A NOTE ON A DISCRIMINANT INEQUALITY

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The following conjecture has been investigated recently by Mordell [1]. "Let $z_1, \ldots, z_n$ be any set of n complex numbers. Then

$$(1) \quad \prod_{1 \leq r,\, s \leq n;\, r \neq s} |z_r - z_s| \leq n^n \left\{ n^{-1} \sum_{r=1}^{n} |z_r|^2 \right\}^{\frac{1}{2}n(n-1)},$$

the equality sign being necessary in the case when the z's are at the vertices of a regular n-sided polygon with center at the origin." For brevity, we shall write

$$\Delta_n = \prod_{1 \leq r < s \leq n} |z_r - z_s|^2 = \prod_{1 \leq r,\, s \leq n;\, r \neq s} |z_r - z_s| .$$

By homogeneity considerations it is clear that we may assume that $\sum_{1 \leq r \leq n} |z_r|^2$ is a constant, say n. In the course of his paper, Professor Mordell verified the inequality (1), under the additional conditions

$$|z_r| = 1, \qquad (r = 1, 2, \ldots, n),$$

using the method of Lagrange multipliers. However, Dr. Erdős informs me that this result is really due to Pólya (see e.g. I. Schur, Math. Zeitschrift 1 (1918), 385). Since the proof is short, it seems worthwhile to outline the argument here. He considered the Vandermonde determinant

$$|A| = |z_j^{i-1}| = \pm \prod_{1 \leq r < s \leq n} (z_r - z_s)$$

and applied Hadamard's inequality to obtain

$$|A|^2 \leq (\sum 1^2)(\sum_{1 \leq r \leq n} |z_r|^2) \ldots (\sum_{1 \leq r \leq n} |z_r^{n-1}|^2) = n^n$$

immediately.

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To obtain some upper bound for Δ_n, we consider

$$\sum_{1\leq r,\ s\leq n;\ r\neq s} |z_r - z_s|^2$$

$$= \sum_{r\neq s} (|z_r|^2 + |z_s|^2) - \sum_{r\neq s} z_r \bar{z}_s - \sum_{r\neq s} \bar{z}_r z_s$$

$$= 2(n-1) \sum_{1\leq r\leq n} |z_r|^2 - 2(\sum_{1\leq r\leq n} z_r)(\sum_{1\leq s\leq n} \bar{z}_s)$$

$$+ 2\sum_{1\leq r\leq n} |z_r|^2$$

$$= 2n \sum_{1\leq r\leq n} |z_r|^2 - 2 |\sum_{1\leq r\leq n} z_r|^2 .$$

When expressed in this form, it is obvious that

$$\sum_{1\leq r< s\leq n} |z_r - z_s|^2 \leq n^2 .$$

Now, applying the inequality of the arithmetic and geometric means, we have

$$(2) \qquad \Delta_n \leq \{2/n(n-1) \sum_{1\leq r< s\leq n} |z_r - z_s|^2\}^{\frac{1}{2}n(n-1)}$$

$$\leq \{2n/(n-1)\}^{\frac{1}{2}n(n-1)} ,$$

which for $n = 3$ gives $\Delta_3 \leq 3^3$ with strict inequality, unless

$$z_1 + z_2 + z_3 = 0,\ |z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| = \sqrt{3} .$$

Thus the conjecture is true for $n = 3$. For $n = 4$ or 5, nothing is known but for $n \geq 6$, the following example shows that it is false.

Take $z_n = 0$ and $z_k = \{n/(n-1)\}^{\frac{1}{2}} \exp\{2\pi ik/(n-1)\}$, $(k = 1, 2, \ldots, n-1)$. Then $\sum_{1\leq r\leq n} |z_r|^2 = n$ and

$$\Delta_n = |z_1 \ldots z_{n-1}|^2 \prod_{1\leq r< s\leq n-1} |z_r - z_s|^2 .$$

By means of Sylvester's determinant, or otherwise, we see that

$$\prod_{1\leq r< s\leq n-1} |z_r - z_s|^2 = (n-1)^{n-1} \{n/(n-1)\}^{\frac{1}{2}(n-1)(n-2)}$$

and so

$$\Delta_n = n^{n-1} \{1 + 1/(n-1)\}^{\frac{1}{2}(n-1)(n-2)} .$$

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Since Δ_n/n^n is an increasing function of n for $n > 2$, it is easy to verify that it is > 1 for $n \geq 6$ and < 1 for $n = 5$.

This raises the question of how to modify the original conjecture. A paper of Mulholland [2] on an analogous integral inequality suggests that if $\sum_{1 \leq r \leq n} |z_r|^2 = n$, then

$$(\Delta_n)^{1/n(n-1)} \leq c_n$$

for some constant c_n satisfying

$$\overline{\lim}\, c_n = (2/\sqrt{e})^{\frac{1}{2}} = 1.10\ldots \quad .$$

By considering a more complicated example than the one above, in which the n points are distributed on p rays of a circle with m of the points on each ray ($n = mp$) and by varying the parameters suitably, I can show that

$$\overline{\lim}\, c_n \geq 1.05\ldots \quad .$$

From (2) we know that $\overline{\lim}\, c_n \leq 2^{\frac{1}{2}}$ and it would be interesting to determine its exact value.

REFERENCES

1. L.J. Mordell, On a discriminant inequality, Canadian J. of Math., (in course of publication).

2. H.P. Mulholland, Inequalities between the geometric mean difference and the polar moments of a plane distribution, J. London Math. Soc. 33 (1958), 260-270.

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