed on a more lengthy route.

Jacques Tits read an initial draft of these results and made some helpful comments which have been incorporated in this paper.

The proof of Tits given in Serre [2] II 1.7 enables one to establish the following useful variant of a Theorem of Tits.

THEOREM 1. *Suppose that (G, B, N, S) is a Tits system and $\tilde{G}$ is a group which has subgroups $\tilde{B}$ and $\tilde{N}$ with the property:*

(i) *the set $\tilde{N} \cup \tilde{B}$ generates $\tilde{G}$ and $\tilde{N} \cap \tilde{B}$ is a normal subgroup of $\tilde{N}$.*

If there exists a group homomorphism $\tilde{\phi}$ of $\tilde{G}$ onto G with the properties:

(ii) *$\tilde{\phi}$ induces a natural isomorphism of $\tilde{N}/\tilde{N} \cap \tilde{B} = \tilde{W}$ onto W and an inverse image $\tilde{S}$ of S under $\tilde{\phi}$ is a set of generators for $\tilde{N}$ modulo $\tilde{N} \cap \tilde{B}$;*

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(iii) $\tilde{\phi}$ *induces an isomorphism of* $\tilde{B}$ *onto* B *and of* $\tilde{G}_{\tilde{s}}$ *onto* G_s, *where*

$$\tilde{G}_s = \tilde{B}\,\langle \tilde{s} \rangle\,\tilde{B} \qquad \textit{for every } \tilde{s} \textit{ in } \tilde{S};$$

then G *is isomorphic to* $\tilde{G}$ *under the group isomorphism* $\tilde{\phi}$.

PROOF. We assume that the above described homomorphism $\tilde{\phi}$ exists. It is well known that $(W, S \bmod N \cap B)$ is a Coxeter system. Hence, by (ii) above, $(\tilde{W}, \tilde{S} \bmod \tilde{N} \cap \tilde{B})$ is a Coxeter system. We shall first show that $(\tilde{G}, \tilde{B}, \tilde{N}, \tilde{S})$ is a Tits system.

($\tilde{T}_1$) is just (i) above;
($\tilde{T}_2$) follows from (ii) above;
($\tilde{T}_4$) follows from (ii) and (iii) above;
($\tilde{T}_5$) follows from (iii) above, since it takes place in $\tilde{G}_{\tilde{s}}$;
($\tilde{T}_7$) is proved as in Serre [2] p. 94 by means of (iii) above.

$\tilde{G}$ is the disjoint union of the double coscts $\tilde{B}\tilde{w}\tilde{B}$, where $\tilde{w}$ varies over $\tilde{W}$, since $\tilde{G}$ is a Tits system. If $\tilde{b}\tilde{w}\tilde{b}'$ is mapped by $\tilde{\phi}$ onto $\tilde{1}$, where $\tilde{b}$ and $\tilde{b}'$ belong to $\tilde{B}$ and $\tilde{w}$ belongs to $\tilde{W}$, then $bwb' = 1$ and w belongs to B, where

$$\tilde{\phi}(\tilde{b}) = b, \qquad \tilde{\phi}(\tilde{b}') = b', \qquad \tilde{\phi}(\tilde{w}) = w,$$

by (ii) and (iii) above. Hence it follows from (iii) above that $\tilde{w}$ and $\tilde{b}\tilde{w}\tilde{b}'$ belong to $\tilde{B}$, which also implies that $\tilde{b}\tilde{w}\tilde{b}' = \tilde{1}$. So $\tilde{\phi}$ is injective.

Let (G, B, N, S) be a Tits system. We now form the free product $\bar{G}$ of the standard parabolic subgroups of rank 1 amalgamating B. Then there is obviously a natural group homomorphism ϕ of $\bar{G}$ onto G. The next result gives some information on the kernel of ϕ.

