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ROOTS OF DEHN TWISTS ABOUT SEPARATING CURVES

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Abstract

Let C be a curve in a closed orientable surface F of genus $g \geq 2$ that separates F into subsurfaces $\widetilde{F}_i$ of genera g_i, for $i = 1, 2$. We study the set of roots in Mod(F) of the Dehn twist t_C about C. All roots arise from pairs of C_{n_i}-actions on the $\widetilde{F}_i$, where $n = \text{lcm}(n_1, n_2)$ is the degree of the root, that satisfy a certain compatibility condition. The C_{n_i}-actions are of a kind that we call nestled actions, and we classify them using tuples that we call data sets. The compatibility condition can be expressed by a simple formula, allowing a classification of all roots of t_C by compatible pairs of data sets. We use these data set pairs to classify all roots for $g = 2$ and $g = 3$. We show that there is always a root of degree at least $2g^2 + 2g$, while $n \leq 4g^2 + 2g$. We also give some additional applications.

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1. Introduction

Let F be a closed orientable surface of genus $g \geq 2$ and C be a simple closed curve in F. Let t_C denote a left-handed Dehn twist about C.

When C is a nonseparating curve, the existence of roots of t_C is not so apparent. In their paper [5], Margalit and Schleimer showed the existence of such roots by finding elegant examples of roots of t_C whose degree is $2g + 1$ on a surface of genus $g + 1$. This motivated an earlier collaborative work with McCullough [6] in which we derived necessary and sufficient conditions for the existence of a root of degree n. As immediate applications of the main theorem in the paper, we showed that roots of even degree cannot exist and that $n \leq 2g + 1$. The latter shows that the Margalit–Schleimer roots achieve the maximum value of n among all the roots for a given genus.

Suppose that C is a curve that separates F into subsurfaces $\widetilde{F}_i$ of genera g_i for $i = 1, 2$, where $g_1 \geq g_2$. (For convenience, we will denote this by $F = F_1 \#_C F_2$, where the F_i are the closed surfaces of genus g_i obtained by coning the $\widetilde{F}_i$.) It is evident that roots of t_C exist. As a simple example, we can obtain a square root of t_C by rotating one of the subsurfaces $\widetilde{F}_i$ on either side of C by an angle π, producing a half-twist near C. As in the case for nonseparating curves, a natural question is whether we can give necessary and sufficient conditions for the existence of a root of t_C of degree n. In this paper, we derive such conditions and apply them to obtain information about the possible degrees.

We will use a special class of C_n-actions. A nestled (n, ℓ)-action is defined to be an orientation-preserving C_n-action on an oriented surface F for which the points fixed by at least one nontrivial element of C_n form $\ell + 1$ orbits, one of which is a distinguished point fixed by all elements. In terms of the quotient orbifold, there are $\ell + 1$ cone points, one of which is a distinguished cone point of order n. Nestled (n, ℓ)-actions are called equivalent if they are conjugate by a homeomorphism taking the distinguished fixed point of one to that of the other. The term *nestled* is motivated by the fact that in our context, these actions appear as portions of larger actions, nestled, so to speak, inside them. The equivalency of two such actions will be given by the existence of a conjugating homeomorphism that also satisfies an additional condition on their distinguished fixed points.

Two equivalence classes of actions will form a *compatible pair* if the turning angles of their representative actions around their distinguished fixed points add up to $2\pi/n$. The key topological idea in our theory is defining nestled (n_i, ℓ_i)-actions on the subsurfaces $\tilde{F}_i$ for $i = 1, 2$ so that they form a compatible pair, thus giving a root of degree $n = \mathrm{lcm}(n_1, n_2)$. Conversely, for each root of degree n, we reverse this argument to produce a corresponding compatible pair.

THEREOM 3.4. *Let $F = F_1\#_C F_2$ be a closed oriented surface of genus $g \geq 2$. Then the conjugacy classes in* $\mathrm{Mod}(F)$ *of roots of t_C of degree n correspond to the compatible pairs* $([h_1], [h_2])$ *of equivalence classes of nestled (n_i, ℓ_i)-actions h_i on F_i of degree n.*

In Section 4, we introduce the abstract notion of a *data set* of degree n. As in the case of nonseparating curves [6], a *data set of degree n* is basically a tuple that encodes the essential algebraic information required to describe a nestled action. We show that equivalence classes of nestled (n, ℓ)-actions actually correspond to data sets, that is, each class has a corresponding data set representation. We use Thurston's orbifold theory [10, Ch. 13] to prove this result. A good reference for this theory is Scott [9]. Data sets D_i of degree n_i, for $i = 1, 2$, form a *data set pair* (D_1, D_2) when they satisfy the formula $(n/n_1)k_1 + (n/n_2)k_2 \equiv 1 \bmod n$, where the turning angles at the centers of the disks are $(2\pi k_i/n_i) \bmod 2\pi$ and $n = \mathrm{lcm}(n_1, n_2)$. In Theorem 5.2, we show that this number-theoretic condition is an algebraic equivalent of the compatibility condition for actions, thus proving that data set pairs correspond bijectively to conjugacy classes of roots. Theorem 5.2 is essentially a translation of our topological theory of roots to the algebraic language of data sets. An immediate application of Theorem 5.2 is the following corollary.

COROLLARY 5.3. *Suppose that $F = F_1\#_C F_2$. Then there always exists a root of the Dehn twist t_C about C of degree* $\mathrm{lcm}(4g_1, 4g_2 + 2)$.

In Section 6, we classify the roots for the closed orientable surfaces of genus 2 and 3. In Section 7, we obtain some bounds on the orders of spherical nestled

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actions, that is, nestled actions whose quotient orbifolds are topologically spheres. For example, we prove that all nestled (n, ℓ)-actions for $n \geq \frac{2}{3}(2g-1)$ have to be spherical. In Section 8, we use the main theorem and the results obtained in Section 7 to obtain the following upper bound on n.

THEOREM 8.6. *Let $F = F_1 \#_C F_2$ be a closed oriented surface of genus $g \geq 2$. Suppose that n denotes the degree of a root of the Dehn twist t_C about C. Then $n \leq 4g^2 + 2g$.*

We show in Proposition 8.9 that if we have a nestled (n, ℓ)-action on F of odd order, then $n \leq 3g + 3$. Using this result, we refine the upper bound derived in Theorem 8.6 to obtain a sharper upper bound for n for $g \geq 10$. Though Theorem 8.6 gives a better upper bound for n for $g \leq 13$, the bound in Theorem 8.14 seems to provide a considerable improvement for $g \geq 14$.

THEOREM 8.14. *Let $F = F_1 \#_C F_2$ be a closed oriented surface of genus $g \geq 10$. Suppose that n denotes the degree of a root of the Dehn twist t_C about C. Then $n \leq \frac{16}{5}g^2 + 12g + \frac{45}{4}$.*

For $g \geq 14$, in Table 2, we provide calculations which indicate the degree of improvement of this estimate.

2. Nestled (n, ℓ)-actions

We introduce nestled (n, ℓ)-actions in this section and give an example of such an action. We know that an action of a group G on a topological space X is defined as a homomorphism $h : G \to \text{Homeo}(X)$. Since we are interested only in C_n-actions on F, we will fix a generator t for C_n and identify the finite order homeomorphism $h(t) \in \text{Homeo}(F)$ as the generating homeomorphism of the action. For the sake of notational convenience, throughout this section and later, we will use h to also denote the generating homeomorphism $h(t)$ of the nestled action. As mentioned earlier, nestled actions will play a crucial role in the theory we will develop for roots of Dehn twists.

DEFINITION 2.1. An orientation-preserving C_n-action on a surface F of genus at least 1 is said to be a *nestled (n, ℓ)-action* if either $n = 1$, or $n > 1$ and:

(i) the action has at least one fixed point;
(ii) the points fixed by some nontrivial element of C_n form $\ell + 1$ orbits, one of which is a distinguished point fixed by all elements.

This is equivalent to the condition that the quotient orbifold of the action has $\ell + 1$ cone points, one of which is a distinguished cone point of order n.

A nestled (n, ℓ)-action is said to be *trivial* if $n = 1$, that is, if it is the action of the trivial group on F. In this case only, we allow a cone point of order one in the quotient orbifold. The distinguished cone point can then be any point in F, and we require $\ell = 0$.

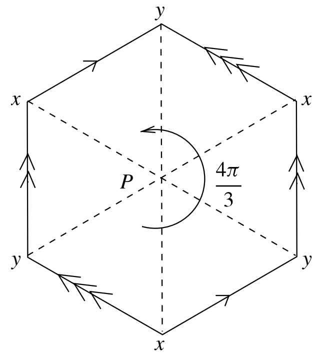

FIGURE 1. A nestled $(2g + 1, 2)$-action for $g = 1$.

DEFINITION 2.2. Assume that F has a fixed orientation and fixed Riemannian metric. Let h be a nestled (n, ℓ)-action on F with a distinguished fixed point P. The *turning angle* $\theta(h)$ for h is the angle of rotation of the induced isomorphism h_* on the tangent space T_P, in the direction of the chosen orientation.

EXAMPLE 2.3 (Margalit and Schleimer [5]). Rotate a regular $(4g + 2)$-gon with opposite sides identified about its center P through an angle $2\pi(g + 1)/2g + 1$ as shown in Figure 1. Identifying the opposite sides of the polygon, we get a C_{2g+1}-action h on the closed orientable surface S_g of genus g with three fixed points denoted by P, x and y. Since the quotient orbifold has three cone points of order $2g + 1$, this defines a nestled $(2g + 1, 2)$-action on S_g. If we choose P as the distinguished fixed point for the action h, then $\theta(h) = 2\pi(g + 1)/2g + 1$.

