THE SPECTRA OF SEMI-NORMAL SINGULAR INTEGRAL OPERATORS

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1. Introduction. Suppose that

(1.1) $\phi(x)$ is measurable and essentially bounded on $[0, 1]$,

and define the bounded self-adjoint operators H and J on the Hilbert space $L^2(0, 1)$ by

(1.2) $(Hf)(x) = xf(x)$ and $(Jf)(x) = (i\pi)^{-1}\int_0^1 \phi(x)\bar{\phi}(t)(t-x)^{-1}f(t)\,dt,$

the integral being a Cauchy principal value

$$\int = \lim_{\epsilon\to 0} \int_{|t-x|>\epsilon} (\ldots)\,dt.$$

It is seen that

(1.3) $HJ - JH = iC$, where $(Cf)(x) = \pi^{-1}\int_0^1 \phi(x)\bar{\phi}(t)f(t)\,dt,$

or, equivalently,

(1.4) $AA^* - A^*A = 2C$, where $A = H + iJ.$

Since $(Cf, f) = \pi^{-1}|(f, \phi)|^2 \geqq 0$, A is semi-normal. (For a discussion of such operators, see [**4**].)

The problem is still open as to whether the spectrum of every semi-normal operator T, that is, an operator satisfying

(1.5) $TT^* - T^*T \geqq 0$ or $\leqq 0$,

but which is not normal, must have positive (planar) measure. In fact, a stronger (and plausible) conjecture is that even

(1.6) $\pi||TT^* - T^*T|| \leqq \text{meas}\,\text{sp}(T)$,

where $\text{sp}(T)$ denotes the spectrum of T; cf. [**4**, p. 51].

In this paper an investigation of the spectrum of the integral operator A will be made. Whether (1.6) must hold even in this special case (with $T = A$) for arbitrary ϕ will remain unsettled. However, its validity will be established if ϕ is restricted to a certain class, for example, if ϕ is Riemann integrable on $[0, 1]$. See the Corollary to Theorem 4.

The following theorems will be proved.

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THEOREM 1. *Let* $\phi(x)$ *satisfy* (1.1) *and let*

$$M_x = \operatorname{ess}\limsup_{t=x} |\phi(t)|^2, \qquad 0 \leqq x \leqq 1. \tag{1.7}$$

If A *is defined by* (1.1), (1.2), *and* (1.4), *then the following inclusion relations hold:*

$$\operatorname{sp}(A) \subset S \equiv \{x + iy\colon 0 \leqq x \leqq 1, -M_x \leqq y \leqq M_x\} \tag{1.8}$$

and

$$\begin{aligned}\operatorname{sp}(A) \supset \{z = x \pm iM_x\colon 0 \leqq x \leqq 1\} + \{z = iy\colon -M_0 \leqq y \leqq M_0\} \\ + \{z = 1 + iy\colon -M_1 \leqq y \leqq M_1\} + \{z = x\colon 0 \leqq x \leqq 1\}.\end{aligned} \tag{1.9}$$

That is, $\operatorname{sp}(A)$ *is contained between the lines* $x = 0$ *and* $x = 1$ *and the graphs* $y = \pm M_x$ $(0 \leqq x \leqq 1)$ *and, in addition,* $\operatorname{sp}(A)$ *contains the graph of the two functions* $y = \pm M_x$ *on* $[0, 1]$, *the vertical segments of the lines* $x = 0$ *and* $x = 1$ *joining these two graphs, and also the interval* $[0, 1]$.

THEOREM 2. *Let* $\phi(x)$ *satisfy* (1.1) *and suppose in addition that*

$$|\phi(x)| > \text{const} > 0 \text{ on some open interval } (a, b) \subset [0, 1]. \tag{1.10}$$

Then every point of (a, b) *is an interior point of* $\operatorname{sp}(A)$; *that is, if* $a < c < b$, *then there exists a disk centred at* c *and lying in* $\operatorname{sp}(A)$. *Moreover, there exist operators* A *satisfying*

$$|\phi(x)| > 0 \quad \text{on } [0, 1] \tag{1.11}$$

and for which $\operatorname{sp}(A)$ *has no interior points whatever* (*so that, in particular,* (1.10) *must be violated*).

THEOREM 3. *Let* $\phi(x)$ *satisfy* (1.1) *and suppose that* $\phi(x)$ *is continuous at some point* c, $0 < c < 1$. *Then the vertical segment consisting of points* $c + iy$, $-|\phi(c)|^2 \leqq y \leqq |\phi(c)|^2$, *is contained in* $\operatorname{sp}(A)$ *and moreover, if* $\phi(c) \neq 0$, *all points* $c + iy$, *where* $-|\phi(c)|^2 < y < |\phi(c)|^2$, *are interior points of* $\operatorname{sp}(A)$.

It is seen that if $\phi(x)$ merely satisfies (1.1), but $\phi(x) \not\equiv 0$ a.e., the assertion of Theorem 1 does not assure that $\operatorname{sp}(A)$ has positive measure. On the other hand, this assertion does follow from Theorem 3 if it is also assumed that $\phi(x)$ has at least one continuity point c with $\phi(c) \neq 0$, or, from Theorem 2, if only (1.10) holds.

THEOREM 4. *Let* $\phi(x)$ *satisfy* (1.1) *and for each* $a > 0$ *let*

$$E_a = \{x \in [0, 1]\colon |\phi(x)| > a\}, \qquad a > 0. \tag{1.12}$$

Suppose that $[0, 1]$ *is the union of disjoint sets* M *and* N; *thus*

$$[0, 1] = M + N, \tag{1.13}$$

where M *is an open set with the property that*

$$E_a \text{ is nowhere dense on } M \text{ for every } a > 0, \tag{1.14}$$

and almost all points of the set N are continuity points of $\phi(x)$*; thus*

(1.15) $$N \subset \{x \in [0, 1]: \phi(x) \text{ continuous}\} + \text{a zero set.}$$

Then

(1.16) $$2 \int_0^1 |\phi(t)|^2 \, dt \leqq \text{meas sp}(A).$$

If, in addition,

(1.17) $$|\phi(x)|^2 = M_x \text{ a.e. on } [0, 1],$$

then equality holds in (1.16), *and in* (1.8), *so that the spectrum of the operator* A *on* $L^2(0, 1)$ *defined by* (1.1), (1.2), *and* (1.4) *is given by*

(1.18) $$\text{sp}(A) = \{x + iy: 0 \leqq x \leqq 1, -M_x \leqq y \leqq M_x\}.$$

Since $\pi(AA^* - A^*A) = 2\pi C$, where C is given by (1.3), and since

(1.19) $$2\pi||C|| = 2\textstyle\int_0^1 |\phi(t)|^2 \, dt,$$

it is seen that in general

(1.20) $$\pi||AA^* - A^*A|| \leqq \text{Area}\{x + iy: 0 \leqq x \leqq 1, -M_x \leqq y \leqq M_x\}.$$

Thus, under the hypotheses (1.13)–(1.15) only of Theorem 4, the inequality (1.16), hence (1.6) with $T = A$, holds. Further, if (1.17) is also assumed, then the asserted equality in (1.16) implies equality in (1.6). Whether relation (1.18) must hold for ϕ arbitrary, that is, for ϕ satisfying only (1.1), will remain undecided. However, the following result holds.