THEOREM 2. *Let* (G, B, N, S) *be a Tits system with* $|S| \geq 2$ *and* G_s, $s \in S$, *be the collection of all standard parabolic subgroups of rank 1 in this Tits system. Suppose that* $\bar{S}$ *is a set of elements, which is in one-to-one correspondence with the set of elements* S *(under the mapping* $\bar{s}$ *goes to* s *for all* s *in* S*), and there is an isomorphism of some group* $\bar{B}$ *onto the group* B. *Further suppose that these two mappings give a natural isomorphism of some group*

$$\bar{G}_{\bar{s}} = \bar{B}\,\langle \bar{s} \rangle\,\bar{B} \qquad \textit{onto} \quad G_s$$

for every s *in* S. *Then these isomorphisms can be extended in a natural way to a homomorphism* ϕ *of the free product with amalgamation (the corresponding pre-Tits system)*

$$\bar{G} = \mathop{*}_{\bar{B}} \bar{G}_{\bar{s}} \qquad \textit{with} \quad \bar{s} \in \bar{S}$$

onto the group G. *Let* $\bar{K}$ *be the kernel of the restriction of* ϕ *to the subgroup* $\langle \bar{s},\ \bar{s} \in \bar{S} \rangle$ *of* $\bar{G}$. *Then the kernel of* ϕ *is the normal subgroup* $\bar{K}^{\bar{G}}$ *of* $\bar{G}$ *generated by* $\bar{K}$. *Also* $\bar{K}^{\bar{G}}$ *is a free group.*

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PROOF. It is clear from the properties of Tits systems and free products with amalgamation that there exists a natural homomorphism ϕ of $\bar{G}$ onto G and

$$\bar{K}^{\bar{G}} \subseteq \text{kernel } \phi.$$

There exists a subgroup $\bar{H}$ of $\bar{B}$ that is mapped isomorphically by ϕ onto $N \cap B$. We define $\bar{N}$ to be the subgroup of $\bar{G}$ which is generated by $\bar{H}$ and all $\bar{s}$ with $\bar{s}$ in $\bar{S}$. Clearly

$$\bar{N} = \underset{\bar{H}}{*} \langle \bar{H}, \bar{s} \rangle \qquad \text{with} \quad \bar{s} \text{ in } \bar{S}$$

and $\bar{N} \cap \bar{B} = \bar{H}$. Further

$$(\bar{s})^{-1} \cdot \bar{N} \cap \bar{B} \cdot \bar{s} \subseteq \bar{N} \cap \bar{B}$$

for every $\bar{s}$ in $\bar{S}$, since a similar result holds in G_s, and so $\bar{N} \cap \bar{B}$ is normal in $\bar{N}$.

We adopt the following notation:

$$\bar{G}/\bar{K}^{\bar{G}} = \tilde{G} \quad , \qquad \bar{B} \cdot \bar{K}^{\bar{G}}/\bar{K}^{\bar{G}} = \tilde{B},$$
$$\bar{N} \cdot \bar{K}^{\bar{G}}/\bar{K}^{\bar{G}} = \tilde{N}, \qquad \bar{S} \cdot \bar{K}^{\bar{G}}/\bar{K}^{\bar{G}} = \tilde{S}.$$

We shall now show that the above forms a Tits system isomorphic to the Tits system (G, B, N, S) under the natural homomorphism $\tilde{\phi}: \tilde{G} \to G$ induced by ϕ. It remains to show that conditions (i) and (ii) of Theorem 1 hold.

(i) Since $\bar{N} \cup \bar{B}$ generate $\bar{G}$, it follows that $\tilde{N} \cup \tilde{B}$ generate $\tilde{G}$. Further

$$y\bar{K}^{\bar{G}} = \bar{b}\bar{K}^{\bar{G}} \qquad \text{for} \quad y \in \bar{N} \qquad \text{and} \quad \bar{b} \in \bar{B}$$

implies that $\phi y = b$ in G (after applying $\tilde{\phi}$), where $\phi \bar{b} = b \in B$. As $\bar{N} = \underset{\bar{H}}{*}\langle \bar{H}, \bar{s} \rangle$ with $\bar{s}$ in $\bar{S}$ and $\bar{H}$ is normal in $\bar{N}$, we have that b belongs to $N \cap B$ and so y belongs to $\bar{H} \cdot \bar{K}^{\bar{G}}$. Hence

$$\tilde{N} \cap \tilde{B} = (\bar{N} \cap \bar{B}) \cdot \bar{K}^{\bar{G}}/\bar{K}^{\bar{G}},$$

which is normal in $\tilde{N}$, since $\bar{N} \cap \bar{B}$ has been shown to be normal in $\bar{N}$.