REMARK 2.4. Every nestled (n, ℓ)-action has an invariant disk around its distinguished fixed point. Indeed, let F be a closed oriented surface with a fixed Riemannian metric ρ, and let h be a nestled (n, ℓ)-action on F with a distinguished fixed point P. Consider the Riemannian metric $\bar{\rho}$ defined by

$$\langle v, w\rangle_{\bar{\rho}} = \frac{1}{n}\sum_{i=1}^{n} \langle {h^i}_*(v), {h^i}_*(w)\rangle_{\rho},$$

where $v, w \in T_P F$. Under this metric $\bar{\rho}$, h is an isometry. Since there exists $\epsilon > 0$ such that $\exp_P : B_\epsilon(0) \subset T_P F \to B_\epsilon(P) \subset F$ is a diffeomorphism, h preserves the disk $B_\epsilon(P)$.

DEFINITION 2.5. Two nestled (n, ℓ)-actions h and h' on F with distinguished fixed points P and P' are *equivalent* if there exists an orientation-preserving homeomorphism $t: F \to F$ such that:

(i) $t(P) = P'$;
(ii) tht^{-1} is isotopic to h' relative to P'.

REMARK 2.6. By definition, equivalent nestled (n, ℓ)-actions h and h' on F are conjugate in $\mathrm{Mod}(F)$. Since conjugate homeomorphisms have the same fixed point data, we have that $\theta(h) = \theta(h')$.

3. Compatible pairs and roots

Suppose that C is a curve that separates a surface F of genus g into two subsurfaces. As mentioned earlier, the central idea is defining compatible nestled actions on the subsurfaces that fit together to give a degree n root of the Dehn twist t_C. We will show in Theorem 3.4 that compatible pairs of equivalent actions correspond bijectively to conjugacy classes of roots of t_C.

NOTATION 3.1. Suppose that C separates a closed orientable surface F into subsurfaces of genera g_1 and g_2, where $g_1 \geq g_2$. Let F_i denote the closed surface obtained by coning the subsurface of genus g_i. We will think of F as $(F_1, C)\#(F_2, C)$, that is, the surface obtained by taking the connected sum of the F_i along C. For convenience, we will denote this by $F = F_1 \#_C F_2$.

DEFINITION 3.2. Equivalence classes $[h_i]$ of nestled (n_i, ℓ_i)-actions h_i on closed oriented surfaces F_i, for $i = 1, 2$, are said to form a *compatible pair* $([h_1], [h_2])$ if $\theta(h_1) + \theta(h_2) = 2\pi/n \bmod 2\pi$.

The integer $n = \mathrm{lcm}(n_1, n_2)$ is called the *degree* of the compatible pair. We may treat $([h_1], [h_2])$ as an unordered pair, since $([h_2], [h_1])$ is a compatible pair if and only if $([h_1], [h_2])$ is.

LEMMA 3.3. *Let F be a compact orientable surface, possibly disconnected. If $h : F \to F$ is a homeomorphism such that h^n is isotopic to id_F, then h is isotopic to a homeomorphism j with $j^n = id_F$.*

PROOF. When F is connected, this is Nielsen's theorem [7]. Suppose that F is not connected. We may assume that h acts transitively on the set of components $F_1, F_2, \ldots, F_\ell$ of F. Choose notation so that $h\,|_{F_i} : F_i \to F_{i+1}$ and $h\,|_{F_{\ell-1}} : F_{\ell-1} \to F_1$. Since $h^n = (h^l)^{n/\ell} \simeq id_F$, Nielsen's theorem implies that $h^\ell\,|_{F_1} \simeq j_1$ where j_1 is a homeomorphism on F_1 with ${j_1}^{n/\ell} = id_{F_1}$. Therefore, $id_{F_1} \simeq j_1 \circ (h^\ell\,|_{F_1})^{-1}$ via an isotopy K_t. Define an isotopy H_t of h by $H_t\,|_{F_i} = h$ for $1 \leq i \leq \ell - 2$ and $H_t\,|_{F_{\ell-1}} = K_t \circ h\,|_{F_{\ell-1}}$. Then $H_1\,|_{F_{\ell-1}} = K_1 \circ h = j_1 \circ (h^\ell\,|_{F_1})^{-1} \circ h$. We see that

$$(H_1\,|_{F_i})^\ell = h^i \circ (j_1 \circ h^{1-\ell}) \circ h^{\ell-1-i} = h^i \circ j_1 \circ h^{-i}$$

and

$$(H_1\,|_{F_i})^n = (H_1\,|_{F_i}{}^\ell)^{n/\ell} = h^i \circ {j_1}^{n/\ell} \circ h^{-i} = h^i \circ h^{-i} = id_{F_i}.$$

The required homeomorphism is $j = H_1$. □

THEOREM 3.4. *Let $F = F_1 \#_C F_2$ be a closed oriented surface of genus $g \geq 2$. Then the conjugacy classes in $\mathrm{Mod}(F)$ of roots of t_C of degree n correspond to the compatible pairs $([h_1], [h_2])$ of equivalence classes of nestled (n_i, ℓ_i)-actions h_i on F_i of degree n.*

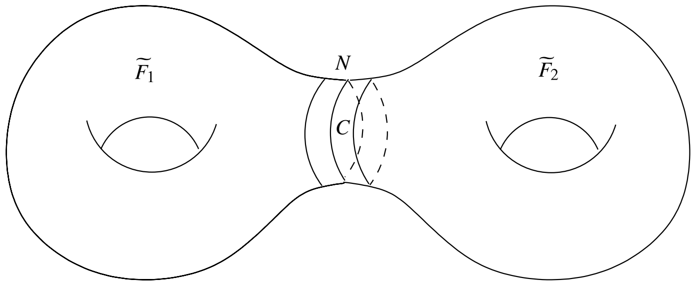

FIGURE 2. The surface F with the separating curve C and the tubular neighborhood N of C.

PROOF. We will first prove that every root of degree n yields a compatible pair of $([h_1], [h_2])$ of degree n.

Fix a closed annulus neighborhood N of C. Let $\widetilde{F}_i$, for $i = 1, 2$, be the components of $\overline{G - N}$, and denote the genus of $\widetilde{F}_i$ by g_i . We fix coordinates on F so that the subsurface $\widetilde{F}_1$ is to the left of C as shown in Figure 2. By isotopy we may assume that $t_C(C) = C$, $t_C(N) = N$, and $t_C|_{\widetilde{F}_i} = id_{\widetilde{F}_i}$ for $i = 1, 2$.

Suppose that h is an nth root of t_C. We have $t_C \simeq ht_Ch^{-1} \simeq t_{h(C)}$, which implies that $h(C)$ is isotopic to C. Changing h by isotopy, we may assume that h preserves C and takes N to N. Put $\widetilde{h_i} = h|_{\widetilde{F}_i}$ for $i = 1, 2$. Since $h^n \simeq t_C$ and both preserve C, there is an isotopy from h^n to t_C preserving C and hence one taking N to N at each time. That is, $\widetilde{h_1}^n$ is isotopic to $id_{\widetilde{F}_1}$ and $\widetilde{h_2}^n$ is isotopic to $id_{\widetilde{F}_2}$. By Lemma 3.3, $\widetilde{h_i}$ is isotopic to a homeomorphism whose nth power is $id_{\widetilde{F}_i}$ for $i = 1, 2$. So we may change $\widetilde{h_i}$ and hence h by isotopy to assume that $\widetilde{h_i}^n = id_{\widetilde{F}_i}$ for $i = 1, 2$.

Let n_i be the smallest positive integer such that $\widetilde{h_i}^{n_i} = id_{\widetilde{F}_i}$ for $i = 1, 2$. Let $s = \mathrm{lcm}(n_1, n_2)$. Clearly, $s \mid n$ since $n_i \mid n$. Also, $h^s = id_{\widetilde{F}_1 \cup \widetilde{F}_2}$ which implies that $h^s = {t_C}^d$ for some integer d. Hence, $(h^s)^{n/s} = ({t_C}^d)^{n/s}$, that is, $h^n = {t_C}^{dn/s}$. We get $t_C = {t_C}^{dn/s}$, which implies that $dn/s = 1$ since no higher power of t_C is isotopic to t_C. Hence, $d = 1$ and $n = s = \mathrm{lcm}(n_1, n_2)$.

Assume for now that h does not interchange the sides of C. We fill in the boundary circles of $\widetilde{F}_1$ and $\widetilde{F}_2$ with disks to obtain the closed orientable surfaces F_1 and F_2 with genera g_1 and g_2 . We then extend $\widetilde{h_i}$ to a homeomorphism h_i on F_i by coning. Thus h_i defines a C_{n_i}-action on F_i where $n_i \mid n$, $C_{n_i} = \langle h_i \mid h_i^{n_i} = 1\rangle$ for $i = 1, 2$, and $\mathrm{lcm}(n_1, n_2) = n$. Since the homeomorphism h_i fixes the center point P_i of the disk $\overline{F_i - \widetilde{F}_i}$, we choose P_i as the distinguished fixed point for h_i. So h_i defines a nestled (n_i, ℓ_i)-action on F_i for some ℓ_i.