COROLLARY OF THEOREM 4. *If* ϕ *is Riemann integrable on* [0, 1], *then* (1.18) *holds.*

The proof of the corollary follows from the observation that the sets M and N can now be chosen to be the empty set and [0, 1], respectively.

2. Some Lemmas.

LEMMA 1. *Let* T *be any bounded operator satisfying* (1.5) *and let* $T = H + iJ$, *where* H *and* J *are self-adjoint. Then the spectra of* H *and* J *are precisely the projections of the spectrum of* T *onto the x-axis and y-axis* (*regarded as real lines*).

Proof. See [**4**, p. 46].

Before formulating the second lemma, some terminology will be needed. Let the spectral resolution of the (arbitrary) self-adjoint operator H be given by

(2.1) $$H = \int \lambda \, dE_\lambda,$$

and for any open interval Δ and bounded operator R on the underlying Hilbert space $\mathfrak{H}$ put $R_\Delta = E(\Delta)RE(\Delta)$. Then if J is also self-adjoint and if

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$HJ - JH = iC$, one has $H_\Delta J_\Delta - J_\Delta H_\Delta = iC_\Delta$, so that if $T = H + iJ$ is semi-normal, so also is T_Δ; cf. [**4**, p. 49]. (Here and below, operators of the form R_Δ are regarded as operators on the Hilbert space $E(\Delta)\mathfrak{H}$.)

LEMMA 2. *Let T satisfy* (1.5) *and let the real number c belong to the open interval Δ. Then, for t real, we have:*

$$\begin{aligned} \sup\{t: c + it \in \operatorname{sp}(T)\} &= \sup\{t: c + it \in \operatorname{sp}(T_\Delta)\}, \\ \inf\{t: c + it \in \operatorname{sp}(T)\} &= \inf\{t: c + it \in \operatorname{sp}(T_\Delta)\}. \end{aligned} \tag{2.2}$$

Proof. It will be clear that it is sufficient to prove only the first relation of (2.2). Suppose that $\operatorname{sp}(T)$ contains some point of the form $c + it$ and let $d = \sup\{t: c + it \in \operatorname{sp}(T)\}$. Since $c + id$ is a boundary point of $\operatorname{sp}(T)$ (cf. [**4**, p. 47]), there exists a sequence of unit vectors $\{x_n\}$ satisfying

$$(H - cI)x_n \to 0 \quad \text{and} \quad (J - dI)x_n \to 0, \quad \text{as } n \to \infty. \tag{2.3}$$

Since Δ is open, c is an interior point of Δ and the first relation of (2.3) implies that $x_n - E(\Delta)x_n \to 0$, so that without loss of generality it can be supposed that $x_n = E(\Delta)x_n$ for all n. It follows from (2.3) that $(H_\Delta - cI_\Delta)x_n \to 0$ and $(J_\Delta - dI_\Delta)x_n \to 0$, so that $c + id \in \operatorname{sp}(T_\Delta)$. Thus, if

$$e = \sup\{t: c + it \in \operatorname{sp}(T_\Delta)\},$$

then $d \leqq e$.

Similarly, if $\operatorname{sp}(T_\Delta)$ contains some point of the form $c + it$, then $c + ie$ is a boundary point of $\operatorname{sp}(T_\Delta)$ and there exist unit vectors $x_n = E(\Delta)x_n$ satisfying

$$(H - cI)x_n \to 0, \qquad E(\Delta)(J - eI)x_n \to 0. \tag{2.4}$$

Since $H_c J_e - J_e H_c = iC$ (where, for any operator R, $R_\lambda \equiv R - \lambda I$), then $(H_c J_e x_n, x_n) - (J_e H_c x_n, x_n) = i(Cx_n, x_n)$, which, by (2.4), tends to 0 as $n \to \infty$. Since $C \geqq 0$ or $C \leqq 0$, then $Cx_n \to 0$. Hence, $H_c y_n \to 0$, where $y_n = (J - eI)x_n$. Since c is an interior point of Δ, it now follows from the second relation of (2.4) (cf. [**4**, p. 54]) that $y_n \to 0$, so that both $(H - cI)x_n \to 0$ and $(J - eI)x_n \to 0$. Hence $c + ie \in \operatorname{sp}(T)$ and, in particular, $e \leqq d$. This completes the proof of Lemma 2.

LEMMA 3. *As above, let $T = H + iJ$ satisfy* (1.5). *Suppose that $a < d < b$, where a and b denote the left and right end points, respectively, of the set* $\operatorname{sp}(H)$, *and let Δ denote either of the intervals $[a, d]$ or $[d, b]$. Let $c = a$ or $c = b$ and, for a real t, suppose that $c + it \in \operatorname{sp}(T_\Delta)$. Then also*

$$c + it \in \operatorname{sp}(T). \tag{2.5}$$

Proof. Since $c + it$ is a boundary point of $\operatorname{sp}(T_\Delta)$ (cf. Lemma 1), there exists a sequence of unit vectors $x_n = E(\Delta)x_n$ satisfying

$$(H - cI)x_n \to 0 \quad \text{and} \quad E(\Delta)(J - tI)x_n \to 0. \tag{2.6}$$

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As in the proof of Lemma 2,

$$(H - cI)(J - tI)x_n \to 0. \tag{2.7}$$

Since c is an end point of $\text{sp}(H)$, it follows from (2.7) and the second relation of (2.6) that $(J - tI)x_n \to 0$. Thus both $(H - cI)x_n \to 0$ and $(J - tI)x_n \to 0$, and hence (2.5) holds.

LEMMA 4. *Let* $\{T_n\}$ *denote a sequence of bounded operators on a Hilbert space* H *which is strongly convergent to* T, $T_n \to T$; *thus*

$$T_n x \to Tx \quad \textit{for all } x \textit{ in } \mathfrak{H}. \tag{2.8}$$

Suppose that $a_n \in \text{sp}(T_n)$ *and that* $a_n \to a$ *as* $n \to \infty$. *In addition, suppose that there is a constant* $\delta > 0$ *for which*

$$(T_n - a_n I)^*(T_n - a_n I) \geqq \delta I \quad \textit{for all } n. \tag{2.9}$$

Finally, suppose that $T_n^* T_n - T_n T_n^* = C_n$ *is completely continuous and that the sequence* $\{C_n\}$ *converges in the uniform norm topology to a (completely continuous) operator* C; *thus*

$$||C_n - C|| \to 0. \tag{2.10}$$

Then

$$a \in \text{sp}(T) \quad (\textit{in fact, } \bar{a} \textit{ is in the point spectrum of } T^*). \tag{2.11}$$

Proof. Since $a_n \in \text{sp}(T_n)$ and (2.9) holds, then there exist unit vectors x_n such that $(T_n - a_n I)^* x_n \to 0$. Since $\{x_n\}$ is a bounded sequence, it has a weakly convergent subsequence, which will also be denoted by $\{x_n\}$, thus there exists a vector x such that $x_n \xrightarrow{w} x$. For any fixed vector y, $T_n y \to Ty$, and hence

$$((T_n - a_n I)^* x_n, y) = (x_n, (T_n - a_n I)y) \to (x, (T - aI)y) = ((T - aI)^* x, y),$$

that is, $(T_n - a_n I)^* x_n \xrightarrow{w} (T - aI)^* x$, and hence $(T - aI)^* x = 0$. If $x \neq 0$, then (2.11) holds, and the proof is complete.