(ii) Since $\tilde{\phi}$ maps $\tilde{N}$ and $\tilde{N} \cap \tilde{B}$ onto N and $N \cap B$ respectively, there is a homomorphism of

$$\tilde{N}/\tilde{N} \cap \tilde{B} \qquad \text{onto} \quad N/N \cap B.$$

This is equivalent to the homomorphism of

$$(\bar{N}/\bar{N} \cap \bar{B})/((\bar{K}^{\bar{G}} \cap \bar{N}) \cdot (\bar{N} \cap \bar{B})/(\bar{N} \cap \bar{B}) \qquad \text{onto} \quad N/N \cap B$$

induced by ϕ. However, ϕ also induces a homomorphism of

$$\bar{N}/\bar{N} \cap \bar{B} = \langle \bar{s}, \bar{s} \in \bar{S} \rangle \cdot \bar{N} \cap \bar{B}/\bar{N} \cap \bar{B} \qquad \text{onto}$$
$$N/N \cap B = \langle s, s \in S \rangle \cdot N \cap B/N \cap B,$$

which has kernel $\bar{K} \cdot \bar{N} \cap \bar{B}/\bar{N} \cap \bar{B}$. Now

$$\bar{K}^{\bar{G}} \cap \bar{N} \supseteq \bar{K}$$

and so the initially given homomorphism of

$$\tilde{N}/\tilde{N} \cap \tilde{B} \quad \text{onto} \quad N/N \cap B$$

is an isomorphism. In fact we have

$$(2.1) \qquad (\bar{K}^{\bar{G}} \cap \bar{N}) \cdot (\bar{N} \cap \bar{B}) = \bar{K} \cdot (\bar{N} \cap \bar{B})$$

Finally, we note that the subgroup $\bar{K}^{\bar{G}}$ is free by the Subgroup Theorem of Hanna Neumann on free products with amalgamation (see for instance Serre [2] p. 56). This is because

$$\bar{B} \cap \bar{K}^{\bar{G}} = \langle 1 \rangle \quad \text{and} \quad \bar{G}_{\bar{s}} \cap \bar{K}^{\bar{G}} = \langle 1 \rangle$$

for every $\bar{s}$ in $\bar{S}$.

COROLLARY. *Let n be an integer greater than 2 and k denote a fixed field. Then the general linear group $GL(n; k)$ is isomorphic to the free product with amalgamation*

$$\bar{G}_n = \bar{G}_{\bar{s}_1} \underset{\bar{B}}{*} \bar{G}_{\bar{s}_2} \underset{\bar{B}}{*} \dots \underset{\bar{B}}{*} \bar{G}_{\bar{s}_{n-1}}$$

modulo the normal subgroup $\bar{K}_n^{\bar{G}_n}$ of $\bar{G}_n$ which is generated by the elements

$$(\bar{s}_i \bar{s}_{i+1})^3 \quad \text{for} \quad 1 \leq i \leq n-2$$

and

$$(\bar{s}_i \bar{s}_j)^2 \quad \text{for} \quad 1 \leq i \leq j-2 \quad \text{and} \quad 3 \leq j \leq n-1.$$

Here B denotes the Borel subgroup of upper triangular matrices and s_i is obtained from the unit matrix by interchanging i-th and $(i+1)$-th rows for $i = 1, 2, \dots, n-1$. This result makes use of the usual Tits system for the general linear group and a well known group presentation for its Weyl group, which is the symmetric group S_n (see for instance Bourbaki [1] Chap. IV p. 24).

We introduce a common property of a Tits system and its corresponding pre-Tits system. A more general version of Lemma 1 can be found in Tits [3] Lemma 13.13.

LEMMA 1. *Let (G, B, N, S) be a Tits system with $|S| \geq 2$ and*

$$w = s_{i_1} s_{i_2} \dots s_{i_m}$$

be a word in the elements of S which modulo $N \cap B$ has length m in the Coxeter group $(W, S \bmod N \cap B)$. If $w^{-1} b_0 w$ belongs to B for some b_0 in B, then there exist elements $b_1, b_2, \dots, b_m$ of B so that

$$s_{i_j}^{-1} b_{j-1} s_{i_j} = b_j \quad \text{for} \quad j = 1, 2, \dots, m.$$

LEMMA 2. *Let (G, B, N, S) be a Tits system with $|S| \geq 2$ and*

$$\bar{G} = \underset{\bar{B}}{*} \bar{G}_{\bar{s}} \quad \text{with} \quad \bar{s} \in \bar{S}$$

be the corresponding pre-Tits system as given in Theorem 2. Let

$$\bar{w} = \bar{s}_{i_1}\bar{s}_{i_2} \ldots \bar{s}_{i_m}$$

be a word in the elements of $\bar{S}$ with $\bar{s}_{i_j} \neq \bar{s}_{i_{j+1}}$ *for every* j. *If* $(\bar{w})^{-1}\cdot\bar{b}_0\cdot\bar{w}$ *belongs to* $\bar{B}$ *for some* $\bar{b}_0$ *in* $\bar{B}$, *then there exists elements* $\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_m$ *of* $\bar{B}$ *so that*

$$(\bar{s}_{i_j})^{-1}\bar{b}_{j-1}\bar{s}_{i_j} = \bar{b}_j \qquad \text{for} \quad j = 1, 2, \ldots, m.$$

The above result can easily be established by induction on m. A similar method enables one to prove the following result.

