Compositio Mathematica **127:** 273–287, 2001.

Class Numbers of CM-Fields with Solvable Normal Closure

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(Received: 24 August 1999; accepted: 11 May 2000)

Abstract. Stark conjectured that given a positive integer h, there are only a finite number of CM-fields L with class number equal to h. We prove this conjecture for fields L of degree $\geqslant 6$ whose normal closure is solvable.

Mathematics Subject Classifications (2000). Primary: 11R42, 11R29; Secondary: 11M20, 20F16.

Key words. class number, CM field, Dedekind zeta function, Siegel zero.

1. Introduction

By a CM-field, we mean a totally complex quadratic extension of a totally real field. In [12], Stark conjectured that given a positive integer h, there are only a finite number of CM-fields L with class number equal to h. He was able to prove that the number of such fields with a fixed degree $\geqslant 6$ is finite. Odlyzko [10] refined this to show that there are only a finite number of such fields of degree $\geqslant 6$ whose maximal totally real subfield K occurs in some tower

$$\mathbb{Q} = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_\ell = K$$

of number fields, with each member normal over its predecessor. In particular, this includes all K which are Galois over $\mathbb{Q}$. Hoffstein–Jochnowitz [5] showed that there is an absolute, effectively computable (large) constant $C > 0$ such that the tower condition above can be replaced with a discriminant bound $d_K > C^{[K:\mathbb{Q}]}$. In this paper, we shall prove Stark's conjecture for CM fields of degree $\geqslant 6$ whose normal closure is solvable.

Stark's conjecture is based on his results about zeros of the Dedekind zeta function near $s = 1$. Let L/K be a Galois extension of number fields. In [12], Stark showed that every simple real zero of the Dedekind zeta function $\zeta_L(s)$ is actually a zero of the Dedekind zeta function $\zeta_N(s)$ of a field $K \subseteq N \subseteq L$ and $[N : K] \leqslant 2$. If the extension L/K is not Galois, there is no reason why such a statement should be true.

https://doi.org/10.1023/A:1017589432526 Published online by Cambridge University Press

However, one case in which it does often happen that simple real zeros come from at most quadratic extensions of the base is when the zero occurs near $s = 1$. The meaning of the word 'near' depends on the context. Stark showed that for any number field L, a zero of $\zeta_L(s)$ which is in the region⋆

$$1 - \frac{1}{4 \log d_L} \leqslant \sigma \leqslant 1, \ |t| \leqslant \frac{1}{4 \log d_L} \tag{1}$$

is necessarily real and simple. As in [7], we shall call such a zero a *Stark zero*.

Now let L/K be an extension, not necessarily Galois. As Stark observed, by choosing $c > 0$ sufficiently small, a zero ρ in the slightly narrower region

$$1 - \frac{c}{n \log d_L} \leqslant \sigma \leqslant 1, \ |t| \leqslant \frac{c}{n \log d_L}$$

where $n = [L : K]$ (for some ground field K) is necessarily a simple zero of $\zeta_M(s)$ where M is the compositum of L and a conjugate field L^σ. Indeed,

$$\log d_M \leqslant 2n \log d_L$$

and so the zero lies in the region

$$1 - \frac{2c}{\log d_M} \leqslant \sigma \leqslant 1, \quad |t| \leqslant \frac{2c}{\log d_M}.$$

If $0 < c < 1/8$, this lies inside the Stark region (1). Hence, it is a simple zero of $\zeta_M(s)$. If we assume Artin's holomorphy conjecture for F/K, where F is some extension of L which is Galois over K, then

$$\frac{\zeta_M(s)\zeta_N(s)}{\zeta_L(s)\zeta_{L^\sigma}(s)}$$

is entire. Here, $N = L \cap L^\sigma$. It follows that $\zeta_N(s)$ has a zero at the same point. Moreover, as the discriminant of N is at most that of L, the same bound as above applies to ρ with d_L replaced with d_N. In particular, ρ is a Stark zero of $\zeta_N(s)$ and so is a simple zero. Now repeating this process if necessary, will yield an at most quadratic extension of K where this zero occurs.

In general, we do not have Artin's conjecture, but it is still useful to be able to deduce the above consequence for zeros. As established by Stark himself [12], this has applications to the growth of class numbers and the finitenes assertions mentioned above. However, the nearer the zero ρ is to $s = 1$, the weaker these growth estimates tend to be. In general, Artin's conjecture can be eliminated only at the cost of narrowing the region in which the zero occurs.

⋆Throughout this paper, σ and t shall denote the real and imaginary parts (respectively) of the complex number s. Thus the region refers to the set of complex numbers $s = \sigma + it$ with σ and t real and satisfying the given inequalities. For a zero ρ of a zeta or L-function, we shall often write $\rho = \beta + i\gamma$ where β and γ are real.

A consequence of our main result is that the conjecture holds for *CM* fields of degree $\geqslant 6$ whose Galois closure is solvable. We get effective lower bounds on the minus part of the class number. These bounds are not best possible, but sufficiently strong to deduce the finiteness theorem.