Suppose then, if possible, that $x = 0$, so that $x_n \xrightarrow{w} 0$. However,

$$(T_n - a_n I)^*(T_n - a_n I)x_n - (T_n - a_n I)(T_n - a_n I)^* x_n = C_n x_n,$$

and since $||C_n - C|| \to 0$ (and C is completely continuous), it follows that $C_n x_n \to 0$. Since $(T_n - a_n I)^* x_n \to 0$, then also $(T_n - a_n I)x_n \to 0$, in contradiction to (2.9). This completes the proof of Lemma 4.

By the *essential spectrum*, es sp(T), of a bounded operator T is meant the complement of the set of complex numbers c for which the range of $T - cI$ is closed and the null spaces of $T - cI$ and $(T - cI)^*$ are both finite-dimensional. (Cf. [**10**; **7**; **6**].) If $B(\mathfrak{H})$ denotes the Banach space of bounded operators on the Hilbert space $\mathfrak{H}$ and if $\mathbf{C}$ denotes the ideal of completely continuous operators on $\mathfrak{H}$, then it is known that the essential spectrum of an operator T on $\mathfrak{H}$ coincides with the spectrum of its natural, homomorphic image in the factor algebra $\mathbf{B}(\mathfrak{H})/\mathbf{C}$; cf. [**1**; **7**].

LEMMA 5. *Let T be a semi-normal operator, so that* (1.5) *holds, with no non-trivial reducing subspace on which T is normal, and suppose also that $TT^* - T^*T = C$ is completely continuous. Then*

$$\text{es sp}(T) = \{c: T_cT_c^* \text{ and } T_c^*T_c \text{ both singular}\}, \tag{2.12}$$

where $T_c \equiv T - cI$.

Proof. Let $c \in \text{es sp}(T)$. Then both $T_cT_c^*$ and $T_c^*T_c$ must be singular. Otherwise, let us say that $T_cT_c^* \geqq \delta I$, where $\delta = \text{const} > 0$. (A similar argument works if $T_c^*T_c \geqq \delta I$.) Then there exists a bounded operator B such that $BT_cT_c^* = T_cT_c^*B = I$ and, since $T_cT_c^* - T_c^*T_c = C$, $BT_c^*T_c = I + D$, where D is completely continuous. Since $T_c^*T_cB^* = I + D^*$, T_c has both a right and left inverse modulo **C** and hence c is not in es sp(T), a contradiction. (The above holds even if T is not semi-normal.)

Next, suppose that both $T_cT_c^*$ and $T_c^*T_c$ are singular. Since T is semi-normal, then $T_cT_c^* - T_c^*T_c$ is semi-definite and it will be clear that there is no loss of generality in supposing that $T_c^*T_c \geqq T_cT_c^*$. There exists a sequence of unit vectors $\{x_n\}$ such that $T_cx_n \to 0$. Since $\{x_n\}$ is bounded, it has a weakly convergent subsequence, which will be denoted by $\{x_n\}$, thus $x_n \xrightarrow{w} x$. But if $x = 0$, then T_c cannot have a left inverse modulo C; for if $BT_c = I + E$, where E is completely continuous, then $x_n \to 0$ (strongly), a contradiction, since the x_n are unit vectors. Thus $c \in \text{es sp}(T)$. On the other hand, if $x \neq 0$, then $T_cx = 0$ and, since $T_c^*T_c \geqq T_cT_c^*$, also $T_c^*x = 0$. Thus T would have a normal reducing subspace, a contradiction.

LEMMA 6. *Let T and T_n $(n = 1, 2, \ldots)$ denote bounded operators on a Hilbert space and suppose that*

$$||T_n - T|| \to 0. \tag{2.13}$$

Then

$$\text{meas sp}(T) \geqq \limsup_{n\to\infty} (\text{meas sp}(T_n)). \tag{2.14}$$

Proof. Let M denote a bounded open set containing all sets sp(T_n) and the (closed) set sp(T) and let M_1 denote the open set $M_1 = M - \text{sp}(T)$. Next, let N be any closed subset of M_1, so that, in particular, N lies at a positive distance from the boundary of M_1. Then

$$\text{sp}(T_n) \cap N \quad \text{is empty for } n \text{ sufficiently large.} \tag{2.15}$$

For, otherwise, there would exist $c_{nk} \in \text{sp}(T_{nk}) \cap N(n_1 < n_2 < \ldots)$, and hence a subsequence $\{m_k\}$ of $\{n_k\}$ such that $c = \lim c_{mk}$ exists. Since N is closed, $c \in N$ and, since $c_{mk} \in \text{sp}(T_{mk})$ and $||T_{mk} - T|| \to 0$, also $c \in \text{sp}(T)$. Thus sp$(T) \cap N$ is not empty, which is impossible. Next, let $\epsilon > 0$ and choose $N = N_\epsilon$ so that meas$(M_1 - N_\epsilon) < \epsilon$. Since sp$(T) = M - M_1$ and since, by (2.15), sp$(T_n) \subset M - N_\epsilon$ for n sufficiently large, it is clear that (2.14) holds.

3. Proof of Theorem 1. First, some properties of the self-adjoint singular integral operator J defined in (1.2) will be noted. If M is defined by

$$M = \operatorname*{ess\,sup}_{[0,1]} |\phi(x)|^2, \tag{3.1}$$

then $\operatorname{sp}(T) = [-M, M]$; cf. [**3**; **5**]. Let $A = H + iJ$ now be defined by (1.1), (1.2), and (1.4), let c be any point of (0, 1), and let Δ denote any open (half-open if $c = 0$ or 1) subinterval of (0, 1) containing c. Then, in the notation following the statement of Lemma 2, consider the (semi-normal) operator A_Δ. It now follows from Lemma 1 that there exist points z_1 and z_2 in $\operatorname{sp}(A_\Delta)$ such that $\operatorname{Re}(z_1)$, $\operatorname{Re}(z_2)$ belong to the closure Δ' of Δ and $\operatorname{Im}(z_1) = M_\Delta$ and $\operatorname{Im}(z_2) = -M_\Delta$, where

$$M_\Delta = \operatorname*{ess\,sup}_{\Delta'} |\phi(x)|^2.$$

According to Lemma 2, both z_1 and z_2 also belong to $\operatorname{sp}(A)$. On letting Δ shrink to the point c, and noting that c is arbitrary in (0, 1), it follows that $\{z = x \pm iM_x : 0 \leqq x \leqq 1\}$ belongs to $\operatorname{sp}(A)$, with M_x defined by (1.7). Further, it is clear from Lemma 1 and from (2.2) of Lemma 2 that (1.8) holds.

