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THE INTEGRAL OF THE SUPREMUM PROCESS OF BROWNIAN MOTION

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Abstract

In this paper we study the integral of the supremum process of standard Brownian motion. We present an explicit formula for the moments of the integral (or area) $\mathcal{A}(T)$ covered by the process in the time interval $[0, T]$. The Laplace transform of $\mathcal{A}(T)$ follows as a consequence. The main proof involves a double Laplace transform of $\mathcal{A}(T)$ and is based on excursion theory and local time for Brownian motion.

Keywords: Brownian motion; supremum process; local time; Brownian areas

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1. Introduction

Let $B(t)$, $t \geq 0$, be a standard Brownian motion. Consider the following associated processes: the supremum process $S(t) = \max_{0\leq s\leq t} B(s)$ and the local time $L(t)$, which can be regarded as a measure of the time $B(t)$ spends at 0 in the interval $[0, t]$; see [10, Chapter VI] for details. It is well known that these two processes, although pathwise quite different, have the same distribution [10, Chapter VI.2],

$$\{S(t)\}_{t\geq 0} \overset{\mathrm{D}}{=} \{L(t)\}_{t\geq 0},$$

where '$\overset{\mathrm{D}}{=}$' denotes equality in distribution.

The purpose of this paper is to study the distribution of the area under $S(t)$ or, equivalently, $L(t)$ over a given time interval $[0, T]$. That is, the integral

$$\mathcal{A}(T) := \int_0^T S(t)\,\mathrm{d}t \overset{\mathrm{D}}{=} \int_0^T L(t)\,\mathrm{d}t. \tag{1.1}$$

For ease of notation, let $\mathcal{A} := \mathcal{A}(1)$.

The area (1.1) appeared as a random parameter when analysing displacements for linear probing hashing. The Laplace transform of $\mathcal{A}$, which is presented in Corollary 2.1, provided the means to prove one of the main theorems in [9].

Note that the usual Brownian scaling

$$\{B(Tt)\}_{t\geq 0} \overset{\mathrm{D}}{=} \{T^{1/2}B(t)\}_{t\geq 0} \quad \text{for any } T > 0,$$

implies the corresponding scaling for the supremum process,

$$\{S(Tt)\}_{t\geq 0} \overset{\mathrm{D}}{=} \{T^{1/2}S(t)\}_{t\geq 0}.$$

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Thus, for $T > 0$,

$$\mathcal{A}(T) = T \int_0^1 S(Tt)\,\mathrm{d}t \overset{\mathrm{D}}{=} T^{3/2}\mathcal{A}, \tag{1.2}$$

and it is enough to study $\mathcal{A}$.

2. Results

Let $\psi(s) := \mathrm{E}(\mathrm{e}^{-s\mathcal{A}})$ denote the Laplace transform of $\mathcal{A}$. An essential part of this paper is devoted to proving the following formula for the Laplace transform of a variation of ψ, or in other words, a *double* Laplace transform of $\mathcal{A}$. Such formulae have already been derived for the integral of $|B(t)|$ and other similar integrals of processes related to Brownian motion; see [8] and the survey [3].

Theorem 2.1. *Let ψ be the Laplace transform of $\mathcal{A}$. For all $\alpha, \lambda > 0$,*

$$\int_0^\infty \psi(\alpha s^{3/2})\mathrm{e}^{-\lambda s}\,\mathrm{d}s = \int_0^\infty \biggl(1 + \frac{3\alpha s}{2\sqrt{2\lambda}}\biggr)^{-2/3} \mathrm{e}^{-\lambda s}\,\mathrm{d}s.$$

Remark 2.1. One of the parameters α and λ in Theorem 2.1 can be eliminated (by setting it equal to 1, for instance) without loss of generality. In fact, for any $\beta > 0$, the formula is preserved by the substitutions $\lambda \mapsto \beta\lambda$, $\alpha \mapsto \beta^{3/2}\alpha$, and $s \mapsto \beta^{-1}s$.

The proof is given in Section 5. It is based on the excursion theory for Brownian motion and is inspired by similar arguments for other Brownian areas; see [8].