The orientation on F restricts to orientations on the F_i, so that we may speak of rotation angles $\theta(h_i)$ for h_i. Then the rotation angle $\theta(h_i) = 2\pi k_i/n_i$ for some k_i with $\gcd(k_i, n_i) = 1$. As seen in Figure 3, the difference in turning angles

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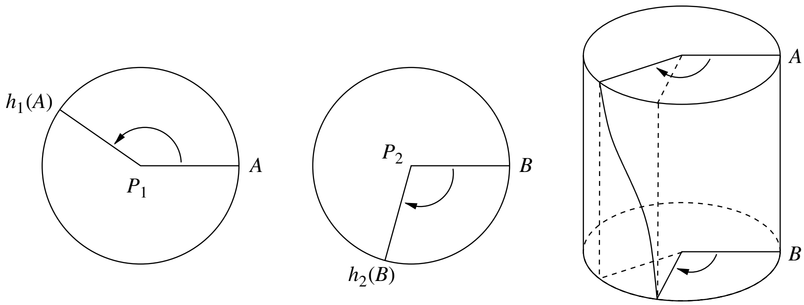

FIGURE 3. The local effect of h_1 and h_2 on disk neighborhoods of P_1 and P_2 in F_1 and F_2, and the effect of h on the neighborhood N of C in F. Only the boundaries of the disk neighborhoods are contained in $\widetilde{F}_i$, where they form the boundary of N. The rotation angle $\theta(h_1)$ is $2\pi k_1/n_1$ and the angle $\theta(h_2)$ is $2\pi k_2/n_2 = 2\pi(1/n - k_1/n_1)$.

equals $2\pi k_2/n_2 - (-2\pi k_1/n_1) = 2\pi/n$, giving $\theta(h_1) + \theta(h_2) \equiv 2\pi/n \bmod 2\pi$. That is, (h_1, h_2) is a compatible pair.

Suppose now that h interchanges the sides of C. Then h must be of even order, say $2n$, and h^2 preserves the sides of C and is of order n. Since the actions of $h^2|_{\widetilde{F}_i}$ on the $\widetilde{F}_i$ are conjugate by $h|_{\widetilde{F_1}\cup\widetilde{F_2}}$, these actions will induce conjugate C_n-actions on the coned surfaces F_i. Consequently, these induced actions will have the same turning angles at the centers P_i of the coned disks of F_i. For this compatible pair of nestled (n_i, ℓ_i)-actions, say (h_1, h_2), associated with h^2, we must have $\theta(h_1) = \theta(h_2) = \pi/n$ and $n_1 = n_2 = n$. If we extend to N using a simple left-handed twist, the twisting angle is $2\pi k/n$, and consequently $h^{2n} = t_C^{2k}$. Other extensions will differ from this by full twists, giving $h^{2n} = t_C^{2k+2jn}$ for some integer j. In any case, h^{2n} cannot equal t_C. This proves that h cannot reverse the sides of C.

Suppose that we have roots h and h' that are conjugate in $\mathrm{Mod}(F)$, that is, there exists $t \in \mathrm{Mod}(F)$ such that $h' = t \circ h \circ t^{-1}$. Then $(h')^n = t \circ h^n \circ t^{-1}$, that is, $t_C = t \circ t_C \circ t^{-1} = t_{t(C)}$. This shows that C and $t(C)$ are isotopic curves. Changing t by isotopy, we may assume that $t(C) = C$ and $t(N) = N$. Let t_i, h_i and h'_i respectively denote the extensions of $t|_{\widetilde{F}_i}$, $h|_{\widetilde{F}_i}$ and $h'|_{\widetilde{F}_i}$ to F_i by coning.

Assume for now that t does not exchange the sides of C. Since t, h and h' all preserve N, we may assume that the isotopy from $t \circ h \circ t^{-1}$ to h' preserves N, and consequently each $t_i \circ h_i \circ t_i^{-1}$ is isotopic to h'_i preserving P_i. Since t_i takes P_i to P_i, h_i and h'_i are equivalent as nestled (n_i, ℓ_i)-actions on F_i, so h and h' produce the same compatible pair $([h_1], [h_2])$.

Suppose that t exchanges the sides of C. Then $g_1 = g_2$, $h'_{3-i} \simeq t_i \circ h_i \circ t_i^{-1}$ and $t_i(P_i) = P_{3-i}$. So the actions h_1 and h'_2 are equivalent, as are the actions h'_1 and h_2. Therefore, the (unordered) compatible pairs for the two roots are the same.

Conversely, given a compatible pair $([h_1], [h_2])$ of equivalence classes of nestled (n_i, ℓ_i)-actions, we can reverse the argument to produce a root h. For let P_i denote the distinguished fixed point of h_i and let p_i denote the corresponding cone point of

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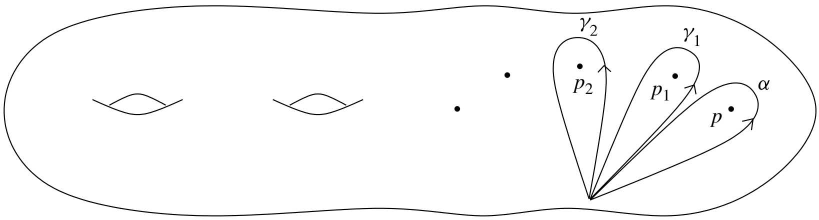

FIGURE 4. The orbifold O.

order n_i in the quotient orbifold O_i. By Remark 2.4, there exists an invariant disk D_i for h_i around p_i. Removing D_i produces the surfaces $\widetilde{F}_i$, and attaching an annulus N produces the surface F of genus g. Condition (ii) on compatible pairs ensures that the rotation angles work correctly to allow an extension of $h_1|_{\widetilde{F}_1} \cup h_2|_{\widetilde{F}_2}$ to an h with h^n being a single Dehn twist about C.

It remains to show that the resulting root h of t_C is determined up to conjugacy in the mapping class group of F. Suppose that $h_i' \in [h_i]$. Let P_i' denote the distinguished fixed point for h_i', and let D_i' be an invariant disk for h_i' around P_i'. Removing the D_i' produces surfaces $\widetilde{F_i'} \cong F_i$, for $i = 1, 2$, and attaching an annulus N' with a $(1/n)$th twist, extends $h_1'|_{\widetilde{F_1'}} \cup h_2'|_{\widetilde{F_2'}}$ to a homeomorphism h' on a surface $F' \cong F$ of genus g. Since $h_i' \in [h_i]$, by definition, there exists t_i such that $t_i(P_i) = P_i'$ and $t_i \circ h_i \circ t_i^{-1} \simeq h_i'$ rel P_i' via an isotopy H_i in $\mathrm{Mod}(F_i')$. Since h_i and h_i' have finite order and are conjugate up to isotopy by t_i, we may assume that $t_i(D_i) = D_i'$ and, identifying F and F' using t, that the isotopy H_i from $t_i \circ h_i \circ t_i^{-1}$ to h_i' is relative to D_i. With respect to this identification, we choose a $k : N \to N$ such that $h'|_N = k \circ h|_N \circ k^{-1}$. Now define $t : F \to F$ by $t|_{\widetilde{F}_i} = h_i|_{\widetilde{F}_i}$, and $t|_N = k$. Then $h' \simeq t \circ h \circ t^{-1}$ via an isotopy H given by $H|_{\widetilde{F}_i} = H_i|_{\widetilde{F}_i}$, and $H|_N = id_N$. □

4. Nestled (n, ℓ)-actions and data sets

In this section, we will introduce the language of data sets of degree n in order to algebraically encode equivalence classes of nestled (n, ℓ)-actions. We will prove that the equivalence classes of nestled (n, ℓ)-actions actually correspond to data sets of length ℓ.

NOTATION 4.1. For a nestled (n, ℓ)-action h on a closed orientable surface F of genus g, we will use the following notation throughout this section. Let O be the quotient orbifold for the action and let $\widetilde{g}$ be the genus of its underlying 2-manifold. Let P be the distinguished fixed point of h and let p be the cone point in O of order n that is its image in O. Let $p_1, \ldots, p_\ell$ be the other possible cone points of O, if any.

Figure 4 shows a generator α of the orbifold fundamental group $\pi_1^{\mathrm{orb}}(O)$ that goes around the point p, and generators γ_i, $1 \le i \le \ell$ going around p_i. Let a_j and b_j, $1 \le j \le \widetilde{g}$, be standard generators of the underlying surface of O, chosen to give the

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following presentation of $\pi_1^{\text{orb}}(\mathcal{O})$:

$$\pi_1^{\text{orb}}(\mathcal{O}) = \Big\langle \alpha, \gamma_1, \ldots, \gamma_\ell, a_1, b_1, \ldots, a_{\widetilde{g}}, b_{\widetilde{g}} \;\Big|\; \alpha^n = \gamma_1^{x_1} = \cdots = \gamma_\ell^{x_\ell} = 1, \alpha\gamma_1 \cdots \gamma_\ell = \prod_{i=1}^{\widetilde{g}} [a_i, b_i] \Big\rangle.$$

With this notation, we develop a set of numerical parameters in order to classify nestled (n, ℓ)-actions.

Remark 4.2. From orbifold covering space theory [10], we have the exact sequence

$$1 \longrightarrow \pi_1(F) \longrightarrow \pi_1^{\text{orb}}(\mathcal{O}) \xrightarrow{\rho} C_n \longrightarrow 1.$$

The homomorphism ρ is obtained by lifting path representatives of elements of $\pi_1^{\text{orb}}(\mathcal{O})$—these do not pass through the cone points so the lifts are uniquely determined.