In [7], it was proved that for an extension L/K of *odd* degree and having solvable normal closure, zeros of the Dedekind zeta function $\zeta_L(s)$ sufficiently near $s = 1$ are also zeros of $\zeta_K(s)$. To state precisely what 'near' means here, we need some notation. Define the function

$$\gamma(r) \;=\; 12^{r-1}3^{1/3}. \tag{2}$$

Let $n = [L : K]$ and $e(n) = \max_{p^\alpha \| n} \alpha$. Then set

$$\delta(n) = (e(n)+1)^2\gamma(e(n)) \;= (e(n)+1)^2 3^{1/3} 12^{e(n)-1}.$$

In [7], Theorem 4.1, it was proved that there is a constant $c_1 > 0$, absolute and effective, such that for any $0 < c < c_1$, and n odd, a zero of $\zeta_L(s)$ in the rectangle

$$1 - \frac{c}{\delta(n)\log d_L} \;\leqslant\; \sigma \;\leqslant\; 1, \qquad |t| \;\leqslant\; \frac{c}{\delta(n)\log d_L}$$

is already a zero of $\zeta_K(s)$. The same constant c_1 will occur in the results of this article.

To get an idea of how much narrower the above rectangle is than the Stark region, note that since

$$e(n) \;\leqslant\; \frac{\log n}{\log 2}$$

we certainly have $\delta(n) \;\ll\; n^4$. For most n, $\delta(n)$ will be much smaller. For example, if n is squarefree,

$$\delta(n) \;\leqslant 4 \cdot 3^{1/3} \;\sim\; 5.7690.$$

Except for a set of n of density zero, $\delta(n) \;\ll\; (\log n)^3$.

In this paper, we study the effect of removing the condition that the degree of L/K be odd. We get a slightly weaker result due to the lack of control of the 2-Sylow subgroup of the normal closure. However, the bound we obtain is sufficient to deduce the finiteness result for class numbers of *CM*-fields.

In Section 2, we establish the main results on zeros. In Section 3, we apply these to Stark's conjecture on the class numbers of CM-fields. Finally, in Section 5, we make some remarks about higher order zeros.

2. Zeros of Dedekind Zeta Functions

THEOREM 2.1. *Let L/K be an extension of degree n whose Galois closure is solvable. Let $0 < c < c_1/2$. Suppose that $\zeta_L(s)$ has a zero β in the range*

$$1 - \frac{c}{n^{e(n)}\delta(n)\log d_L} \leqslant \sigma \leqslant 1, \quad |t| \leqslant \frac{c}{n^{e(n)}\delta(n)\log d_L}.$$

Then, there is a field N with $K \subseteq N \subseteq L$ and $[N:K] \leqslant 2$ with $\zeta_N(\beta) = 0$.

In some cases, we are able to get as good a result as in the case of odd degree. Two such cases are given below.

THEOREM 2.2 *Let L/K be an extension of degree n whose Galois closure is supersolvable. Suppose that $\zeta_L(s)$ has a zero β in the range*

$$1 - \frac{c}{\delta(n)\log d_L} \leqslant \sigma \leqslant 1, \quad |t| \leqslant \frac{c}{\delta(n)\log d_L}. \tag{3}$$

Then, there is a field $K \subseteq N \subseteq L$ with $[N:K] \leqslant 2$ and $\zeta_N(\beta) = 0$.

Remark. This result generalizes the result of Odlyzko–Skinner [11] to extensions whose normal closure is supersolvable. Their result states that if L/K is a radical extension of odd degree, then a Stark zero of $\zeta_L(s)$ is already a Stark zero of $\zeta_K(s)$. Note that we have also eliminated the parity restriction on the degree of the extensions which is imposed in [11].

THEOREM 2.3. *Let L/K be an extension of degree 2^m with solvable normal closure. Suppose that $\zeta_L(s)$ has a zero in the range (3). Then, there is a field $K \subseteq N \subseteq L$ with $[N:K] \leqslant 2$ and $\zeta_N(\beta) = 0$.*

Remark. In all of the above results, the zero β is necessarily real and simple because it lies in the Stark region (1).

Remark. All irreducible characters of a supersolvable group are monomial and so Artin's conjecture is known. Thus, Stark's theorem implies the above result with $\delta(n)$ replaced by n.

We note that the method of [7] actually proves the following result.

PROPOSITION 2.1. *Let $0 < c < c_1/2$. Let L/K be an extension of degree n such that the Galois closure is solvable and there are no intermediate extensions $K \subseteq M \subseteq L$.*

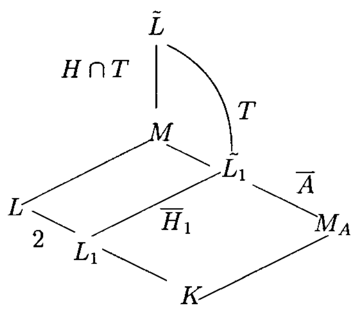
Suppose that $\zeta_L(s)$ has a zero β in the rectangle

$$1-\frac{c}{\delta(n)\log d_L}\leqslant\sigma\leqslant 1,\quad |t|\leqslant\frac{c}{\delta(n)\log d_L}.$$

Then, there is a field $K\subseteq N\subseteq L$ with $[N:K]\leqslant 2$ and $\zeta_N(\beta)=0$.