That the set $\{z = iy : -M_0 \leqq y \leqq M_0\}$ occurring in (1.9) also belongs to $\operatorname{sp}(A)$ follows from Lemma 3. To see this, let $\Delta_\epsilon = [0, \epsilon]$, where $\epsilon > 0$. Then if t is any point in $[-m_\epsilon, m_\epsilon]$, where

$$m_\epsilon = \operatorname*{ess\,sup}_{\Delta_\epsilon} |\phi(x)|^2,$$

there exists some $z \in \operatorname{sp}(A_\Delta)$, hence, by Lemma 3, also $z \in \operatorname{sp}(A)$, satisfying $\operatorname{Im}(z) = t$. By Lemma 1, $\operatorname{Re}(z) \in \Delta_\epsilon$. Since $[-M_0, M_0] \subset [-m_\epsilon, m_\epsilon]$, it is clear that if $-M_0 \leqq t \leqq M_0$, there exist $z_n = c_n + it \in \operatorname{sp}(A)$ with $c_n \to 0$. Hence $it \in \operatorname{sp}(A)$, and thus the set $\{z = iy : -M_0 \leqq y \leqq M_0\}$ belongs to $\operatorname{sp}(A)$. A similar argument shows that the set $\{z = 1 + iy : -M_1 \leqq y \leqq M_1\}$ also belongs to $\operatorname{sp}(A)$.

It remains to be proved that

$$[0, 1] \subset \operatorname{sp}(A). \tag{3.2}$$

The argument will be somewhat complicated and another lemma will be needed before continuing with the proof.

LEMMA 7. *Suppose that*

$$\phi(x) \text{ is of class } C^2 \text{ and that } |\phi(x)| > 0 \text{ on } [0, 1], \tag{3.3}$$

and let A be defined by (1.1), (1.2), *and* (1.4). *Then*

$$\begin{aligned} \text{es sp}(A) = \{z = x \pm i|\phi(x)|^2 : 0 \leqq x \leqq 1\} \\ + \{z = iy : -|\phi(0)|^2 \leqq y \leqq |\phi(0)|^2\} \\ + \{z = 1 + iy : -|\phi(1)|^2 \leqq y \leqq |\phi(1)|^2\}. \end{aligned} \tag{3.4}$$

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Also

$$\text{(3.5)} \qquad \text{sp}(A) = \{x + iy \colon 0 \leqq x \leqq 1, \ -|\phi(x)|^2 \leqq y \leqq |\phi(x)|^2\},$$

that is, sp(A) *is the region between the graphs* $y = \pm|\phi(x)|^2$ *over* $0 \leqq x \leqq 1$ *and between the lines* $x = 0$ *and* $x = 1$.

Proof. The assertion (3.4) is essentially a result of Schwartz [**7**]. Actually, the singular integral operators considered by Schwartz are Cauchy principal values of the form

$$\lim_{\epsilon \to 0} \int (\ldots)(t + i\epsilon - x)^{-1}\, dt,$$

but his methods involving the theory of commutative Banach algebras apply equally well (and almost verbatim) in the present case to yield (3.4). The details will be omitted. (Relation (3.4) holds even if C^2 is replaced by C^0.)

It is clear from (3.3) that the set of (3.4) is a piecewise C^2 simple closed curve. The assertion of (3.5) is that sp(A) consists of this curve together with its interior. Since A is semi-normal, its spectrum cannot be precisely the above simple closed curve unless A is normal; see [**8**; **9**; **11**]. However, if A is normal, then clearly $\phi(x) \equiv 0$, in contradiction with the second relation of (3.3).

Consequently, there exists at least one point of sp(A) in the interior of the curve (3.4). Since $|\phi(x)| > 0$, it is clear that A has no non-trivial normal reducing subspaces. (In fact, it follows from the Weierstrass approximation theorem that $L^2(0, 1)$ is the least space containing the range of C in (1.3) and (1.4) and which is invariant under the multiplication operator $H = x$.) Hence, by Lemma 5, relation (2.12) (with $T = A$) holds. But (even for any bounded operator) at any boundary point c of sp(A), both $A_c^*A_c$ and $A_cA_c^*$ (where $A_c \equiv A - cI$) are singular. It now follows that the entire interior of the set (3.4) is in sp(A), that is, relation (3.5) holds, and the proof of Lemma 7 is complete.

Next, consider the finite interval Hilbert transform Q defined on $L^2(0, 1)$ by

$$\text{(3.6)} \qquad (Qf)(x) = (i\pi)^{-1} \int_0^1 (t - x)^{-1} f(t)\, dt.$$

For any $a > 0$, define $\phi_a(x)$ on $[0, 1]$ by

$$\text{(3.7)} \qquad \phi_a(x) = \begin{cases} \phi(x) & \text{if } |\phi(x)| \geqq a, \\ a & \text{if } |\phi(x)| < a, \end{cases}$$

and let the self-adjoint singular integral operator J_a be defined by

$$\text{(3.8)} \qquad (J_a f)(x) = (i\pi)^{-1} \int_0^1 \phi_a(x) \bar{\phi}_a(t)(t - x)^{-1} f(t)\, dt.$$

It is clear that

$$(3.9)\qquad Jf - J_af = \phi Q(\bar{\phi} f) - \phi_a Q(\bar{\phi}_a f) = (\phi - \phi_a)Q(\bar{\phi} f) + \phi_a Q((\bar{\phi} - \bar{\phi}_a)f),$$

and hence

$$||(J - J_a)f|| \leqq 2a \sup_{[0,1]} |\phi(x)|\; ||f||,$$

so that, in particular, $||J - J_a|| \to 0$ as $a \to 0$. Hence, if $A_a = H + iJ_a$, H being the multiplication operator x on $L^2(0, 1)$, then

$$(3.10)\qquad ||A_a - A|| \to 0 \quad \text{as } a \to 0.$$

It will be shown that for each $a > 0$,

$$(3.11)\qquad [0, 1] \subset \mathrm{sp}(A_a).$$

Relation (3.2) then follows from (3.10) and (3.11).

In order to prove (3.11), let $0 \leqq c \leqq 1$ and, as above, let $A_a = H + iJ_a$, where J_a is defined by (3.8). Then if $B_a = A_a - cI$ one has (cf. (1.4))

$$(3.12)\qquad B_a B_a^* = (H - cI)^2 + J_a^2 + C_a,$$

where

$$(3.13)\qquad (C_a f)(x) = \pi^{-1} \int_0^1 \phi_a(x)\bar{\phi}_a(t)d(t)\, dt.$$

Now suppose that $c \notin \mathrm{sp}(A_a)$, so that, in particular,

$$(3.14)\qquad (A_a - cI)(A_a - cI)^* \geqq \delta I$$

holds for some constant $\delta > 0$.