For $1 \le i \le \ell$, the preimage of p_i consists of n/x_i points cyclically permuted by h, where x_i is the order of the stabilizer of each point in the preimage of p_i. Each of the points has stabilizer generated by h^{n/x_i}. Its rotation angles must be the same at all points of the orbit, since its action at one point is conjugate by a power of h to its action at each other point. So the rotation angle at each point is of the form $2\pi c_i'/x_i$, where c_i' is a residue class modulo x_i and $\gcd(c_i', x_i) = 1$. Lifting the γ_i, we have that $\rho_1(\gamma_i) = h^{(n/x_i)c_i}$ where $c_i c_i' \equiv 1 \bmod x_i$.

Since C_n is abelian, we have that $\rho(\prod_{i=1}^{\widetilde{g}}[a_i, b_i]) = 1$, so

$$1 = \rho_i(\alpha\gamma_1 \cdots \gamma_\ell) = t^{a+(n/x_1)c_1+\cdots+(n/x_i)c_i},$$

giving

$$a + \sum_{i=1}^{\ell} \frac{n}{x_i} c_i \equiv 0 \mod n.$$

Thus, we obtain a collection of numerical parameters $D = (n, \widetilde{g}, a; (c_1, x_1), \ldots, (c_\ell, x_\ell))$ satisfying certain number-theoretic conditions.

We call the collection of numerical parameters obtained in Remark 4.2 a *data set*, which we formalize in the following definition.

Definition 4.3. A *data set* is a tuple $D = (n, \widetilde{g}, a; (c_1, x_1), \ldots, (c_\ell, x_\ell))$ where $n, \widetilde{g}$ and the x_i are integers, a is a residue class modulo n, and each c_i is a residue class modulo x_i, such that:

(i) $n \ge 1, \widetilde{g} \ge 0$, each $x_i > 1$, and each x_i divides n;
(ii) $\gcd(a, n) = \gcd(c_i, x_i) = 1$;
(iii) $a + \sum_{i=1}^{\ell}(n/x_i)c_i \equiv 0 \bmod n$.

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The number n is called the *degree* of the data set and the number ℓ is called the length of the data set. If $n = 1$, then we require that $a = 1$, and the data set is $D = (1, \widetilde{g}, 1;)$. The integer g defined by

$$g = \widetilde{g}n + \frac{1}{2}(1 - n) + \frac{1}{2}\sum_{i=1}^{\ell}\frac{n}{x_i}(x_i - 1),$$

is called the *genus* of the data set. We consider two data sets to be the same if they differ by reordering the pairs $(c_1, x_1), \ldots, (c_\ell, x_\ell)$.

Remark 4.4. For any data set $D = (n, \widetilde{g}, a; (c_1, x_1), \ldots, (c_\ell, x_\ell))$, we have $\mathrm{lcm}\{x_1, x_2, \ldots, x_n\} = n$. To see this, put $k = \mathrm{lcm}\{x_1, x_2, \ldots, x_\ell\}$. Since each $x_i \mid n$, we have $k \mid n$. So it remains to show that $n \mid k$. Condition (iii) implies that

$$\frac{ak}{k} + \sum_{i=1}^{\ell}\frac{n(k/x_i)}{k}c_i \equiv 0 \mod n.$$

Multiplying by k,

$$ak + n\sum_{i=1}^{\ell}(k/x_i)c_i \equiv 0 \mod n.$$

Since $\gcd(a, n) = 1$, we have $n \mid k$.

With this notation, we are ready to establish the key property of data sets.

Proposition 4.5. *Data sets of degree n, genus g and length ℓ correspond to equivalence classes of nestled (n, ℓ)-actions on closed orientable surfaces of genus g.*

Proof. Let h be a nestled (n, ℓ)-action. From Remark 4.2, it is apparent that h yields a data set $D = (n, \widetilde{g}, a; (c_1, x_1), \ldots, (c_\ell, x_\ell))$ of degree n and length ℓ. The fact that the data set D has genus equal to g follows easily from the multiplicativity of the orbifold covering $F \to \mathcal{O}$:

$$\frac{2 - 2g}{n} = 2 - 2\widetilde{g} + \Big(\frac{1}{n} - 1\Big) + \sum_{i=1}^{\ell}\Big(\frac{1}{x_i} - 1\Big). \tag{4.1}$$

Consider another nestled (n, ℓ)-action h' in the equivalence class of h with a distinguished fixed point P'. Then by definition there exists an orientation-preserving homeomorphism $t \in \mathrm{Mod}(F)$ such that $t(P) = P'$ and $th't^{-1}$ is isotopic to h relative to P. Therefore, the two actions will have the same fixed point data and hence produce the same data set D.

Conversely, given a data set $D = (n, \widetilde{g}, a; (c_1, x_1), \ldots, (c_\ell, x_\ell))$, we can reverse the argument to produce an equivalence class of a nestled (n, ℓ)-action h on a surface F of genus g. We construct the orbifold $\mathcal{O}$ and representation $\rho : \pi_1^{\mathrm{orb}}(\mathcal{O}) \to C_n$. Any finite subgroup of $\pi_1^{\mathrm{orb}}(\mathcal{O})$ is conjugate to one of the cyclic subgroups generated by α or a γ_i, so condition (ii) in the definition of the data set ensures that the kernel of ρ is

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torsion-free. Therefore the orbifold covering $F \to O$ corresponding to the kernel is a manifold, and calculation of the Euler characteristic shows that F has genus g.

It remains to show that the resulting action on F is determined up to our equivalence in $\mathrm{Mod}(F)$. Suppose that two actions h and h' on F with distinguished fixed points P and P' have the same data set D. D encodes the fixed-point data of the periodic transformations h. By a result of Nielsen [7] (see also Edmonds [2, Theorem 1.3]), h and h' have to be conjugate by an orientation-preserving homeomorphism t. As in the proof of Theorem 1.1 in [6], t may be chosen so that it preserves $t(P) = P'$. Thus D determines h up to equivalence. □

Proposition 4.5 enables us to view equivalence classes of nestled (n, ℓ)-actions simply as data sets.

NOTATION 4.6. We will denote a data set of degree n and genus g by $D_{n,g,i}$, where i is an index that can be used to distinguish data sets with the same values of (n, ℓ). The trivial data set $D = \{1, g, 1; \}$, for any g, will be denoted by $D_{1,g}$.

EXAMPLE 4.7. The following are examples of data sets that represent nestled $(n, 2)$-actions, for every $g \geq 1$ and n equal to $2g + 1$, $4g$ and $4g + 2$.

(i) $D_{2g+1,g,1} = (2g + 1, 0, 1; (g, 2g + 1), (g, 2g + 1))$.
(ii) $D_{4g,g,1} = (4g, 0, 1; (1, 2), (2g - 1, 4g))$.
(iii) $D_{4g+2,g,1} = (4g + 2, 0, 1; (1, 2), (g, 2g + 1))$.

REMARK 4.8. For the data set $D = (n, \widetilde{g}, a; (c_1, x_1), \ldots, (c_n, x_\ell))$ associated with a nestled (n, ℓ)-action, Equation (4.1) in the proof of Proposition 4.5 gives the inequality

$$\frac{1 - 2g}{n} = -(\ell - 1) - 2\widetilde{g} + \sum_{i=1}^{\ell} \frac{1}{x_i} \leq -(\ell - 1) + \sum_{i=1}^{\ell} \frac{1}{x_i}\,. \tag{4.2}$$

REMARK 4.9. There exists no nontrivial action with $\ell = 0$. Suppose that we assume the contrary. Using Notation 4.1,

$$\pi_1^{\mathrm{orb}}(O) = \Big\langle \alpha, a_1, b_1, \ldots, a_{\widetilde{g}}, b_{\widetilde{g}} \,\Big|\, \alpha^n = 1, \alpha = \prod_{j=1}^{\widetilde{g}} [a_j, b_j] \Big\rangle.$$

Since C_n is abelian, $\rho(\alpha) = \rho(\prod_{j=1}^{\widetilde{g}} [a_j, b_j]) = 1$, which is impossible since ρ has torsion-free kernel.

5. Data set pairs and roots

By Theorem 3.4, each conjugacy class of a root of t_C in $\mathrm{Mod}(F)$ corresponds to a compatible pair $([h_1], [h_2])$ of (equivalence classes of) nestled actions, and by Proposition 4.5, such a pair determines a pair (D_1, D_2) of data sets. To determine which pairs of data sets arise, we must replace the geometric compatibility condition in Theorem 3.4 by an algebraic compatibility condition on the corresponding data sets.

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Definition 5.1. Two data sets $D_1 = (n_1, \widetilde{g}_1, a_1; (c_{11}, x_{11}), \ldots, (c_{1\ell}, x_{1\ell}))$ and $D_2 = (n_2, \widetilde{g}_2, a_2; (c_{21}, x_{21}), \ldots, (c_{2m}, x_{2m}))$ are said to form a *data set pair* (D_1, D_2) if

$$\frac{n}{n_1}k_1 + \frac{n}{n_2}k_2 \equiv 1 \mod n \tag{5.1}$$

where $n = \mathrm{lcm}(n_1, n_2)$ and $a_i k_i \equiv 1 \bmod n_i$. Note that although the k_i are only defined modulo n_i, the expressions $(n/n_i)k_i$ are well defined modulo n. The integer n is called the *degree* of the data set pair and $g = g_1 + g_2$ is called the *genus* of the data set pair. We consider (D_1, D_2) to be an unordered pair, that is, (D_1, D_2) and (D_2, D_1) are equivalent as compatible pairs.

We can now reformulate Theorem 3.4 in terms of data sets.