Proof of Theorem 2.1. We will proceed by induction on the degree n of L/K. Denote by $\tilde{L}$ the normal closure of L/K. Let H be the subgroup $\mathrm{Gal}(\tilde{L}/L)$. If H is maximal, then the result follows from Proposition 2.1. Thus, we may suppose that $H\subsetneq H_1$ and we may suppose that H_1 is a maximal subgroup of $G=\mathrm{Gal}(\tilde{L}/K)$. Denote by L_1 the fixed field of H_1. Let $m=[L:L_1]$. Then $m<n$ and we can apply the induction hypothesis to L/L_1. We get a field N_1 contained in L and of degree at most 2 over L_1 such that $\zeta_{N_1}(\beta)=0$. If $n_1=[N_1:K]<n$, we can apply the induction hypothesis again to the extension N_1/K. Then, we produce an extension N of K with $[N:K]\leqslant 2$ and $\zeta_N(\beta)=0$. Thus, we are done by the induction hypothesis except if $n_1=n$, i.e. $L=N_1$, so we now consider this case.

Denote by $\tilde{L}_1$ the normal closure of L_1/K. Note that L is not contained in $\tilde{L}_1$ (as $\mathrm{Gal}(\tilde{L}/\tilde{L}_1)$ is a normal subgroup of G and is therefore not contained in H). Thus the compositum $M=L\tilde{L}_1$ is of degree 2 over $\tilde{L}_1$. Now replacing L with its Galois conjugates, we see that the compositum of the fields $L^\sigma\tilde{L}_1$ is a $(2,2,\cdots,2)$ extension of $\tilde{L}_1$ containing L and which is Galois over K. Hence, it is equal to $\tilde{L}$. Thus, $\tilde{L}/\tilde{L}_1$ is an elementary Abelian 2-extension. Denote the Galois group of this extension by T. We see that $H\cap T$ is a subgroup of T of index 2 and M is its fixed field.



Consider $\overline{G}=G/T=\mathrm{Gal}(\tilde{L}_1/K)$. Then $\overline{H_1}\ =\ H_1/T$ is a maximal subgroup of $\overline{G}$. Let $\overline{A}$ be a minimal normal subgroup of $\overline{G}$. As G is solvable, so is $\overline{G}$ and so, $\overline{A}$ is elementary Abelian. If $\overline{A}=\{1\}$, then $\tilde{L}_1=L_1$ and $M=L$. In this case, $\zeta_M(s)$ has a simple zero at $s=\beta$. Let us show that this is always the case.

Thus, we may assume that $\overline{A}\neq\{1\}$. Then $\overline{A}\ \simeq\ (\mathbb{Z}/p)^r$ for some prime p. Then $\overline{G}\ =\ \overline{H_1}\cdot\overline{A}$ (semidirect product). Note that $n=2[L_1:K]=2|\overline{A}|=2p^r$.

Hence, lifting to G, we have $G=H_1A$ and $H_1\cap A\ =\ T$. Note that $H_1=HT$ and that $H\cap T$ is normal in H_1. Let us set $\overline{H}\ =\ H/H\cap T$. Then,

$$\overline{H}\ \simeq\ HT/T\ \simeq\ H_1/T\ \simeq\overline{H_1}.$$

https://doi.org/10.1023/A:1017589432526 Published online by Cambridge University Press

We proceed as in [7] and produce a simple zero of $\zeta_M(s)$. Let us briefly recall the argument. We have $\mathrm{Gal}(M/L) = \overline{H} \simeq \overline{H_1}$. Moreover,

$$\overline{H_1} \hookrightarrow \mathrm{Aut}(\overline{A}) \simeq \mathrm{GL}_r(\mathbb{Z}/p)$$

is an irreducible solvable subgroup of $\mathrm{GL}_r(\mathbb{Z}/p)$. Hence, by a result of Dixon [1], we know that there is a subnormal Abelian subgroup B (say) of $\overline{H}$ of index bounded by $\gamma(r)$. Setting Y to be the fixed field of B, we have the following diagram.

$$\begin{array}{rc} & \tilde{L} \\ H \cap T & \Big| \\ & M \\ B & \Big| \\ & Y \\ \leqslant \gamma(r) & \Big| \\ & L \end{array}$$

If $a_1, \cdots, a_r$ are elements of G which project in G/T to generators of $\overline{A}$, then

$$H \bigcap (\cap_i \, a_i H a_i^{-1}) \subseteq H \cap T.$$

To see this, let us set

$$H' = H_1 \bigcap (\cap_i \, a_i H_1 a_i^{-1}).$$

For any $a \in A$, if $x \in H_1 \cap aH_1a^{-1}$ then xT commutes with aT. Indeed, writing $x = aya^{-1}$ for some $y \in H_1$, we see that

$$xy^{-1} = aya^{-1}y^{-1} \in A \cap H_1 = T.$$

Hence, $xT = yT$. Thus, the image of H' in $\overline{H_1} = H_1/T$ commutes with $\overline{A}$. In fact, it is easily seen that

$$H'T/T = \{xT \in \overline{H_1} : xT \text{ commutes with } \overline{A}\}.$$

It follows that $\overline{H_1}$ normalizes $H'T/T$ and so $H'T/T$ is normal in $\overline{G} = \overline{H_1} \cdot \overline{A}$ and is contained in $\overline{H_1}$. But as $\tilde{L}_1$ is the normal closure of L_1/K, this forces $H'T/T = \{1\}$ (as $\overline{A} \neq \{1\}$). Thus, $H' \subseteq T$ and

$$H \bigcap (\cap_i a_i H a_i^{-1}) \subseteq H \cap T$$

as claimed.