Next, define $A_n = H + iJ_n$ ($n = 1, 2, \ldots$), where H is as given in (1.2) and

$$(3.15)\qquad (J_n f)(x) = (i\pi)^{-1} \int_0^1 \phi_n(x)\bar{\phi}_n(t)(t - x)^{-1} f(t)\, dt,$$

where the ϕ_n are defined below. These functions ϕ_n should satisfy

$$(3.16)\qquad \phi_n \in C^2 \text{ on } [0, 1], \quad a \leqq |\phi_n(x)| \leqq \text{const}, \quad \phi_n(x) \to \phi_a(x) \text{ a.e.}$$

In view of (3.16), one has

$$(3.17)\qquad A_n f \to A_a f, \quad A_n^* f \to A_a^* f, \quad \text{as } n \to \infty, \qquad \text{for each } f \in L^2(0, 1),$$

that is, the sequences $\{A_n\}$ and $\{A_n^*\}$ converge strongly to A and A^*, respectively. This can be seen from a relation similar to (3.9). In fact, $(A_n - A_a)f = i(J_n - J_a)f$ and

$$(J_n - J_a)f = (\phi_n - \phi_a)Q(\bar{\phi}_n f) + \phi_a Q((\bar{\phi}_n - \bar{\phi}_a)f).$$

Using $(a + b)^2 \leqq 2(a^2 + b^2)$, one obtains

$$(3.18) \quad \int_0^1 |(J_n - J_a)f|^2\,dt \leqq 2\int_0^1 |\phi_n - \phi_a|^2 |Q(\bar{\phi}_n f)|^2\,dt + 2\int_0^1 |\phi_a|^2 |Q((\bar{\phi}_n - \bar{\phi}_a)f)|^2\,dt.$$

The second integral on the right of the inequality is majorized by

$$\text{const}||Q((\bar{\phi}_n - \bar{\phi}_a)f)||^2 \leqq \text{const}||(\bar{\phi}_n - \bar{\phi}_a)f||^2. \text{ But } \phi_n \to \phi_a \text{ a.e.}$$

and also $|(\bar{\phi}_n - \bar{\phi}_a)f|^2 \leqq \text{const}|f|^2$, and hence by Lebesgue's dominated convergence theorem $||(\bar{\phi}_n - \bar{\phi}_a)f|| \to 0$ as $n \to \infty$. Thus, in order to prove (3.17), it is sufficient to show that the first integral on the right of the inequality (3.18) also tends to 0. But this integral equals

$$(3.19) \quad \int_0^1 |\phi_n - \phi_a|^2(|h_n|^2 - |h|^2)\,dt + \int_0^1 |\phi_n - \phi_a|^2 |h|^2\,dt,$$

where $h_n = Q(\bar{\phi}_n f)$ and $h = Q(\bar{\phi}_a f)$. Since $||\bar{\phi}_n f - \bar{\phi}_a f|| \to 0$, then $||h_n - h|| \to 0$. Since $|\phi_n - \phi_a| \leqq$ const, the first integral of (3.19) tends to 0. That the last integral of (3.19) also tends to 0 follows from another application of the dominated convergence theorem, and thus (3.17) is now established.

Next, it will be shown that there exists a constant $\gamma > 0$ such that

$$(3.20) \quad (A_n - cI)(A_n - cI)^* \geqq \gamma I.$$

By a relation similar to (3.12),

$$(3.21) \quad ||(A_n - cI)^* f||^2 = ||(H - cI)f||^2 + ||J_n f||^2 + (C_n f, f), \qquad f \in L^2(0, 1),$$

where $(C_n f)(x) = \pi^{-1}\phi_n(f, \phi_n)$. Let g_n be defined by

$$(3.22) \quad g_n = (\bar{\phi}_n/\bar{\phi}_a)f.$$

Then $J_n f = \phi_n Q(\bar{\phi}_n f)$ and $J_a g_n = \phi_a Q(\bar{\phi}_a g_n)$, so that by (3.22),

$$(3.23) \quad J_n f = (\phi_n/\phi_a)J_a g_n.$$

Now $C_n f = \pi^{-1}\phi_n(f, \phi_n)$ and $C_a f = \pi^{-1}\phi_a(f, \phi_a)$, and hence

$$(C_n f, f) = \pi^{-1}|(f, \phi_n)|^2 = \pi^{-1}|(g_n, \phi_a)|^2 = (C_a g_n, g_n).$$

Consequently,

$$||(A_n - cI)^* f||^2 = \int (x - c)^2 |\bar{\phi}_a/\bar{\phi}_n|^2 |g_n|^2\,dt + \int |\phi_n/\phi_a|^2 |J_a g_n|^2\,dt + (C_a g_n, g_n).$$

Furthermore,

$$(3.24) \quad ||(A_a - cI)^* g_n||^2 = \int (x - c)^2 |g_n|^2\,dt + \int |J_a g_n|^2\,dt + (C_a g_n, g_n).$$

Since $0 < a \leqq |\phi_a(x)|$, $|\phi_n(x)| \leqq$ const, it is clear that there exist positive constants α, β, where $0 < \alpha < 1 < \beta$, such that, for all x, one has

$$\alpha \leqq |\phi_a(x)/\phi_n(x)|, \; |\phi_n(x)/\phi_a(x)| \leqq \beta,$$

and hence

$$(3.25)\qquad \alpha^2||(A_a - cI)^*g_n||^2 \leqq ||(A_n - cI)^*f||^2 \leqq \beta^2||(A_a - cI)^*g_n||^2.$$

In view of (3.22),

$$(3.26)\qquad \alpha||f|| \leqq ||g_n|| \leqq \beta||f||.$$

Relations (3.25), (3.26), and (3.14) now yield (3.20).

As noted above,

$$\pi(C_nf - C_af) = \phi_n(f, \phi_n) - \phi_a(f, \phi_a) = \phi_n(f, \phi_n - \phi_a) + (\phi_n - \phi_a)(f, \phi_a),$$

and hence $\pi||C_nf - C_af|| \leqq \text{const}||\phi_n - \phi_a|| \, ||f||$. Thus $||C_n - C|| \to 0$ as $n \to \infty$. Now, for $0 \leqq c \leqq 1$, it follows from (3.5) of Lemma 7 (with ϕ, A replaced by ϕ_n, A_n) that $c \in \text{sp}(A_n)$, hence $\bar{c} = c \in \text{sp}(A_n^*)$. Thus Lemma 4 can be applied, with the T_n, T, C_n, and C corresponding to the present A_n^*, A_a^*, $2C_n$, and $2C$, respectively, and $a_n = c$. It follows that $c \in \text{sp}(A_a)$. Thus the assumption that $c \in \text{sp}(A_a)$ implies (3.14), which, in turn, implies that $c \in \text{sp}(A_a)$, a contradiction. Thus, it follows that, indeed, $c \in \text{sp}(A_a)$ and, since c is arbitrary in [0, 1], also $[0, 1] \subset \text{sp}(A_a)$. As noted earlier, (3.2) follows, and the proof of Theorem 1 is now complete.