LEMMA 3. *Let* (G, B, N, S) *be a Tits system with* $|S| \geq 2$ *and*

$$\bar{G} = \underset{\bar{B}}{*}\, \bar{G}_{\bar{s}} \qquad \text{with} \quad \bar{s} \in \bar{S}$$

be the corresponding pre-Tits system as given in Theorem 2. Further let

$$w(s_i) = s_{i_1}s_{i_2} \ldots s_{i_m} \qquad \text{and} \quad s_{i_0}w(s_i)$$

be words in the elements of S *which modulo* $N \cap B$ *have length* m *and* $m + 1$ *respectively in the Coxeter group* $(W, S \bmod N \cap B)$. *Then for given* $\bar{b}$ *in* $\bar{B}$ *there exist* $\bar{b}'$ *and* $\bar{b}''$ *of* $\bar{B}$ *so that*

$$\bar{s}_{i_0}\bar{b}w(\bar{s}_i) = \bar{b}'\bar{s}_{i_0}w(\bar{s}_i)\bar{b}'' \tag{3.1}$$

holds in $\bar{G}$, *where*

$$w(\bar{s}_i) = \bar{s}_{i_1}\bar{s}_{i_2} \ldots \bar{s}_{i_m}.$$

PROOF. (a) We use the unique representation for the elements of a free product with amalgamation. Suppose that equation (3.1) holds. Then we have that

$$\bar{b}'\bar{s}_{i_0} = \bar{s}_{i_0}\bar{b}_1 \qquad \text{for some } \bar{b}_1 \in \bar{B},$$

that is,

$$\bar{b}' \in \bar{s}_{i_0}\bar{B}(\bar{s}_{i_0})^{-1}.$$

Substitute this in equation (3.1):

$$\bar{b}'' \in (w(\bar{s}_i))^{-1}\bar{B}w(\bar{s}_i).$$

Now it follows that (3.1) holds if we can show that

$$\bar{B} = (\bar{B} \cap (\bar{s}_{i_0})^{-1}\bar{B}\bar{s}_{i_0})\cdot(\bar{B} \cap w(\bar{s}_i)\bar{B}(w(\bar{s}_i))^{-1}).$$

(b) Here we show the above equality holds. By Axiom (T_7), we have that

$$B = (B \cap s_{i_0}^{-1}Bs_{i_0})\cdot(B \cap w(s_i)B(w(s_i))^{-1}).$$

However, G_s is isomorphic to $\bar{G}_{\bar{s}}$ for every s in S. Under every such isomorphism, B is mapped onto $\bar{B}$. Now the required result follows on applying Lemmas 1 and 2 to

$$B \cap w(s_i)B(w(s_i))^{-1} \qquad \text{and} \quad \bar{B} \cap w(\bar{s}_i)\bar{B}(w(\bar{s}_i))^{-1}$$

respectively.

LEMMA 4. *Let* (G, B, N, S) *be a Tits system and*

$$\bar{G} = \underset{\bar{B}}{*}\, \bar{G}_{\bar{s}} \qquad \text{with} \quad \bar{s} \in \bar{S}$$

be the corresponding pre-Tits system. Then every element of $\bar{G}$ *belongs to some subset of the form*

$$\bar{B}w(\bar{s})\bar{B}\cdot\bar{E},$$

where $\bar{E}$ *is the subgroup of* $\bar{G}$ generated by all elements of the form $\bar{b}\bar{x}(\bar{b})^{-1}$ *with* $\bar{b}$ *belonging to* $\bar{B}$ *and* $\bar{x}$ *belonging to* $\bar{K}$. *Further* $w(s)$ *varies over the collection of all distinct reduced words in the Coxeter group* $(W, S \bmod N \cap B)$.