Thus, M is the compositum of at most $r+1$ conjugates of L and so $\log d_M \leqslant |B|\gamma(r)(r+1)\log d_L$.

On the other hand, $|B| \log d_Y \leqslant \log d_M$ and so $\log d_Y \leqslant \gamma(r)(r+1)\log d_L$. Thus,

$$1 - \frac{c}{(r+1)\log d_Y} \leqslant 1 - \frac{c}{\delta(n)\log d_L}$$

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and so β is a simple zero of $\zeta_Y(s)$ as it lies in the region (1). Next, using the fact that M/Y is Abelian, the zero-free region of [8], Proposition (3.8), and the conductor bound of [7], Proposition (3.2), we deduce that β is a simple zero of $\zeta_M(s)$ also.

Suppose M/K is not Galois. Let $\sigma \in \mathrm{Gal}(\tilde{L}/K)$ be such that $M \neq M^\sigma$. As the zeta function of a conjugate field has the same zeros, we get a simple zero of $\zeta_{M^\sigma}(s)$ as well. Putting these together, we get a zero of the zeta function of the compositum extension MM^σ. Note that as M and M^σ are quadratic over $\tilde{L}_1$, we have $\log d_{MM^\sigma} \leqslant 4 \log d_M$. Thus, if

$$\beta \geqslant 1 - \frac{c}{\log d_M}$$

for a $0 < c < 1/16$, then it follows that

$$\beta \geqslant 1 - \frac{4c}{\log d_{MM^\sigma}}$$

and it is a simple zero of MM^σ. This forces the zeta function of the intersection $M \cap M^\sigma = \tilde{L}_1$ to have a simple zero at $s = \beta$ since

$$\frac{\zeta_{MM^\sigma}(s)\zeta_{\tilde{L}_1}(s)}{\zeta_M(s)\zeta_{M^\sigma}(s)}$$

is entire. Indeed, as MM^σ is a biquadratic extension of $\tilde{L}_1$, there is a quadratic extension M' (say) distinct from M and M^σ with $\tilde{L}_1 \subseteq M' \subseteq MM^\sigma$ and the above quotient of zeta functions is $\zeta_{M'}(s)/\zeta_{\tilde{L}_1}(s)$ and this is equal to the Hecke L-function of the extension $M'/\tilde{L}_1$.

But $\tilde{L}_1/L_1$ is Galois, and so there is a field $L_1 \subseteq \tilde{N}_1 \subseteq \tilde{L}_1$ with $[\tilde{N}_1 : L_1] \leqslant 2$ and $\zeta_{\tilde{N}_1}(\beta) = 0$. If $\tilde{N}_1 = L_1$, then β is a Stark zero of $\zeta_{L_1}(s)$ and we are in the maximal case. Thus the result of Proposition 2.1 implies the result. If $[\tilde{N}_1 : L_1] = 2$, then $\tilde{N}_1 \cap L = L_1$ and then, the compositum $L\tilde{N}_1$ would be a subfield of M whose zeta function vanishes to order $\geqslant 2$ at β. By the Aramata–Brauer theorem, $\zeta_{L\tilde{N}_1}(s)$ divides $\zeta_M(s)$ (that is, $\zeta_M(s)/\zeta_{L\tilde{N}_1}(s)$ is entire). Thus, $\mathrm{ord}_{s=\beta}\zeta_M(s) \geqslant 2$ contradicting the simplicity of the zero of $\zeta_M(s)$. Hence, we must have

$$\beta \leqslant 1 - \frac{1}{16 \log d_M}.$$

Now, we have as before, the inequality $\log d_M \leqslant |\overline{H}|(r+1)\log d_L$. Since

$$\overline{H} \simeq \overline{H_1} \hookrightarrow \mathrm{Aut}(\overline{A}) \simeq \mathrm{GL}_r(\mathbb{Z}/p)$$

it follows that

$$|\overline{H}| \leqslant p^{r^2} = |\overline{A}|^r \leqslant n^{e(n)}.$$

Hence,

$$\beta \leqslant 1 - \frac{1}{16n^{e(n)}(r+1)\log d_L}$$

contradicting the hypothesis of the theorem. This contradiction establishes that M/K is in fact Galois. But in this case, by Stark's theorem, there exists $K \subseteq N \subseteq M$ with $[N:K] \leqslant 2$ such that $\zeta_N(\beta) = 0$ and such that any subfield of M whose zeta function vanishes at $s = \beta$ must contain N. Thus, $K \subseteq N \subseteq L$ and we are done.