4. Proof of Theorem 2. Suppose that $0 < c < 1$ and that $|\phi(x)| \geqq \text{const} > 0$ on (a, b), where $a < c < b$. Then it will be shown that there exists a disk about c lying in $\text{sp}(A)$. (That c lies in $\text{sp}(A)$ was proved in Theorem 1 above.) Suppose the contrary, so that c is a boundary point of $\text{sp}(A)$, and hence

$$(4.1)\qquad (A - cI)^*g_n \to 0$$

holds for a sequence of unit vectors $\{g_n\}$. A relation similar to (3.21) holds for $||(A - cI)^*g_n||^2$; thus

$$(4.2)\qquad ||(A - cI)^*g_n||^2 = \int (x - c)^2|g_n|^2\,dt + \int |Jg_n|^2\,dt + \pi^{-1}|(g_n, \phi)|^2.$$

By (4.1) and (4.2) it follows that $\int (x - c)^2|g_n|^2\,dt \to 0$. Hence

$$\int_{[0,1]-(a,b)} |g_n|^2\,dt \to 0,$$

and hence there is no loss of generality in supposing that $g_n = 0$ outside (a, b).

Now, in place of the sequence of operators A_n of § 3, consider the single operator B where

$$(4.3)\qquad (Bf)(x) = xf(x) + i(J_\psi f)(x),$$
$$(J_\psi f)(x) = (i\pi)^{-1}\int_0^1 \psi(x)\bar{\psi}(t)(t - x)^{-1}f(t)\,dt,$$

and where $\psi(x)$ is of class C^2 on [0, 1] and $|\psi(x)| \geqq \text{const} > 0$ on [0, 1].

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(For instance, one could take $\psi(x) \equiv 1$.) Corresponding to (3.22), define now f_n by

$$f_n = (\bar{\phi}/\bar{\psi})g_n. \tag{4.4}$$

Clearly, $(f_n, \psi) = (g_n, \phi)$, and hence

$$||(B - cI)^*f_n||^2 = \int (x - c)^2|f_n|^2\, dt + \int |J_\psi f_n|^2\, dt + \pi^{-1}|(g_n, \phi)|^2. \tag{4.5}$$

Since all terms on the right of (4.2) tend to 0 as $n \to \infty$, it is clear (in view of (4.4)) that at least the first and third terms on the right of (4.5) also tend to 0. But (cf. (3.23)) $Jg_n = (\phi/\psi)J_\psi f_n$ and hence, since $|\psi| \leqq$ const on [0, 1], we have

$$\int |\phi|^2|J_\psi f_n|^2\, dt \to 0, \qquad n \to \infty. \tag{4.6}$$

However,

$$\int_{[0,1]-(a,b)} |J_\psi f_n|^2\, dt \to 0. \tag{4.7}$$

To see this, recall that $(H - cI)g_n \to 0$ ($H = x$ on $L^2(0, 1)$), and hence $(H - cI)f_n \to 0$. But (cf. (1.3)), $(H - cI)J_\psi - J_\psi(H - cI) = iC_\psi$, where $(C_\psi f)(x) = \pi^{-1}\psi(f, \psi)$. In particular, $C_\psi \geqq 0$. Since

$$((H - cI)J_\psi f_n, f_n) - (J_\psi(H - cI)f_n, f_n) = i||C_\psi^{\frac{1}{2}}f_n||^2 \to 0,$$

it is clear that $(H - cI)J_\psi f_n \to 0$, that is, $\int (x - c)^2|J_\psi f_n|^2\, dt \to 0$. Since $c \in (a, b)$, (4.7) follows. It now follows from (4.6), (4.7), and (1.10) that $||J_\psi f_n|| \to 0$. Thus, by (4.5),

$$||(B - cI)^*f_n|| \to 0, \tag{4.8}$$

where B is defined by (4.3). It is clear that $0 <$ const $\leqq ||f_n|| \leqq$ const. Since $(B - cI)(B - cI)^* \geqq (B - cI)^*(B - cI)$ (cf. (1.4)), it follows from Lemma 5 that c is in the essential spectrum of B. But, since ψ is of class C^2, and since $|\psi(x)| \geqq$ const > 0 on [0, 1], in particular on (a, b), this is impossible by Lemma 7 (with ϕ replaced by ψ), a contradiction. Thus, the hypothesis that c is a boundary point of sp(A) in untenable. Thus, since c belongs to sp(A), c is an interior point of sp(A), as was to be shown.

5. An example. In this section, the proof of Theorem 2 will be completed by giving an example of a function $\phi(x)$ satisfying (1.11) for which the spectrum of the associated operator A defined by (1.1), (1.2), and (1.4) has no interior points.

Let $0 < \beta < 1$ and define a Cantor set C_1 (nowhere dense perfect set) of positive measure on [0, 1] by putting $C_1 = [0, 1] - \sum I_n$, where $\{I_n\}$ is a sequence of open intervals defined as follows. Let I_1 be an open interval of length $|I_1| = \frac{1}{2}\beta$ and in the centre of [0, 1]. Next, remove open intervals I_2 and I_3 from the centre of each of the two remaining closed intervals and suppose that $|I_2| + |I_3| = \frac{1}{4}\beta$. Similarly, remove open intervals I_4, I_5, I_6, and I_7 of total length $\beta/8$ from the centre of each of the remaining four (closed) intervals. Continuing in this way, one obtains a Cantor set C_1 of measure $|C_1| = 1 - \beta$. Next, one proceeds with a similar construction on each of the removed intervals $I_2, I_3, \ldots$. (Actually these intervals are open, but this is of no consequence.) This procedure is repeated indefinitely in such a way that the sum of the measures of the Cantor sets thereby constructed is 1. One thus obtains a sequence of disjoint nowhere dense sets $\{C_n\}$ $(n = 1, 2, \ldots)$ such that $[0, 1] = \sum C_n + Z$, where Z is a set of measure 0 and C_n and Z are pairwise disjoint. Then define ϕ (a.e.) by

$$\phi(x) = 1/n \quad \text{on } C_n \qquad (n = 1, 2, \ldots). \tag{5.1}$$

If E_a is defined by (1.12), it is clear that E_a, being the union of a finite number of the C_ns, is nowhere dense on [0, 1]. That $\mathrm{sp}(A)$ cannot have any interior points follows readily from (1.8).