PROOF. Every element of $\bar{G}$ (which does not belong to $\bar{B}$) is a finite product of the form

$$\prod \bar{b}_0\bar{s}\bar{b}_1,$$

where every $\bar{b}_0$, $\bar{b}_1 \in \bar{B}$ and every $\bar{s} \in \bar{S}$. We proceed by induction on the number of elements in this product. The required result is trivially true for a product with one element.

By the induction hypothesis, it is necessary to consider an expression of the form

$$\bar{g} = \bar{b}_0\bar{s}_0\bar{b}_1\cdot\bar{b}_2w_1(\bar{s})\bar{b}_3'\cdot\bar{e},$$

where every $\bar{b}_0$, $\bar{b}_1$, $\bar{b}_2$, $\bar{b}_3' \in \bar{B}$ and $\bar{s}_0 \in \bar{S}$ and $\bar{e} \in \bar{E}$ and $w_1(s)$ is a reduced word in the Coxeter group $(W, S \bmod N \cap B)$. Now there are two possibilities:

(i) $s_0w_1(s)$ is a reduced word in the Coxeter group;

(ii) $w_1(\bar{s}) = \bar{s}_0w_2(\bar{s})\cdot\bar{x}_1\bar{b}_3''$ with $\bar{x}_1 \in \bar{K}$, $\bar{b}_3'' \in \bar{B}$ and $s_0w_2(s)$ is a reduced word in the Coxeter group, by Proposition 4 of Bourbaki [1] Chap. IV p. 15 and equation (2.1) in the proof of Theorem 2.

If case (i) occurs, then the required result follows from Lemma 3. Suppose we are now in case (ii). Then

$$\bar{g} = \bar{b}_0\bar{s}_0\bar{b}_1\cdot\bar{b}_2\bar{s}_0w_2(\bar{s})\bar{x}_1\bar{b}_3\cdot\bar{e},$$

with $\bar{b}_3 \in \bar{B}$. Now

$$\bar{b}_0\bar{s}_0\bar{b}_1\cdot\bar{b}_2\bar{s}_0 = \bar{b}_4 \qquad \text{or} \quad \bar{b}_4\bar{s}_0\bar{b}_5,$$

where $\bar{b}_4$, $\bar{b}_5 \in \bar{B}$, since Axiom (T_5) holds in $\bar{B}\langle\bar{s}_0\rangle\bar{B}$. Hence either

$$\bar{g} = \bar{b}_4w_2(\bar{s})\bar{x}_1\bar{b}_3\cdot\bar{e}$$

or

$$\bar{g} = \bar{b}_4\bar{s}_0\bar{b}_5 w_2(\bar{s})\bar{x}_1\bar{b}_3\cdot\bar{e}$$
$$= \bar{b}_6 w_1(\bar{s})\bar{b}_7\cdot\bar{e}' \qquad \text{by Lemma 3,}$$

where $\bar{b}_6$, $\bar{b}_7 \in \bar{B}$ and $\bar{e}' \in \bar{E}$. They are both of the required form.

We are now able to give a useful set of generators for the kernel $\bar{K}^{\bar{G}}$ of the natural homomorphism of a pre-Tits system onto its Tits system.

THEOREM 3. *Let* $\{\bar{b}_\lambda, \lambda \in \Lambda\}$ *be a collection of elements of* $\bar{B}$ *which form a complete set of left coset representatives relative to the centraliser of* $\bar{K}$ *in* $\bar{B}$. *Then*

$$\bar{K}^{\bar{G}} = \{\bar{b}_\lambda \bar{x}(\bar{b}_\lambda)^{-1}; \quad \bar{x} \in \bar{K}, \quad \lambda \in \Lambda\}.$$

PROOF. By Lemma 4,

$$\bar{G} = \bigcup_{w(\bar{s})} (\bar{B}w(\bar{s})\bar{B}\cdot\bar{E}),$$

where the union is taken over all distinct reduced words $w(s)$ in the Coxeter group $(W, S \bmod N \cap B)$. By Théorème 1 of Bourbaki [1] Chap. IV, p. 25,

$$\bigcup_{w(\bar{s})} \bar{B}w(\bar{s})\bar{B}$$

is a complete set of left coset representatives of $\bar{G}$ relative to $\bar{K}^{\bar{G}}$. Hence

$$\left(\bigcup_{w(\bar{s})} \bar{B}w(\bar{s})\bar{B}\right) \cap \bar{K}^{\bar{G}} = \langle \bar{1} \rangle.$$

This result, the initially given equality and the fact that $\bar{E} \subseteq \bar{K}^{\bar{G}}$ enables us to deduce that $\bar{E} = \bar{K}^{\bar{G}}$.