Proof of Theorem 2.2. We proceed exactly as in the proof of Theorem 2.1 up to the point where we produce a simple zero of $\zeta_M(s)$. The case $p = 2$ is included in Theorem 2.3 so we may restrict ourselves here to $p \neq 2$. As G is supersolvable, it contains a normal subgroup of odd order divisible by p whose projection to $\overline{G}$ contains $\overline{A}$. Call this subgroup S. Now T and S are two normal subgroups of G of coprime order. In particular, they are disjoint and this implies that they commute. In particular, S commutes with $H \cap T$ But this means that $H \cap T$ is normalized by $H_1 S = G$. This forces $H \cap T = 1$ and so M/K is Galois. Hence, by Stark's theorem ([12], Theorem 3) the simple zero β of $\zeta_M(s)$ comes from a zero of $\zeta_N(s)$ where $K \subseteq N \subseteq M$, $[N:K] \leqslant 2$. By the uniqueness of N, it follows that $N \subseteq L$.

Proof of Theorem 2.3. We proceed as in the proof of Theorem 2.1 to obtain a simple zero of $\zeta_M(s)$. (We use the same notation as in the proof of Theorem 2.1.) Now we observe that $H \cap T$ acts on A by conjugation, and this action factors through the quotient $\overline{A} = A/T$. Thus, we get a map $H \cap T \longrightarrow \mathrm{GL}_r(\mathbb{Z}/2)$. As $H \cap T$ is a 2-group, it must have a fixed vector. This means that $H \cap T$ has a normalizer which is strictly larger than H_1. But H_1 is maximal and so as above, this forces $H \cap T = 1$ and M/K is Galois. We are done as above.

3. Applications to Class Numbers

We begin by proving a slight strengthening of Theorem 2.1. We will be considering the class $\mathfrak{S}$ of number fields M which have solvable normal closure over $\mathbb{Q}$. We define a subclass $\mathfrak{S}_{D,R}$ of fields for which there is a chain $\mathbb{Q} = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$ such that for each $1 \leqslant i \leqslant t$, the extension M_i/M_{i-1} has solvable normal closure and

$$\max_i e([M_{i+1} : M_i]) \leqslant R.$$

and

$$\max_i [M_{i+1} : M_i] \leqslant D.$$

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PROPOSITION 3.1. *Let* $M \in \mathfrak{S}_{D,R}$. *Let* $0 < c < c_1/54$. *Suppose there is a real* β *in the range*

$$1 - \frac{c}{(2D)^{R+1}(R+1)^2\gamma(R)\log d_M} \leqslant \beta \leqslant 1, \tag{4}$$

such that $\zeta_M(\beta) = 0$. *Then there is a quadratic field* F *contained in* M *such that* $\zeta_F(\beta) = 0$. *If further,* M *is complex,* M_{t-1} *is real,* $[M : M_{t-1}] = 2$ *and* $\zeta_{M_{t-1}}(\beta) \neq 0$, *then* F *is complex.*

Proof. We proceed as in [12], Lemma 10, using Theorem 2.1. Let $K \subseteq L \subseteq M$ be fields with L/K and M/L having solvable normal closure. Suppose that $\zeta_M(\beta) = 0$ with β in the range (4). To prove the first part, it suffices to show that there is a field N contained in M such that $N = K$ or $[N : K] = 2$ and such that $\zeta_N(\beta) = 0$. This is because we can shorten the sequence by one field and in doing this, we do not increase D or R. Moreover,

$$\log d_N \leqslant \log d_M, \quad [N : \mathbb{Q}] \leqslant [M : \mathbb{Q}].$$

Hence, we can proceed inductively.

By Theorem 2.1, there is a field

$$L \subseteq P \subseteq M$$

with $[P : L] \leqslant 2$ and $\zeta_P(\beta) = 0$. If $P = L$, then we reapply Theorem 2.1 to get N. Hence, we may suppose that $[P : L] = 2$.

Let $\tilde{M}$ be the normal closure of M/L and $\tilde{L}$ the normal closure of L/K. Set $\tilde{P} = P\tilde{L}$. Consider the various conjugate fields $\{\tilde{P}^\sigma\}$ as σ ranges over embeddings $\sigma : \tilde{P} \longrightarrow \overline{\mathbb{Q}}$ which are the identity on K. Let P_1 be the compositum of the $\tilde{P}^\sigma$. Then P_1/K is Galois and it is a $(2, 2, \cdots, 2)$ extension of $\tilde{L}$. As $\mathrm{Gal}(\tilde{L}/K)$ is solvable, it follows that $\mathrm{Gal}(P_1/K)$ is solvable. Thus P/K has solvable normal closure and so by Theorem 2.1, the field N exists. Notice that because $[P : K] = 2[L : K]$, $e([P : K]) \leqslant r + 1$, where $r = e([L : K])$. Then, with $n = [L : K]$, $\delta(2n) \leqslant 27\delta(n)$ and

$$1 - \frac{27c}{(2n)^{e(n)+1}\delta(2n)\log d_P} \leqslant 1 - \frac{c}{(2D)^{R+1}(R+1)^2\gamma(R)\log d_P} \leqslant \beta \leqslant 1.$$

Since $27c < c_1/2$, Theorem 2.1 applies.