Theorem 5.2. *Let $F = F_1 \#_C F_2$ be a closed oriented surface of genus $g \geq 2$. Then data set pairs (D_1, D_2) of degree n and genus g, where D_1 is a data set of genus g_1 and D_2 is a data set of genus g_2, correspond to the conjugacy classes in* $\mathrm{Mod}(F)$ *of roots of t_C of degree n.*

Proof. Let h denote the conjugacy class of a root of t_C of degree n with compatible pair representation $([h_1], [h_2])$. From Proposition 4.5, the h_i correspond to data sets $D_i = (n_i, \widetilde{g}_i, a_i; (c_{i1}, x_{i1}), \ldots, (c_{i\ell_i}, x_{1\ell_i}))$. So it suffices to show that the geometric condition $\theta(h_1) + \theta(h_2) = 2\pi/n$ in Definition 3.2 is equivalent to the condition $(n/n_1)k_1 + (n/n_2)k_2 \equiv 1 \bmod n$ in Definition 5.1.

As in the proof of Proposition 3.4, let P_i denote the center of the filling disk of the subsurface $\widetilde{F}_i$ of genus g_i. Choosing P_i as the distinguished fixed point of h_i, we get that $\theta(h_i) = 2\pi k_i/n_i$, where $\gcd(k_i, n_i) = 1$ and $a_i k_i \equiv 1 \bmod n_i$. Since $h^n = t_C$, the left-hand twisting angle along N is $2\pi/n$, which equals $2\pi k_2/n_2 - (-2\pi k_1/n_1) = 2\pi/n$, giving $(n/n_1)k_1 + (n/n_2)k_2 \equiv 1 \bmod n$. The converse is just a matter of reversing the argument. □

Corollary 5.3. *Suppose that $F = F_1 \#_C F_2$. Then there always exists a root of the Dehn twist t_C about C of degree* $\mathrm{lcm}(4g_1, 4g_2 + 2)$.

Proof. As in Theorem 5.2, let $\widetilde{F}_i$ denote the subsurfaces obtained by cutting F along C, and let F_i denote the surfaces obtained by adding disks to the F_i. Let $n_1 = 4g_1$ and $n_2 = 4g_2 + 2$. From Example 4.7, for any residue class a_i modulo n_i with $\gcd(a_i, n_i) = 1$, the data set $D_1 = (n_1, 0, a_1; (-a_1, 2g_1), (a_1, 4g_1))$ defines a nestled $(n_1, 2)$-action on a surface F_1 of genus g_1, and the data set $D_2 = (n_2, 0, a_2; (a_2, 2), (a_2 g_2, 2g_2 + 1))$ defines a nestled $(n_2, 2)$-action on F_2 of genus g_2.

Let k_i denote the inverse of a_i modulo n_i and let $n = \mathrm{lcm}(n_1, n_2)$. We will now show that the a_i can be selected so that Equation (5.1) is satisfied. In other words, this will prove that D_1 and D_2 form a data set pair (D_1, D_2). Since n/n_1 and n/n_2 are relatively prime, there always exist integers p and q such that

$$\frac{n}{n_1}p + \frac{n}{n_2}q = 1.$$

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In particular, since n/n_1 and n/n_2 are not both odd, by [6, Lemma 7.1], p and q can be chosen so that $\gcd(p, n_1) = \gcd(q, n_2) = 1$. Let k_1 be the residue class of p modulo n_1 and let k_2 be the residue class of q modulo n_2. Taking modulo n,

$$\frac{n}{n_1}k_1 + \frac{n}{n_2}k_2 \equiv 1 \mod n.$$

Therefore, by Theorem 5.2, there exists a root of t_C of order $\mathrm{lcm}(4g_1, 4g_2 + 2)$. □

Corollary 5.4. *Let $F = F_1 \#_C F_2$ be a closed oriented surface of genus $g \geq 2$. Suppose that M denotes the maximum degree of a root of the Dehn twist t_C about C. Then $2g^2 + 2g \leq M$.*

Proof. If g is even, then Corollary 5.3 with $g_1 = g_2 = g/2$ gives a root of degree $\mathrm{lcm}(2g, 2g + 1) = 2g(2g + 1)$. If g is odd, then $g_1 = (g + 1)/2$ and $g_2 = (g - 1)/2$ gives a root of degree $\mathrm{lcm}(2(g + 1), 2g) = 2g(g + 1)$. □

6. Classification of roots for the closed orientable surfaces of genus 2 and 3

6.1. Surface of genus 2. Let $F = F_1 \#_C F_2$ be the closed orientable surface of genus 2. Up to homeomorphism, there is a unique curve C that separates F into two subsurfaces of genus 1. Given a root of t_C, the process described in the proof of Theorem 3.4 produces orientation-preserving C_{n_i}-actions on the tori F_i, for $i = 1, 2$, with $n = \mathrm{lcm}(n_1, n_2)$.

If a cyclic group C_n acts faithfully on a surface F fixing a point x_0, then the map $C_n \longrightarrow \mathrm{Aut}(\pi_1(F, x_0))$ is a monomorphism [1, Theorem 2, page 43]. We also know that the group of orientation-preserving automorphisms $\mathrm{Aut}^+(\pi_1(F_i, x_0)) \cong \mathrm{Aut}^+(\mathbb{Z} \times \mathbb{Z}) \cong \mathrm{SL}(2, \mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$. Since any element of finite order of an amalgamated product $A *_C B$ is conjugate to one of the groups A or B [4], it can only be of order 2, 3, 4 or 6. Taking the least common multiple of any two of these orders gives 12 as the only other possibility for the order of a root of t_C. We summarize these inferences in the following corollary.

Corollary 6.1. *Let F be the closed orientable surface of genus 2 and C a separating curve in F. Then a root of a Dehn twist t_C about C can only be of degree 2, 3, 4, 6, or 12.*

By Theorem 5.2, classifying compatible pairs of C_{n_i}-actions on F_i is equivalent to classifying all data set pairs of genus 2. Given below are the data set pairs that represent each conjugacy class of roots. For $n = 2$:

(i) $(D_{2,1,1}, D_{1,1})$, where $D_{2,1,1} = (2, 0, 1; (1, 2), (1, 2), (1, 2))$.

For $n = 3$:

(i) $(D_{3,1,1}, D_{1,1})$, where $D_{3,1,1} = (3, 0, 1; (1, 3), (1, 3))$;
(ii) $(D_{3,1,2}, D_{3,1,2})$, where $D_{3,1,2} = (3, 0, 2; (2, 3), (2, 3))$.

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For $n = 4$:

(i) $(D_{4,1,1}, D_{1,1})$, where $D_{4,1,1} = (4, 0, 1; (1, 2), (1, 4))$;
(ii) $(D_{4,1,2}, D_{2,1,1})$, where $D_{4,1,2} = (4, 0, 3; (1, 2), (3, 4))$.

For $n = 6$:

(i) $(D_{6,1,1}, D_{1,1})$, where $D_{6,1,1} = (6, 0, 1; (1, 2), (1, 3))$;
(ii) $(D_{6,1,2}, D_{3,1,1})$, where $D_{6,1,2} = (6, 0, 5; (1, 2), (2, 3))$;
(iii) $(D_{3,1,2}, D_{2,1,1})$.

For $n = 12$:

(i) $(D_{6,1,2}, D_{4,1,1})$;
(ii) $(D_{4,1,2}, D_{3,1,1})$.

It can be shown using elementary calculations that these are the only possible roots for the various orders. For example, when $n = 12$, the condition $\mathrm{lcm}(n_1, n_2) = 12$ would imply that the set $\{n_1, n_2\}$ can be either $\{6, 4\}$ or $\{4, 3\}$. When $n_1 = 6$ and $n_2 = 4$, the data set pair condition gives $2k_1 + 3k_2 \equiv 1 \bmod 12$. Since k_i is a residue modulo n_i, the only possible solution to this equation is $k_1 = 5$ and $k_2 = 1$. This would imply that $a_1 = 5$ and $a_2 = 1$ since a_i is the inverse of k_i modulo n_i. Geometrically, this represents the root h of t_C whose twisting angle on one side is $2\pi k_1/n_1 = 5\pi/3$ and on the other side of C is $2\pi k_2/n_2 = \pi/2$. Each data set D_i in the data set pair (D_1, D_2) is then uniquely determined by condition (iii) (for data sets) and the formula for calculating the genus g_i. Similar calculations can be used to determine all the data set pairs for the surface of genus 3.

6.2. Surface of genus 3. Let $F = F_1 \#_C F_2$ be the closed orientable surface of genus 3. Then (up to homeomorphism), F has a unique curve that separates the surface into two subsurfaces of genera 2 and 1. As in the classification of roots of in the genus 2 case, it suffices to classify pairs of compatible pairs of nestled (n_i, ℓ_i)-actions on surfaces F_i, for $i = 1, 2$. The various nestled (n_2, ℓ_2)-actions on the torus F_2 have already been classified in the genus 2 case. So it remains to classify all possible (n_1, ℓ_1)-actions on the surface F_1 of genus 2 and then determine how many of these actions form compatible pairs with nestled (n_2, ℓ_2)-actions on F_2. By Remark 8.1, we have that $n_1 \leq 10$ and $n_2 \leq 6$. Therefore, Theorem 5.2 would imply that classifying compatible nestled (n_i, ℓ_i)-actions on the F_i is equivalent to determining all possible data sets pairs $(D_{n_1,2,i}, D_{n_2,1,j})$, where $n_1 \leq 10$ and $n_2 \leq 6$. Given below are the data set pairs that represent roots of various degrees that were determined by programming the number-theoretic conditions for data set and their pairs in software [8] written for the GAP programming language. For $n = 2$:

(i) $(D_{1,2}, D_{2,1,1})$;
(ii) $(D_{2,2,1}, D_{1,1})$, where $D_{2,2,1} = (2, 0, 1; (1, 2), (1, 2), (1, 2), (1, 2), (1, 2))$;
(iii) $(D_{2,2,2}, D_{1,1})$, where $D_{2,2,2} = (2, 1, 1; (1, 2))$.