Finally, we consider a characterisation of Tits systems in terms of pre-Tits systems. This is similar in aim to the results of Tits [4].

Suppose that we are given a system of groups (in the sense of Tits [4])

$$B, N, (P_i)_{i \in I},$$

which are subsets of some universal set. Take $N \cap P_i = N_i$ for every i. Further suppose that the following properties hold:

(i) $P_i \cap P_j = B$ if $i \neq j$;
(ii) $H = N \cap B$ is a normal subgroup of N and one puts $W = N/H$;
(iii) the group N_i/H is of the order 2 with elements H and s_iH and one puts $\{s_i, i \in I\} = S$;
(iv) $P_i = B \cup Bs_iB$ for every i;
(v) (W, SH) is a Coxeter group.

We take $|I| \geq 2$.

Let $\mathbf{G} = *_{\bar{B}} \mathbf{P}_i$, where $\mathbf{P}_i$ is isomorphic to P_i for every i and $\mathbf{P}_i \cap \mathbf{P}_j = \mathbf{B}$ which is naturally isomorphic to B. Take the inverse image of s_i under the above isomorphisms to be $\mathbf{s}_i$ for all $i \in I$ and

$$\mathbf{S} = \{\mathbf{s}_i,\ i \in I\}.$$

Under the above given isomorphism of $\mathbf{B}$ onto B we have a subgroup $\mathbf{H}$ of $\mathbf{B}$ which is mapped isomorphically onto $N \cap B = H$. Let

$$\mathbf{N} = \langle \mathbf{H},\ \mathbf{s}_i;\ i \in I \rangle$$

in $\mathbf{G}$. Then

$$\mathbf{N} = \underset{\mathbf{H}}{*} \langle \mathbf{H},\ \mathbf{s}_i \rangle \qquad \text{with} \quad i \in I.$$

There exists a natural homomorphism of

$$\langle \mathbf{s}_i,\ i \in I \rangle \qquad \text{onto} \qquad \langle s_i,\ i \in I \rangle.$$

Let $\mathbf{K}$ denote the kernel of this homomorphism, while $\mathbf{K}^{\mathbf{G}}$ denotes the normal subgroup of $\mathbf{G}$ generated by $\mathbf{K}$. We take

$$\mathbf{W} = \mathbf{N}/\mathbf{H} \simeq * \langle \mathbf{s}_i;\ \mathbf{s}_i^2 \rangle \qquad \text{with} \qquad i \in I,$$

since $\mathbf{H}$ is normal in $\mathbf{N}$ by condition (ii).

We formulate the following further conditions.

C(a) $\mathbf{P}_i \cap \mathbf{K}^{\mathbf{G}} = \langle \mathbf{1} \rangle$ for every $i \in I$.

C(b) $\mathbf{N} \cdot \mathbf{K}^{\mathbf{G}}/\mathbf{K}^{\mathbf{G}}$ is naturally isomorphic to N and

$$(\mathbf{N} \cdot \mathbf{K}^{\mathbf{G}}/\mathbf{K}^{\mathbf{G}}) \cap (\mathbf{B} \cdot \mathbf{K}^{\mathbf{G}}/\mathbf{K}^{\mathbf{G}}) \subseteq \mathbf{H} \cdot \mathbf{K}^{\mathbf{G}}/\mathbf{K}^{\mathbf{G}}.$$

These two conditions ensure that the group $\mathbf{G}/\mathbf{K}^{\mathbf{G}}$ has subgroups naturally isomorphic to P_i, B and N and has "Weyl group" naturally isomorphic to $W = N/H$. We are thus able to perform the obvious identifications.

C(c) If wH belongs to W and s_i belongs to S such that

$$l(s_i wH) = l(wH) + 1.$$

then in $\mathbf{G}/\mathbf{K}^{\mathbf{G}}$ we have that

$$B = B_{wH} \cdot B_{s_i H}.$$

C(d) B is not normal in P_i for every i.

Together with Theorem 2 the above given material gives

THEOREM 4. $(\mathbf{G}/\mathbf{K}^{\mathbf{G}}, B, N, S)$ *is a Tits system if and only if conditions (i)–(v) and C(a)–C(d) hold. If these conditions do hold, then* $\mathbf{G}$ *is the corresponding pre-Tits system.*

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