It is interesting to observe that (1.8) can be improved in this case to the assertion

$$\mathrm{sp}(A) = \{x + iy: 0 \leqq x \leqq 1, -M_x \leqq y \leqq M_x\}. \tag{5.2}$$

In order to see this, define $\phi_n(x)$ on [0, 1] by

$$\phi_n(x) = \begin{cases} \phi(x) & \text{if } \phi(x) > 1/n, \\ 0 & \text{otherwise,} \end{cases} \tag{5.3}$$

and define the self-adjoint singular integral operators J_n by

$$(J_nf)(x) = (i\pi)^{-1} \int_0^1 \phi_n(x)\bar{\phi}_n(t)(t - x)^{-1}f(t)\, dt. \tag{5.4}$$

It is clear that for each fixed n, $\phi_n(x) > 0$ on $S_n = \sum_{k=1}^n C_k$ and $\phi_n(x) = 0$ a.e. on $[0, 1] - \sum_{k=1}^n C_k$, so that, in particular, J_n leaves invariant $\mathfrak{H}_n = L^2(S_n)$. Thus $J_n = J_n/\mathfrak{H}_n \oplus 0/\mathfrak{H}_n^{\perp}$. The second term denotes the 0 operator on $\mathfrak{H}_n^{\perp} = L^2(0, 1) \ominus \mathfrak{H}_n$. Since $\mathfrak{H}_n$ is obviously invariant under the multiplication operator $H = x$, then $\mathfrak{H}_n$ reduces $A_n = H + iJ_n$ and thus one has the representation

$$A_n = B_n \oplus x/\mathfrak{H}_n^{\perp}, \qquad \text{where } B_n = A_n/\mathfrak{H}_n. \tag{5.5}$$

It is clear (cf. the argument following (3.9)) that

$$||A - A_n|| \to 0, \qquad n \to \infty, \tag{5.6}$$

and that

$$\mathrm{sp}(A_n) = \mathrm{sp}(B_n) + \mathrm{sp}(x/\mathfrak{H}_n^{\perp}). \tag{5.7}$$

Since $\mathrm{sp}(\mathrm{Re}(B_n))$ is the closure of $\sum_{k=1}^n C_k$, which is nowhere dense on [0, 1], it follows from [**4**, p. 54, Theorem 3.7.1] that

$$\pi||B_nB_n^* - B_n^*B_n|| \leqq \mathrm{meas}\,\mathrm{sp}(B_n). \tag{5.8}$$

Further, it is clear that the left side of (5.8) is unchanged if B_n is replaced by A_n and, from (5.7), that meas sp(B_n) = meas sp(A_n). (Note that the measure here is that of the plane.) Consequently,

$$(5.9) \qquad \pi||A_nA_n^* - A_n^*A_n|| \leqq \text{meas sp}(A_n).$$

In view of (5.6) and Lemma 6,

$$(5.10) \qquad \pi||AA^* - A^*A|| \leqq \text{meas sp}(A).$$

Hence (cf. 1.19)),

$$(5.11) \quad \text{meas sp}(A) \geqq 2\int_0^1 |\phi(t)|^2\, dt$$
$$= \text{Area}\,\{x + iy\colon 0 \leqq x \leqq 1,\ -|\phi(x)|^2 \leqq y \leqq |\phi(x)|^2\}.$$

Since, in the present example, $M_x = |\phi(x)|^2$ a.e., (5.2) now follows from (1.8) and the fact that the set S is closed.

6. Proof of Theorem 3. Let $0 < c < 1$ and suppose that ϕ is continuous at c and that $\phi(c) \neq 0$. Then $|\phi(x)| > 0$ on some open interval containing c and, by Theorem 2, there exists a disk about c belonging to sp(A). It follows in particular that for t real and $|t|$ sufficiently small, all points $c + it$ are interior points of sp(A). It will be shown that if $d = \sup\{s\colon c + it$ are interior points of sp(A) for $0 \leqq t \leqq s\}$, then

$$(6.1) \qquad d = |\phi(c)|^2.$$

The argument with "sup" replaced by "inf" and $|\phi|^2$, $[0, s]$ replaced by $-|\phi|^2$, $[s, 0]$ is similar. Thus, in order to complete the proof of Theorem 3 it is sufficient to show that (6.1) holds.

First, it is clear that $c + id$ is in the boundary of sp(A) and (cf. (1.8)) that $d \leqq |\phi(c)|^2$. It will be shown that the supposition that (6.1) does not hold, that is, $d < |\phi(c)|^2$, leads to a contradiction. The argument will be essentially that used in § 4.

To see this, note that if $z = c + id$, then $(A - zI)(A - zI)^*$ is singular, so that there exists a sequence $\{g_n\}$ of unit vectors for which

$$(6.2) \qquad ||(A - zI)^*g_n|| \to 0.$$

Corresponding to (4.2), this becomes

$$(6.3) \quad ||(A - zI)^*g_n||^2 = \int (x - c)^2|g_n|^2\, dt + \int |(J - dI)g_n|^2\, dt + \pi^{-1}|(g_n, \phi)|^2.$$

Let $\psi(x)$ denote the constant function defined by

$$(6.4) \qquad \psi(x) \equiv \phi(c)$$

and define J_ψ and B as in (4.3) and f_n by (4.4). In view of (6.2), the first integral of (6.3) tends to 0 and it follows as in § 4 that there is no loss of generality in supposing that the $g_n(x) = 0$ outside of open intervals (a_n, b_n)

containing c and such that $b_n - a_n \to 0$ as $n \to \infty$. A simple calculation yields

$$(6.5) \qquad (J - dI)g_n = (\phi/\psi)(J_\psi - dI)f_n + d(\phi/\psi - \bar{\psi}/\bar{\phi})f_n.$$

Since $b_n - a_n \to 0$ and since ϕ is continuous at $x = c$, it can be supposed that $\phi(x) \neq 0$ on (a_n, b_n). (At points of [0, 1], where ϕ may be 0, f_n is also 0 and one interprets $(\psi/\phi)f_n$ as 0.)

By (6.3) and (6.5) and noting again that $(f_n, \psi) = (g_n, \phi)$, one has

$$(6.6) \qquad ||(A - zI)^*g_n||^2 = \int (x - c)^2|(\bar{\psi}/\bar{\phi})f_n|^2\,dt + \int |(\phi/\psi)(J_\psi - dI)f_n + d(\phi/\psi - \bar{\psi}/\bar{\phi})f_n|^2\,dt + \pi^{-1}|(f_n, \psi)|^2.$$

Since, by (4.4) and the definition of g_n, also $f_n = 0$ outside (a_n, b_n) and since $(\phi/\psi - \bar{\psi}/\bar{\phi}) = (|\phi|^2 - |\psi|^2)/\psi\bar{\phi}$, it is clear from (6.4) that

$$(6.7) \qquad \int |(\phi/\psi - \bar{\psi}/\bar{\phi})f_n|^2\,dt \to 0 \quad \text{as } n \to \infty,$$

and hence

$$(6.8) \qquad ||(A - zI)^*g_n||^2 = \int (x - c)^2|(\bar{\psi}/\bar{\phi})f_n|^2\,dt + \int|\phi/\psi(J_\psi - dI)f_n|^2\,dt + \pi^{-1}|(f_n, \psi)|^2 + o(1),$$

where $o(1)$ denotes a term which tends to 0 as $n \to \infty$. But, corresponding to (4.5),

$$(6.9) \quad ||(B - zI)^*f_n||^2 = \int (x - c)^2|f_n|^2\,dt + \int |(J_\psi - dI)f_n|^2\,dt + \pi^{-1}|(f_n, \psi)|^2.$$

The argument of § 4 shows that a relation similar to (4.7) but with J_ψ replaced by $J_\psi - dI$ holds, where now (a, b) is some fixed interval containing c on which $|\phi(x)| > \text{const} > 0$.