For $n = 3$:

(i) $(D_{1,2}, D_{3,1,1})$;

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(ii) $(D_{3,2,1}, D_{1,1})$, where $D_{3,2,1} = (3, 0, 1; (2, 3), (2, 3), (1, 3))$;
(iii) $(D_{3,2,2}, D_{1,1})$, where $D_{3,2,2} = (3, 0, 2; (1, 3), (1, 3), (2, 3))$.

For $n = 4$:

(i) $(D_{1,2}, D_{4,1,1})$;
(ii) $(D_{4,2,1}, D_{1,1})$, where $D_{4,2,1} = (4, 0, 1; (1, 2), (1, 2), (3, 4))$;
(iii) $(D_{4,2,2}, D_{4,1,1})$, where $D_{4,2,2} = (4, 0, 3; (1, 2), (1, 2), (2, 4))$.

For $n = 5$:

(i) $(D_{5,2,1}, D_{1,1})$, where $D_{5,2,1} = (5, 0, 1; (1, 5), (3, 5))$;
(ii) $(D_{5,2,2}, D_{1,1})$, where $D_{5,2,2} = (5, 0, 1; (2, 5), (2, 5))$.

For $n = 6$:

(i) $(D_{1,2}, D_{6,1,2})$;
(ii) $(D_{6,2,1}, D_{1,1})$, where $D_{6,2,1} = (6, 0, 1; (2, 3), (1, 6))$;
(iii) $(D_{2,2,1}, D_{3,1,2})$;
(iv) $(D_{2,2,2}, D_{3,1,2})$;
(v) $(D_{3,2,2}, D_{2,1,1})$;
(vi) $(D_{3,2,1}, D_{6,1,2})$;
(vii) $(D_{6,2,2}, D_{3,1,1})$, where $D_{6,2,2} = (6, 0, 5; (1, 3), (5, 6))$.

For $n = 8$:

(i) $(D_{8,2,1}, D_{1,1})$, where $D_{8,2,1} = (8, 0, 1; (1, 2), (3, 8))$;
(ii) $(D_{8,2,2}, D_{2,1,1})$, where $D_{8,2,2} = (8, 0, 5; (1, 2), (7, 8))$;
(iii) $(D_{8,2,3}, D_{4,1,1})$, where $D_{8,2,3} = (8, 0, 7; (1, 2), (5, 8))$;
(iv) $(D_{8,2,4}, D_{4,1,2})$, where $D_{8,2,4} = (8, 0, 3; (1, 2), (1, 8))$.

For $n = 10$:

(i) $(D_{10,2,1}, D_{1,1})$, where $D_{10,2,1} = (10, 0, 1; (1, 2), (2, 5))$;
(ii) $(D_{5,2,3}, D_{2,1,1})$, where $D_{5,2,3} = (5, 0, 3; (1, 5), (1, 5))$;
(iii) $(D_{5,2,4}, D_{2,1,1})$, where $D_{5,2,4} = (5, 0, 3; (3, 5), (4, 5))$.

For $n = 12$:
(i) $(D_{4,2,2}, D_{3,1,1})$;
(ii) $(D_{3,2,1}, D_{4,1,2})$;
(iii) $(D_{4,2,1}, D_{6,1,2})$;
(iv) $(D_{6,2,2}, D_{4,1,1})$.
For $n = 15$:
(i) $(D_{5,2,5}, D_{3,1,2})$, where $D_{5,2,5} = (5, 0, 3; (1, 5), (1, 5))$;
(ii) $(D_{5,2,6}, D_{3,1,2})$, where $D_{5,2,6} = (5, 0, 3; (3, 5), (4, 5))$.
For $n = 20$:
(i) $(D_{5,2,5}, D_{4,1,1})$, where $D_{5,2,5} = (5, 0, 4; (4, 5), (2, 5))$;
(ii) $(D_{5,2,6}, D_{4,1,1})$, where $D_{5,2,6} = (5, 0, 4; (3, 5), (3, 5))$;
(iii) $(D_{10,2,1}, D_{4,1,2})$, where $D_{10,2,1} = (10, 0, 7; (1, 2), (4, 5))$.

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For $n = 24$:
(i) $(D_{8,2,4}, D_{3,1,2})$;
(ii) $(D_{8,2,3}, D_{6,1,1})$.
For $n = 30$:
(i) $(D_{10,2,2}, D_{3,1,1})$, where $D_{10,2,2} = (10, 0, 9; (1, 2), (3, 5))$;
(ii) $(D_{5,2,7}, D_{6,1,2})$, where $D_{5,2,7} = (5, 0, 1; (1, 5), (3, 5))$;
(iii) $(D_{5,2,8}, D_{6,1,2})$, where $D_{5,2,8} = (5, 0, 1; (2, 5), (2, 5))$.
As in the earlier genus 2 case, it can be shown using elementary calculations that these are the only possible roots up to conjugacy for the various orders. For example, when $n = 15$, since $n_1 \leq 10$ and $n_2 \leq 6$, we would have that $\{n_1, n_2\} = \{3, 5\}$. Since there is no C_5-action on the torus, we have that $n_1 = 5$. When $n_1 = 5$ and $n_2 = 3$, the data set pair condition gives $3k_1 + 5k_2 \equiv 1 \mod 15$, where k_i is a residue modulo n_i. The only solution to this equivalence is $k_1 = k_2 = 2$, which would imply that $a_1 = 3$ and $a_2 = 2$. The data set pairs satisfying these conditions are $(D_{5,2,5}, D_{3,1,2})$ and $(D_{5,2,6}, D_{3,1,2})$. Using similar calculations, we can determine all the other possible data set pairs.

7. Spherical nestled actions

A *spherical action* is simply a nestled (n, ℓ)-action whose quotient orbifold is topologically a sphere. We will show in Proposition 7.3 that nestled (n, ℓ)-actions must be spherical when n is sufficiently large relative to g. This means that in order to derive bounds on n, it suffices to restrict attention to spherical actions. We will also derive several other results on spherical actions which we will use in later sections.

Definition 7.1. A nontrivial nestled (n, ℓ)-action is said to be *spherical* if the underlying manifold of its quotient orbifold is topologically a sphere.

Example 7.2. The actions in Examples 2.3 and 4.7 are spherical actions.

Proposition 7.3. *If $n > \frac{2}{3}(2g - 1)$, then every nestled (n, ℓ)-action on F is spherical.*

Proof. Let $D = (n, \widetilde{g}, a; (c_1, x_1), \ldots, (c_n, x_\ell))$ be the data set associated with a nestled (n, ℓ)-action on F. Equation (4.2) gives

$$\widetilde{g} = \frac{1}{2} + \frac{2g - 1}{2n} - \frac{\ell}{2} + \frac{1}{2} \sum_{i=1}^{\ell} \frac{1}{x_i}. \tag{7.1}$$

Each $x_i \geq 2$, and by Remark 4.9, we must have $\ell \geq 1$, so this becomes

$$\widetilde{g} \leq \frac{1}{2} + \frac{2g - 1}{2n} - \frac{\ell}{4} \leq \frac{1}{4} + \frac{2g - 1}{2n}.$$

That is, $\widetilde{g} \geq 1$ can hold only when $n \leq (4g - 2)/3$. □

Remark 7.4. There exists no spherical nestled $(n, 1)$-action on the surface of genus $g \geq 1$. Suppose we assume to the contrary that $\ell = 1$. Then Equation (4.1) would

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imply that

$$\frac{1-2g}{n} = \frac{1}{x_1}.$$

This is impossible since $x_1 > 0$ and $g \geq 1$.

PROPOSITION 7.5. *Suppose that a surface F of genus g has a spherical nested (n, ℓ)-action. Write the prime factorization of n as $n = p^a {q_1}^{a_1} \cdots {q_k}^{a_k}$ where $p^a > {q_i}^{a_i}$ for each $i \geq 1$, and write q for* $\min\{p, q_1, \ldots, q_k\}$. *If*

$$n > \frac{2g-1}{2 - \frac{2}{q} - \frac{1}{p^a}},$$

then $\ell = 2$.

PROOF. Each $x_i \geq q$, and by Proposition 4.4, at least one $x_i \geq p^a$. Using Equation (7.1),

$$0 = \frac{1}{2} + \frac{2g-1}{2n} - \frac{\ell}{2} + \frac{1}{2}\sum_{i=1}^{\ell} \frac{1}{x_i} \leq \frac{1}{2} + \frac{1}{2p^a} + \frac{2g-1}{2n} - \frac{\ell}{2} + \frac{\ell-1}{2q}$$

$$\ell \leq 1 + \frac{q}{(q-1)p^a} + \frac{q}{q-1}\Big(\frac{2g-1}{n}\Big).$$

The right-hand side of the latter inequality is less than 3 when the inequality in the proposition holds. Therefore, by Remark 7.4, $\ell = 2$. □

COROLLARY 7.6. *Suppose that a surface F of genus g has a spherical nested (n, ℓ)-action, $\ell \geq 2$.*

(i) *If $n = 2$, then $\ell = 2g + 1$. In particular, there does not exist a spherical nested* $(2, 2)$*-action.*
(ii) *If $n = 3$, then $\ell = g + 1$. There exists a spherical nested* $(3, 2)$*-action if and only if $g = 1$.*
(iii) *If n is even, $n \geq 4$, and $n > \frac{4}{3}(2g - 1)$, then $\ell = 2$.*
(iv) *If n is odd, $n \geq 5$, and $n > \frac{15}{17}(2g - 1)$, then $\ell = 2$.*

PROOF. For (i), an Euler characteristic calculation shows that $\ell = 2g + 1$ when $n = 2$. These are exactly the hyperelliptic actions.