For the second part of the Proposition, consider the sequence of fields

$$\mathbb{Q} \subseteq F \subseteq FM_1 \subseteq FM_2 \subseteq \cdots \subseteq FM_{t-1} \subseteq FM_t = M.$$

Each extension has solvable normal closure over its predecessor. Hence, by the Theorem of Uchida [13] and Van der Waall [14] (see also [9]), β is also a zero of $\zeta_E(s)$, $E = FM_{t-1}$. But this means that $E \neq M_{t-1}$ as we are assuming that $\zeta_{M_{t-1}}(\beta) \neq 0$. Also, as we are assuming that $[M : M_{t-1}] = 2$, it follows that $E = M$. As we are assuming that M is complex and M_{t-1} is real, it follows that F is complex. This proves the Proposition.

PROPOSITION 3.2. *Let K be a totally real field of degree n with solvable normal closure over $\mathbb{Q}$. Let L be a totally complex quadratic extension of K. Let* $0 < c < c_1/54$. *Suppose β is a real zero of $\zeta_L(s)/\zeta_K(s)$ in the range*

$$1 - \frac{c}{n^{e(n)}\delta(n)\log d_L} \leqslant \beta \leqslant 1$$

where $n = [L : \mathbb{Q}]$. Then, there is a complex quadratic field F contained in L such that $\zeta_F(\beta) = 0$.

Remark. The zero β is necessarily simple as it lies in the Stark region (1) and so $\zeta_K(\beta) \neq 0$.

Proof. Let $\tilde{K}$ denote the normal closure of $K/\mathbb{Q}$ and set $M = \tilde{K}L$. It is easy to show as above that the normal closure of $L/\mathbb{Q}$ is solvable. By Theorem 2.1, there is a field N with $\mathbb{Q} \subseteq N \subseteq L$ and $[N : \mathbb{Q}] = 2$, $\zeta_N(\beta) = 0$. It only remains to check that it is complex. This is clear, for N is not contained in K. Thus $KN = L$ and as L is complex, so is N. (Note that N is not contained in K as $\zeta_K(\beta) \neq 0$.) □

Now using Proposition 3.1 and proceeding as in the proof of Stark [12], Theorem 1, we deduce the following.

THEOREM 3.1. *Let K be a field of degree n in $\mathfrak{S}$ and denote by ρ_K the residue at $s = 1$ of $\zeta_K(s)$. Then*

$$\rho_K > c\frac{1}{n^{e(n)}\delta(n)} d_K^{-1/n}$$

for an effective absolute constant $c > 0$.

For the next result, let K be a totally real field in $\mathfrak{S}$ and let L be a quadratic extension of K which is totally complex. Denote by $h(L)$ the class number of L.

THEOREM 3.2. *Let K be a totally real field of degree n in $\mathfrak{S}$ and let L be a quadratic extension of K which is totally complex. Writing $d_L = d_K^2 f$, we have*

$$h(L) > \frac{c}{n(2n)^{e(2n)}\delta(n)} c_8^n f^{\frac{1}{2}(1-\frac{1}{n})}$$

where $c > 0$ and $c_8 > 1$ are effective and absolute constants. For any $\varepsilon > 0$, there is an effective constant $c(\varepsilon) > 0$ depending only on ε, such that

$$h(L) \geqslant \frac{c(\varepsilon)^n}{n(2n)^{e(2n)}\delta(n)} d_K^{\frac{1}{2}-\frac{1}{n}-\varepsilon} f^{\frac{1}{2}-\frac{1}{2n}}.$$

Proof. Denote by β_0 the possible real zero of $\zeta_L(s)$ in the region

$$1 - \frac{c}{n^{e(n)}\delta(n)\log d_L} \leqslant \sigma \leqslant 1.$$

Write $d_L = d_K^2 f$. For any σ_1 with

$$1 + \frac{1}{4\log d_L} = \sigma_0 \leqslant \sigma_1 \leqslant 2$$

and

$$\beta_1 = \begin{cases} \beta_0 & \text{if it exists} \\ 1 - \frac{c}{n^{e(n)}\delta(n)\log d_L} & \text{otherwise} \end{cases}$$

we have, as in [12] and [10], §4,

$$h(L) \geqslant c_3^{-1}(1-\beta_1) f^{1/2} d_K^{\frac{1}{2}-\frac{1}{2}(\sigma_1 - 1)} (2\pi)^{-n} \zeta_K(\sigma_1)^{-1}. \tag{5}$$

By Theorem 2.1, it follows that for $0 < c < c_1/54$, $\zeta_L(s)$ has no real zeros in the region

$$\max\left(1 - \frac{c}{(2n)^{e(2n)}\delta(n)\log d_L}, 1 - \frac{c_2}{d_L^{1/2n}}\right) \leqslant \sigma \leqslant 1.$$

(Indeed, if $\zeta_L(s)$ has a real zero in the interval

$$1 - \frac{c}{(2n)^{e(2n)}\delta(n)\log d_L} \leqslant \sigma \leqslant 1$$

then Theorem 2.1 implies that it comes from a quadratic extension N (say) of $\mathbb{Q}$. Note that $L/\mathbb{Q}$ has solvable normal closure. But such a zero must lie in the interval

$$1 - \frac{c_2}{d_N^{1/2}} \leqslant \sigma \leqslant 1.$$

Now using the estimate $d_L \geqslant d_N^n$, the result follows.) In particular, it follows that

$$1 - \beta_1 > \frac{c_4^{-1}}{(2n)^{e(2n)}\delta(n)} \min\left(d_K^{-1/n} f^{-1/2n}, (\log d_L)^{-1}\right).$$