Relations (6.2) and (6.8) now imply that $(B - zI)^*f_n \to 0$. As in § 4, this leads to a contradiction. For, since ψ is of class C^2, this implies by Lemma 5 that z is in the essential spectrum of B. But $d < |\psi(c)|^2 = |\phi(c)|^2$ and hence this is impossible (by Lemma 7). This completes the proof of Theorem 3.

7. Proof of Theorem 4. Let $\phi_a(x)$ be defined on [0, 1] by

$$(7.1) \qquad \phi_a(x) = \begin{cases} \phi(x) & \text{if } x \in N + (M \cap E_a), \\ 0 & \text{otherwise, that is, if } x \in M - E_a. \end{cases}$$

Let J_a and A_a be defined by (1.1), (1.2), and (1.4) with ϕ replaced by ϕ_a. Since

$$(7.2) \qquad \phi_a(x) \to \phi(x) \quad \text{uniformly on } [0, 1] \quad \text{as } a \to 0,$$

it is clear (cf. § 3) that $||J_a - J|| \to 0$ as $a \to 0$, and hence

$$(7.3) \qquad ||A - A_a|| \to 0 \quad \text{as } a \to 0.$$

If $\mathfrak{H}_a = L^2(N + (M \cap E_a))$, then $\mathfrak{H}_a^\perp = L^2(M - E_a)$ and, since $\phi_a(x) = 0$ on $M - E_a$, it is clear that J_a leaves $\mathfrak{H}_a^\perp$ invariant, hence also $\mathfrak{H}_a$. Since

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$H = x$ also leaves these spaces invariant, these spaces reduce A_a. Thus, if $B_a = A_a/\mathfrak{H}_a$, then

$$A_a = B_a \oplus x/\mathfrak{H}_a{}^{\perp} \quad \text{on } L^2(0, 1) = \mathfrak{H}_a \oplus \mathfrak{H}_a{}^{\perp}. \tag{7.4}$$

Let $H = x = \int \lambda\, dE_\lambda$. If C_a is defined by (1.4) with A replaced by A_a, thus $C_a f = \pi^{-1}\phi_a(f, \phi_a)$, then it is clear (cf. (1.19)) that

$$2\pi||E(P)CE(P)|| = 2\int_P |\phi(t)|^2\, dt \tag{7.5}$$

holds for any Borel set P. Since almost all points of the set $N_a = N \cap E_a$ are continuity points of both ϕ and ϕ_a, it follows from Theorem 3 that the segment L_x: $x + iy$, with $-|\phi(x)|^2 \leqq y \leqq |\phi(x)|^2$, belongs to $\mathrm{sp}(A_a)$ for almost all x in N_a. But, by (7.4), $\mathrm{sp}(A_a) = \mathrm{sp}(B_a) + \mathrm{sp}(x/\mathfrak{H}_a{}^{\perp})$ and hence, for almost all x in N_a, L_x is also in $\mathrm{sp}(B_a)$. Thus,

$$2\int_{N_a} |\phi(t)|^2\, dt \leqq \mathrm{meas}\{z \in \mathrm{sp}(B_a)\colon \mathrm{Re}(z) \in N_a\}. \tag{7.6}$$

Since $\mathrm{sp}(\mathrm{Re}(B_a))$ is the closure of the set $N + (M \cap E_a)$ and since E_a is nowhere dense on M, it follows that the set $\{x \in \mathrm{sp}(\mathrm{Re}(B_a))\colon x \in M \cap E_a\}$ is nowhere dense on M. If $M = \sum I_n$ is the canonical decomposition of M as the union of disjoint open intervals $\{I_n\}$, then it follows from the argument in [**4**, pp. 54–55], that if $f \in \mathfrak{H}_a$ and if $C_a' = C_a/\mathfrak{H}_a$, then

$$(2\pi)^{\frac{1}{2}}||(C_a')^{\frac{1}{2}}E(I_n)f|| \leqq [\mathrm{meas}\{z\colon z \in \mathrm{sp}(B_a) \text{ and } \mathrm{Re}(z) \in I_n\}]^{\frac{1}{2}}||E(I_n)f||.$$

Since $(2\pi)^{\frac{1}{2}}||C_a'E(M)f|| \leqq (2\pi)^{\frac{1}{2}} \sum ||C_a'E(I_n)f||$, an application of the Schwarz inequality together with $||E(M)C_a'E(M)|| = ||(C_a')^{\frac{1}{2}}E(M)||^2$ yields

$$(2\pi)||E(M)C_a'E(M)|| \leqq \mathrm{meas}\{z\colon z \in \mathrm{sp}(B_a) \text{ and } \mathrm{Re}(z) \in M\}.$$

Hence, by a relation for C_a' corresponding to (7.5),

$$2\int_{M \cap E_a} |\phi(t)|^2\, dt \leqq \mathrm{meas}\{z\colon z \in \mathrm{sp}(B_a) \text{ and } \mathrm{Re}(z) \in M\}. \tag{7.7}$$

Combining (7.6) and (7.7) and noting (7.4), one obtains

$$2\int_{E_a} |\phi(t)|^2\, dt \leqq \mathrm{meas}\,\mathrm{sp}(B_a) = \mathrm{meas}\,\mathrm{sp}(A_a). \tag{7.8}$$

Relation (1.16) now follows from (7.8), (7.3), and Lemma 6.

In case (1.17) is also assumed, then $2\int_0^1 |\phi(t)|^2\, dt$ is the area of the set S in (1.8). It follows from (1.8) and (1.16) that equality must hold in (1.16) and, since S is closed, also in (1.8). This completes the proof of Theorem 4.

References

1. F. V. Atkinson, *The normal solubility of linear equations in normed spaces*, Mat. Sb. (N.S.) *28* (*70*) (1951), 3–14.

2. L. A. Coburn, *Weyl's theorem for nonnormal operators*, Michigan Math. J. *13* (1966), 285–288.

3. J. D. Pincus, *Commutators, generalized eigenfunction expansions and singular integral operators*, Trans. Amer. Math. Soc. *121* (1966), 358–377.

4. C. R. Putnam, *Commutation properties of Hilbert space operators and related topics* Ergebnisse der Math. und ihrer Grenzgebiete, Band 36 (Springer-Verlag, New York, 1967).
5. M. Rosenblum, *A spectral theory for self-adjoint singular integral operators*, Amer. J. Math. *88* (1966), 314–328.
6. M. Schechter, *Invariance of the essential spectrum*, Bull. Amer. Math. Soc. *71* (1965), 365–367.
7. J. Schwartz, *Some results on the spectra and spectral resolutions of a class of singular integral operators*, Comm. Pure Appl. Math. *15* (1962), 75–90.
8. J. G. Stampfli, *Hyponormal operators and spectral density*, Trans. Amer. Math. Soc. *117* (1965), 469–476.
9. ——— *Minimal range theorems for operators with thin spectra*, Pacific J. Math. *23* (1967), 601–612.
10. F. Wolf, *On the invariance of the essential spectrum under a change of boundary conditions of partial differential boundary operators*, Indag. Math. *21* (1959), 142–147.
11. T. Yoshino, *Spectral resolution of a hyponormal operator with the spectrum on a curve*, Tôhoku Math. J. *19* (1967), 86–97.

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