For (ii), when $n = 3$, an Euler characteristic calculation shows that $\ell = g + 1$, and as seen in Section 6, there is a nested $(3, 2)$-action on the torus.

For (iii), suppose first that $n = 6$. In Proposition 7.5 we have $q = 2$ and $p^a = 3$, giving the conclusion that if $6 > \frac{3}{2}(2g - 1)$, then $\ell = 2$. The condition $6 > \frac{3}{2}(2g - 1)$ holds exactly when $g \leq 2$, so (iii) is true in this case. One can check that there exist nested $(6, 2)$-actions exactly when $g \leq 2$. For the cases of (iii) other than $n = 6$, we have $q = 2$ and $p^a \geq 4$, and Proposition 7.5 gives the result.

For (iv), we have $q \geq 3$ and $p^a \geq 5$. Again Proposition 7.5 gives the result. □

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8. Bounds on the degree of a root

In this section, we use Theorem 5.2 and the results derived in Section 7 to derive some results on the degree n of a root. Among the results obtained are a general upper bound for n in Theorem 8.6, which is later refined in Theorem 8.14 to obtain a sharper upper bound which is stable in the sense that it applies once the genus is sufficiently large. In Table 2, we give data which indicate the degree of improvement of the stable upper bound for $g \geq 14$. However, it is worth mentioning here that Theorem 8.6 does provide a better bound for $g \leq 13$. As in Notation 3.1, we will assume throughout this section that $g_1 \geq g_2$ whenever $F = F_1 \#_C F_2$.

Remark 8.1. It is a well-known fact [3] that the maximum order for an automorphism of a surface of genus g is $4g + 2$. In Example 4.7, we showed that a nestled action of order $4g + 2$ always exists.

Proposition 8.2. *There exists no nestled $(4g + 1, \ell)$-action.*

Proof. By Proposition 7.3, a nestled $(4g + 1, \ell)$-action must be spherical, and by Proposition 7.5, $\ell = 2$. Therefore, Equation (4.1) from the proof of Proposition 4.5 simplifies to give

$$\frac{2g+2}{4g+1} = \frac{1}{x_1} + \frac{1}{x_2}.$$

Without loss of generality, we may assume that $x_1 \leq x_2$. Since $x_i \mid 4g + 1$, $x_i \geq 3$. If $x_1 = 3$, then

$$x_2 = \frac{3(4g+1)}{2g+5} = 3\Big(2 - \frac{9}{2g+5}\Big).$$

Since $x_2 = 3$ is the only integer solution for x_2, Proposition 4.4 would imply that $n = 3$, which contradicts the fact that $n = 4g + 1$. If $x_1 \geq 4$, then we would have that

$$\frac{1}{2} < \frac{2+2g}{4g+1} = \frac{1}{x_1} + \frac{1}{x_2} \leq \frac{1}{2},$$

which is not possible. □

Proposition 8.3. *Let $F = F_1 \#_C F_2$ be a closed oriented surface of genus $g \geq 2$. Let (D_1, D_2) be a data set pair corresponding to a root of t_C of degree n, and let n_i be the degree of D_i for $i = 1, 2$. Then the n_i cannot both satisfy $n_i \equiv 2 \bmod 4$.*

Proof. Suppose for contradiction that both n_i satisfy $n_i \equiv 2 \bmod 4$. Let a_i denote the a-value of D_i, and let k_i denote the inverse of a_i modulo n_i. Since $\gcd(k_i, n_i) = 1$, the k_i must be odd. Also the fact that $\gcd(n_1, n_2) = 2k$ for some odd integer k implies that n/n_i is odd. From Equation (5.1) for the data set pair (D_1, D_2), we must have that

$$\frac{n}{n_1}k_1 + \frac{n}{n_2}k_2 \equiv 1 \quad \bmod n,$$

which is impossible since $(n/n_1)k_1 + (n/n_2)k_2$ and n are even. □

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Proposition 8.4. *Let $F = F_1 \#_C F_2$ be a closed oriented surface of genus $g \geq 2$. Suppose that $M(g_1, g_2)$ denotes the maximum degree of a root of the Dehn twist t_C about C. Then $M(g_1, g_2) \leq 16g_1g_2 + 4(2g_1 - g_2) - 2$.*

Proof. Let n be the order of a root of t_C, given by a data set pair (D_1, D_2). We have $n = \text{lcm}(n_1, n_2)$, where n_i is the degree of D_i. By Remark 8.1, each $n_i \leq 4g_i + 2$. By Proposition 8.2, neither $n_i = 4g_i + 1$ nor, by Proposition 8.3, can we have both $n_1 = 4g_1 + 2$ and $n_2 = 4g_2 + 2$. If both $n_1 = 4g_1$ and $n_2 = 4g_2$, then

$$\text{lcm}(n_1, n_2) = 4\,\text{lcm}(g_1, g_2) \leq 4g_1g_2 \leq 16g_1g_2 + 4(2g_1 - g_2) - 2.$$

In general, since $g_1 \geq g_2$, we have that

$$\begin{aligned} M(g_1, g_2) &\leq \max\{(4g_1 + 2)(4g_2 - 1), (4g_1 - 1)(4g_2 + 2)\} \\ &= 16g_1g_2 + 4(2g_1 - g_2) - 2. \end{aligned}$$

□

Notation 8.5. We will denote the upper bound $16g_1g_2 + 4(2g_1 - g_2) - 2$ derived in Proposition 8.4 by $U(g_1, g_2)$.

Theorem 8.6. *Let $F = F_1 \#_C F_2$ be a closed oriented surface of genus $g \geq 2$. Suppose that n denotes the degree of a root of the Dehn twist t_C about C. Then $n \leq 4g^2 + 2g$.*

Proof. Since $g_2 = g - g_1$, we have that

$$16g_1g_2 + 4(2g_1 - g_2) - 2 = -16g_1^2 + g_1(16g + 12) - (4g + 2),$$

which has its maximum when $g_1 = \frac{1}{8}(4g + 3)$. The fact that g_1 is an integer implies that when g is even, $g_1 = g_2 = g/2$, and when g is odd, $g_1 = (g + 1)/2$ and $g_2 = (g - 1)/2$. So Proposition 8.4 tells us that when g is even, $n \leq M(g/2, g/2) \leq 4g^2 + 2g - 2$, and when g is odd, $n \leq M((g + 1)/2, (g - 1)/2) \leq 4g^2 + 2g$.

Notation 8.7. We will denote the upper bound $4g^2 + 2g$ derived in Theorem 8.6 by $U(g)$.

Notation 8.8. Let $F = F_1 \#_C F_2$ be a closed oriented surface of genus $g \geq 2$. We will denote the realizable maximum degree of a root coming from compatible pairs of spherical nestled $(n, 2)$-actions on the F_i by $m(g_1, g_2)$, and the maximum over all such pairs of genera (g_1, g_2) (that is, $\max\{m(g_1, g_2) \mid g_1 + g_2 = g\}$) by $m(g)$.

For $14 \leq g \leq 35$, Table 1 shows the genus pairs (g_1, g_2) for which $m(g_1, g_2) = m(g)$ and the upper bound $U(g)$. The last column gives the ratio $m(g)/U(g)$. These computations were made using software [8] written for the GAP programming language.

The following proposition and its subsequent corollary will be used later in Proposition 8.11 to derive a sharper upper bound for $M(g_1, g_2)$ than the $U(g_1, g_2)$ obtained in Proposition 8.4. Finally, in Theorem 8.14, we will use Proposition 8.4 and some elementary calculus to derive an upper bound for n that is significantly sharper than $U(g)$.

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Table 1. The data seems to indicate that for large genera the ratio $m(g)/U(g)$ stabilizes to the range 0.79–0.82.

g	$m(g_1, g_2) = m(g)$	$U(g_1, g_2)$	$m(g_1, g_2)/U(g_1, g_2)$	$U(g)$	$m(g)/U(g)$
14	$m(8, 6) = 714$	806	0.89	812	0.88
15	$m(9, 6) = 798$	910	0.88	930	0.86
16	$m(10, 6) = 858$	1014	0.85	1056	0.81
17	$m(11, 6) = 966$	1118	0.86	1190	0.81
18	$m(10, 8) = 1122$	1326	0.85	1332	0.84
19	$m(10, 9) = 1254$	1482	0.85	1482	0.85
20	$m(12, 8) = 1326$	1598	0.83	1640	0.81
21	$m(11, 10) = 1518$	1806	0.84	1806	0.84
22	$m(12, 10) = 1650$	1974	0.84	1980	0.83
23	$m(12, 11) = 1794$	2162	0.83	2162	0.83
24	$m(12, 12) = 1950$	2350	0.83	2352	0.83
25	$m(15, 10) = 2046$	2478	0.83	2550	0.80
26	$m(14, 12) = 2262$	2750	0.82	2756	0.82
27	$m(15, 12) = 2418$	2950	0.82	2970	0.81
28	$m(16, 12) = 2550$	3150	0.81	3192	0.80
29	$m(17, 12) = 2730$	3350	0.81	3422	0.80
30	$m(16, 14) = 2958$	3654	0.81	3660	0.81
31	$m(16, 15) = 3162$	3906	0.81	3906	0.81
32	$m(18, 14) = 3306$	4118	0.80	4160	0.79
33	$m(17, 16) = 3570$	4422	0.81	4422	0.81
34	$m(18, 16) = 3774$	4686	0.81	4692	0.80
35	$m(18, 17) = 3990$	4970	0.80	4970	0.80

Proposition 8.9. *Suppose that we have a nestled (n, ℓ)-action on a surface F of genus g, where n is a positive odd integer. Then $n \leq 3g + 3$.*

Proof. From Remark 7.4, we have that $\ell \neq 1$. When $\ell \geq 2$, the proposition follows from Corollary 7.6. Let $D = (n, \widetilde{g}, a; (c_1, x_1), (c_2, x_2))$ be a data set for the nestled $(n, 2)$-action on F. Since n is odd and $x_i \mid n$, we have that $x_i \geq 3$. If $x_1 \geq 3$, then Remark 4.4 implies that $x_2 \geq \frac{n}{3}$. So Equation (4.2) gives the inequality

$$\frac{1 - 2g}{n} \leq -1 + \frac{1}{3} + \frac{3}{n},$$

which upon simplification gives $n \leq 3g + 3$.