It is easy to deduce that

$$1 - \beta_1 > \frac{c_4^{-1}}{n(2n)^{e(2n)}\delta(n)} d_K^{-1/n} f^{-1/2n}.$$

Thus

$$h(L) \geqslant \frac{c_5^{-1}}{n(2n)^{e(2n)}\delta(n)} f^{(n-1)/2n} \frac{d_K^{\frac{1}{2}-\frac{1}{2}(\sigma_1-1)-\frac{1}{n}}}{\zeta_K(\sigma_1)(2\pi)^n}.$$

By Odlyzko [10], pp. 284–285, we deduce that for some effective, absolute constants

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$c_6, c_7 > 0$,

$$h(L) \geqslant \frac{c_6^{-1}}{n(2n)^{e(2n)}\delta(n)} f^{(n-1)/2n}(1+c_7)^n.$$

This proves the first lower bound. For the second, take $\sigma_1 = 1 + 2\varepsilon$ in (5) and proceed as in Stark [12], (32). This gives the second bound.

COROLLARY 3.1. *Given h, the set of (complex) CM fields L in $\mathfrak{S}$, and $[L^+ : \mathbb{Q}] > 2$ and having $h(L) = h$ is finite and can be determined effectively.*

Proof. Since $n = [L^+ : \mathbb{Q}] > 1$, the first bound of Theorem 3.2 and the condition $h(L) = h$ bound n and f. Here, we are using the fact that

$$n^{e(n)} \ll \exp\{c(\log n)^2\} = \mathbf{o}(c_8^n).$$

In particular, given L^+, there are only a finite number of possibilities for L. Since $[L^+ : \mathbb{Q}] > 2$, the second estimate of Theorem 3.2 bounds the discriminant of L^+. Thus the number of possibilities for L^+ are finite. All of this is effective.

Remark. It should be possible to get an effective estimate for the *number* of L with $h(L) = h$ as a function of h.

4. Examples

We note that if $[L : K] \leqslant 4$, then the Galois closure is necessarily solvable over K. We deduce from Corollary 3.1 the following.

PROPOSITION 4.1. *As L ranges over the set*

$$\{L \text{ a CM field}, [L : \mathbb{Q}] \leqslant 8, [L : \mathbb{Q}] \neq 4\}$$

we have $h(L) \longrightarrow \infty$ effectively.

Proof. If $[L : \mathbb{Q}] \geqslant 6$, this follows from Corollary 3.1. If $[L : \mathbb{Q}] = 2$, this follows from the work of Goldfeld [3] combined with that of Gross and Zagier [4].

5. Remarks on Higher-Order Zeros

We make some remarks on the generalization of Stark's theorem on simple zeros to zeros of higher order. Most of our remarks require the assumption of Artin's conjecture. However, we will see how in the solvable case, the full strength of the conjecture is not needed.

Let L/K be a Galois extension. For each zero ρ of $\zeta_L(s)$, we wish to associate a subfield L_ρ, normal over K such that

(1) $\mathrm{ord}_{s=\rho}\zeta_{L_\rho}(s) = \mathrm{ord}_{s=\rho}\zeta_L(s)$
(2) if $K \subseteq M \subseteq L$ and $\mathrm{ord}_{s=\rho}\zeta_M(s) = \mathrm{ord}_{s=\rho}\zeta_L(s)$, then $M \supseteq L_\rho$.

If we assume Artin's conjecture, the existence of L_ρ is immediate. Indeed, we have the factorization $\zeta_L(s) = \prod L(s, \chi)^{\chi(1)}$ where the product ranges over the irreducible characters of $\mathrm{Gal}(L/K)$. Consider the set $Z_\rho = \{\chi : L(\rho, \chi) = 0\}$. Define $H_\rho = \cap_{\chi \in Z_\rho} \ker \chi$. Then H_ρ is a normal subgroup of G and we let L_ρ denote its fixed field. It is easy to see that properties (1) and (2) above are satisfied.

We do not need the full strength of Artin's conjecture to establish the existence of L_ρ. Consider the following result which we have already used several times. Let M_1 and M_2 be two subfields of L with $L = M_1 M_2$ and $K = M_1 \cap M_2$. Suppose that L is contained in a field $\tilde{L}$ (say) which is Galois over K. Assume Artin's conjecture for $\tilde{L}/K$. Then the quotient

$$\frac{\zeta_L(s)\zeta_K(s)}{\zeta_{M_1}(s)\zeta_{M_2}(s)}$$

is entire, a fact we have already used several times, and which according to [12] is due to Brauer. In the case that L/K has solvable normal closure, the above suffices to establish the existence of L_ρ. This follows from the following.