Corollary 8.10. *Suppose that we have a spherical nestled $(4g - N, 2)$-action on an F of genus g, where N is a positive odd integer. Then $g \leq N + 3$.*

Proposition 8.11. *Let $F = F_1 \#_C F_2$ be a closed oriented surface of genus $g \geq 2$. Suppose that $M(g_1, g_2)$ denotes the maximum order of a root of the Dehn twist t_C about C. Then given a positive odd integer N, we have that $M(g_1, g_2) \leq 16g_1g_2 + 4(2g_1 - Ng_2) - 2N$ whenever both $g_i > N + 3$.*

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Proof. By Remark 8.1, each $n_i \leq 4g_i + 2$. From Propositions 8.2 and 8.3, we know that $n_i \neq 4g_i + 1$ and that n_i cannot both be $4g_i + 2$. Suppose that the n_i are not both even. If $\ell_i > 2$, then from Corollary 7.6 we have that $n_i \leq \frac{15}{17}(2g_i - 1)$. If $\ell_i = 2$, then Corollary 8.10 tells us that for all $g_i > N + 3$, there exists no spherical nestled $(4g_i - N, 2)$-action on F. In particular, if $g_i > N + 3$, then from Proposition 7.3, $n_i \leq \frac{2}{3}(2g_i - 1) \leq \frac{15}{17}(2g_i - 1)$. So for all ℓ, if $g_i > N + 3$, then $n_i \leq \frac{15}{17}(2g_i - 1)$. We can see that $\frac{15}{17}(2g_i - 1) \leq 4g_i - N$ whenever $g_i \geq \frac{1}{38}(17N - 15)$. Therefore, if $g_i > \max\{N + 3, \frac{1}{38}(17N - 15)\} = N + 3$, then

$$\begin{aligned} M(g_1, g_2) &\leq \max\{(4g_1 - N)(4g_2 + 2), (4g_1 + 2)(4g_2 - N)\} \\ &= 16g_1g_2 + 4\max\{(2g_1 - Ng_2), (2g_2 - Ng_1)\} - 2N \\ &= 16g_1g_2 + 4(2g_1 - Ng_2) - 2N. \end{aligned}$$

Suppose that both the n_i are even. Then from Propositions 8.2 and 8.3,

$$M(g_1, g_2) \leq \operatorname{lcm}(4g_1 + 2, 4g_2) \leq 8g_1g_2 + 4g_2.$$

We need to show that

$$8g_1g_2 + 4g_2 \leq 16g_1g_2 + 4(2g_1 - Ng_2) - 2N.$$

Since $g_1 > N + 3$,

$$\begin{aligned} &(16g_1g_2 + 4(2g_1 - Ng_2) - 2N) - (8g_1g_2 + 4g_2) \\ &\quad = 8g_1g_2 + 8g_1 - 4(N + 1)g_2 - 2N > 8g_1g_2 + 8g_1 + 4(g_1 - 2)g_2 + 2(g_1 - 3) \\ &\quad = 12g_1g_2 + 10g_1 - 8g_2 - 6 > 0. \end{aligned}$$ □

Remark 8.12. Since $g_1 \geq g_2$ by assumption, the condition $g_i > N + 3$ in the hypothesis of Proposition 8.11 can be replaced by $g_2 > N + 3$. The fact that N is an odd integer would imply that both $g_i \geq 5$ and consequently $g \geq 10$.

Notation 8.13. Let $F = F_1 \#_C F_2$ be a closed oriented surface of genus $g \geq 2$. We will denote the upper bound $16g_1g_2 + 4(2g_1 - Ng_2) - 2N$ derived in Proposition 8.11 by $\mathrm{SU}(g_1, g_2, N)$. From Remark 8.12, we have that $g_i \geq 5$ and $N < g_2 - 3$. Hence $\min\{\mathrm{SU}(g_1, g_2, N) \mid N \text{ odd},\ g_i \geq 5, \text{ and } 1 \leq N < g_2 - 3\}$ is a well-defined positive integer and we denote this by $\mathrm{SU}(g_1, g_2)$.

Theorem 8.14. *Let $F = F_1 \#_C F_2$ be a closed oriented surface of genus $g \geq 10$. Suppose that n denotes the degree of a root of the Dehn twist t_C about C. Then $n \leq \frac{16}{5}g^2 + 12g + \frac{45}{4}$.*

Proof. From Theorem 8.11, given a positive odd integer N, we have that $M(g_1, g_2) \leq 16g_1g_2 + 4(2g_1 - Ng_2) - 2N$ whenever both $g_i > N + 3$. Since $g_1 \geq g_2$, it suffices to assume that $g_2 > N + 3$, that is, $N < g_2 - 3$. Consequently, $N \leq g_2 - 5$ when N is odd, and $N \leq g_2 - 4$ when N is even. Therefore, for any g_2,

$$\mathrm{SU}(g_1, g_2) \leq \mathrm{SU}(g - g_2, g_2, g_2 - 5) = -20g_2^2 + 2(8g + 5)g_2 + 8g + 10.$$

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TABLE 2. This data illustrates that the stable bound $SU(g)$ is significantly closer to $m(g)$ when compared with $U(g)$. The data seems to indicate that for large genera the ratio $m(g)/SU(g)$ stabilizes to the range 0.90–0.92.

g	$m(g_1, g_2) = m(g)$	$SU(g_1, g_2)$	$m(g_1, g_2)/SU(g_1, g_2)$	$SU(g)$	$m(g)/SU(g)$
14	$m(8, 6) = 714$	806	0.89	806	0.89
15	$m(9, 6) = 798$	910	0.88	911	0.88
16	$m(10, 6) = 858$	1014	0.85	1022	0.84
17	$m(11, 6) = 966$	1118	0.86	1140	0.85
18	$m(10, 8) = 1122$	1258	0.89	1264	0.89
19	$m(10, 9) = 1254$	1330	0.94	1394	0.90
20	$m(12, 8) = 1326$	1530	0.87	1531	0.87
21	$m(11, 10) = 1518$	1638	0.93	1674	0.91
22	$m(12, 10) = 1650$	1806	0.91	1824	0.90
23	$m(12, 11) = 1794$	1886	0.95	1980	0.91
24	$m(12, 12) = 1950$	2050	0.95	2142	0.91
25	$m(15, 10) = 2046$	2310	0.89	2311	0.89
26	$m(14, 12) = 2262$	2450	0.92	2486	0.91
27	$m(15, 12) = 2418$	2650	0.91	2668	0.91
28	$m(16, 12) = 2550$	2850	0.89	2856	0.89
29	$m(17, 12) = 2730$	3050	0.90	3050	0.90
30	$m(16, 14) = 2958$	3190	0.93	3251	0.91
31	$m(16, 15) = 3162$	3286	0.96	3458	0.91
32	$m(18, 14) = 3306$	3654	0.90	3672	0.90
33	$m(17, 16) = 3570$	3762	0.95	3892	0.92
34	$m(18, 16) = 3774$	4026	0.94	4118	0.92
35	$m(18, 17) = 3990$	4130	0.97	4351	0.92

Since $-20g_2^2 + 2(8g + 5)g_2 + 8g + 10$ has its maximum when $g_2 = \frac{2}{5}g + \frac{1}{4}$, from Proposition 8.11, we have that

$$n \leq M(\tfrac{3}{5}g - \tfrac{1}{4}, \tfrac{2}{5}g + \tfrac{1}{4}) \leq \tfrac{16}{5}g^2 + 12g + \tfrac{45}{4}. \qquad \square$$

NOTATION 8.15. We will denote the upper bound $\frac{16}{5}g^2 + 12g + \frac{45}{4}$ derived in Theorem 8.14 by $SU(g)$.

For $14 \leq g \leq 35$, Table 2 gives $SU(g_1, g_2)$, $m(g_1, g_2)/SU(g_1, g_2)$, and the ratio $m(g)/SU(g)$.

Based on the observable data in Tables 1 and 2, we make the following conjecture.

CONJECTURE 8.16. *Let $F = F_1\#_C F_2$ be a closed oriented surface of genus $g \geq 2$. Then for sufficiently large values of g the ratio $m(g)/U(g)$ stabilizes to the range* 0.79–0.82, *while the ratio $m(g)/SU(g)$ stabilizes to the range* 0.90–0.92.

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