PROPOSITION 5.1. *Let L/K have solvable normal closure. Let M_1 and M_2 be two subfields of L containing K, with the property that*

$$\mathrm{ord}_{s=\rho}\zeta_{M_1}(s) = \mathrm{ord}_{s=\rho}\zeta_{M_2}(s) = \mathrm{ord}_{s=\rho}\zeta_L(s).$$

Suppose that the quotient

$$\frac{\zeta_{M_1 M_2}(s)\zeta_{M_1 \cap M_2}(s)}{\zeta_{M_1}(s)\zeta_{M_2}(s)}$$

is entire. Then

$$\mathrm{ord}_{s=\rho}\zeta_{M_1 \cap M_2}(s) = \mathrm{ord}_{s=\rho}\zeta_L(s).$$

Proof. Let $N = M_1 \cap M_2$ and $M = M_1 M_2$. Then by hypothesis,

$$\frac{\zeta_M(s)\zeta_N(s)}{\zeta_{M_1}(s)\zeta_{M_2}(s)}$$

is entire. Moreover, by the result of Uchida [13] and Van der Waall [14], $\zeta_{M_1}(s)$ divides $\zeta_M(s)$ and $\zeta_M(s)$ divides $\zeta_L(s)$. (Here, and elsewhere in the paper, we say that a function $f(s)$ divides another function $g(s)$ if the quotient $g(s)/f(s)$ is entire.) Thus,

$$\mathrm{ord}_{s=\rho}\zeta_{M_1}(s) \leqslant \mathrm{ord}_{s=\rho}\zeta_M(s) \leqslant \mathrm{ord}_{s=\rho}\zeta_L(s)$$

and thus we have equality throughout. Hence,

$$\mathrm{ord}_{s=\rho}\zeta_N(s) \geqslant \mathrm{ord}_{s=\rho}\zeta_L(s).$$

The reverse inequality also holds (as ζ_N divides ζ_L). □

PROPOSITION 5.2. *Under the hypotheses of Proposition 5.1, there is a subfield L_ρ of L satisfying* (1) *and* (2). *Moreover, L_ρ is normal over K.*

Proof. By Proposition 5.1, the set of subfields M of L with

$$\mathrm{ord}_{s=\rho}\zeta_M(s) = \mathrm{ord}_{s=\rho}\zeta_L(s)$$

is closed under intersections, and thus has a minimal element. To see that L_ρ is normal over K, we can apply Proposition 5.1 to two conjugates of L_ρ.

The full strength of Artin's conjecture gives some information on the structure of $\mathrm{Gal}(L_\rho/K)$. □

PROPOSITION 5.3. *Assume Artin's conjecture for L/K and let $s = \rho$ be a zero of $\zeta_L(s)$ of order r. If L/K has solvable normal closure, then* $\mathrm{Gal}(L_\rho/K)$ *has a normal subgroup of index divisible only by primes $\leqslant r+1$. In general, the index is divisible only by primes $\leqslant (2r+1)$. Moreover, it has an Abelian normal subgroup of index bounded by a function of r alone.*

Proof. If we factor $\zeta_{L_\rho}(s) = \prod L(s,\chi)^{\chi(1)}$, then by Artin's conjecture, it follows that any χ for which $L(\rho,\chi) = 0$ has $\chi(1) \leqslant r$. Now the assertion follows from results of Ito (see Isaacs [6], Corollary (14.6)) and Feit–Thompson ([6], p. 245). The last statement is due to Mal'cev (see Dixon).

PROPOSITION 5.4. *If in the above situation,* $\mathrm{Gal}(L_\rho/K)$ *has an irreducible representation of degree r, then this representation is also faithful.*

Proof. Such a representation would be the unique one whose L-function vanishes at $s = \rho$ and it would be a simple zero. If this character were not faithful then it would contradict the minimality of L_ρ.

Remark. Note that we need only assume Artin's conjecture at the point $s = \rho$.

Next, one can ask whether every subfield of L which is normal over K is of the form L_ρ for some zero ρ of $\zeta_L(s)$. Let M be such a subfield. The question then, is whether there exists ρ such that

(1) $\mathrm{ord}_{s=\rho}\zeta_M(s) = \mathrm{ord}_{s=\rho}\zeta_L(s)$
(2) $\mathrm{ord}_{s=\rho}\zeta_N(s) < \mathrm{ord}_{s=\rho}\zeta_L(s)$ for any proper subfield $K \subseteq N \subset M$ with N/K normal.

If we write

$$\zeta_L(s) = \zeta_M(s)\cdot\frac{\zeta_L(s)}{\zeta_M(s)},$$

condition (1) means that $\zeta_L(s)/\zeta_M(s) \neq 0$. As for (2), this means that

$$\frac{\zeta_M(\rho)}{\zeta_N(\rho)} = \frac{\zeta_L}{\zeta_N}(\rho)\left(\frac{\zeta_L}{\zeta_M}(\rho)\right)^{-1} = 0.$$

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Now, M/K has a faithful irreducible character χ (say). If we take ρ to be a zero of $L(s,\chi)$ then $\zeta_M/\zeta_N(\rho) = 0$ for any N as above. The question then is whether there exists such a zero which is not a zero of ζ_L/ζ_M. We do not know the answer to this question.

Acknowledgements

This paper was written while the author was visiting the Institute of Mathematical Sciences, Chennai. He would like to thank the Director, Prof. R. Ramachandran for the kind invitation to visit and Prof. R. Balasubramanian for several useful discussions. He would also like to thank the referee for a careful reading of the paper. Research partially supported by a grant from NSERC.

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