Theorem 2.2. *The nth moment of $\mathcal{A}$ is*

$$\mathrm{E}(\mathcal{A}^n) = \frac{n!\,\Gamma(n+2/3)}{\Gamma(2/3)\Gamma(3n/2+1)}\biggl(\frac{3\sqrt{2}}{4}\biggr)^n, \qquad n \in \mathbb{N}.$$

Proof. Set $\lambda = 1$ in Theorem 2.1 and denote the left- and right-hand sides by

$$I(\alpha) := \int_0^\infty \psi(\alpha s^{3/2})\mathrm{e}^{-s}\,\mathrm{d}s$$

and

$$J(\alpha) := \int_0^\infty \biggl(1 + \frac{3\alpha s}{2\sqrt{2}}\biggr)^{-2/3} \mathrm{e}^{-s}\,\mathrm{d}s.$$

The integrand of $I(\alpha)$ and all its derivatives with respect to α are dominated by functions of the form $s^K\mathrm{e}^{-s}$, uniformly in $\alpha > 0$. Differentiation of $I(\alpha)$ is therefore allowed indefinitely due to dominated convergence. The same argument applies to $J(\alpha)$.

Also, the dominated convergence theorem shows that integration (with respect to s) can be interchanged with taking the limit $\alpha \to 0+$. Thus,

$$\begin{aligned}
\lim_{\alpha\to 0+} \frac{\mathrm{d}^n I(\alpha)}{\mathrm{d}\alpha^n} &= \lim_{\alpha\to 0+} \int_0^\infty \frac{\mathrm{d}^n}{\mathrm{d}\alpha^n}\psi(\alpha s^{3/2})\mathrm{e}^{-s}\,\mathrm{d}s \\
&= \int_0^\infty \lim_{\alpha\to 0+} (-s^{3/2})^n\,\mathrm{E}(\mathcal{A}^n \exp\{-\alpha s^{3/2}\mathcal{A}\})\mathrm{e}^{-s}\,\mathrm{d}s \\
&= (-1)^n\,\mathrm{E}(\mathcal{A}^n) \int_0^\infty s^{3n/2}\mathrm{e}^{-s}\,\mathrm{d}s \\
&= (-1)^n\Gamma\biggl(\frac{3n}{2}+1\biggr)\mathrm{E}(\mathcal{A}^n)
\end{aligned}$$

TABLE 1: The first four moments of $\mathcal{A}$.

n	$\mathrm{E}(\mathcal{A}^n)$
1	$4/3\sqrt{2\pi}$
2	$\frac{5}{12}$
3	$64/63\sqrt{2\pi}$
4	$\frac{11}{24}$

and

$$\begin{aligned}\lim_{\alpha\to 0+}\frac{\mathrm{d}^n J(\alpha)}{\mathrm{d}\alpha^n} &= \lim_{\alpha\to 0+}\int_0^\infty \frac{\mathrm{d}^n}{\mathrm{d}\alpha^n}\biggl(1+\frac{3\alpha s}{2\sqrt{2}}\biggr)^{-2/3}\mathrm{e}^{-s}\,\mathrm{d}s \\ &= \int_0^\infty \lim_{\alpha\to 0+}\frac{\Gamma(n+2/3)}{\Gamma(2/3)}\biggl(\frac{-3s}{2\sqrt{2}}\biggr)^n\biggl(1+\frac{3\alpha s}{2\sqrt{2}}\biggr)^{-n-2/3}\mathrm{e}^{-s}\,\mathrm{d}s \\ &= \frac{\Gamma(n+2/3)}{\Gamma(2/3)}\biggl(\frac{-3}{2\sqrt{2}}\biggr)^n\int_0^\infty s^n\mathrm{e}^{-s}\,\mathrm{d}s \\ &= \frac{\Gamma(n+2/3)}{\Gamma(2/3)}\biggl(\frac{-3\sqrt{2}}{4}\biggr)^n n!.\end{aligned}$$

The fact that $I(\alpha)=J(\alpha)$ completes the proof.

The first four moments of $\mathcal{A}$ are listed in Table 1. Furthermore, Stirling's formula provides the asymptotic relation

$$\mathrm{E}(\mathcal{A}^n)\sim\frac{2\sqrt{3\pi}}{3\Gamma(2/3)}n^{1/6}\biggl(\frac{n}{3\mathrm{e}}\biggr)^{n/2},\qquad n\to\infty. \tag{2.1}$$

Corollary 2.1. *The Laplace transform of $\mathcal{A}$ is*

$$\psi(s)=\frac{1}{\Gamma(2/3)}\sum_{n=0}^{\infty}\frac{\Gamma(n+2/3)}{\Gamma(3n/2+1)}\biggl(\frac{-3\sqrt{2}s}{4}\biggr)^n.$$

Proof. The corollary follows from the identity

$$\psi(s)=\sum_{n=0}^{\infty}\frac{(-s)^n}{n!}\mathrm{E}(\mathcal{A}^n).$$

Note that the sum converges absolutely for every complex s.

The graph of $\psi(s)$ is shown in Figure 1.

Remark 2.2. The Laplace transform of $\mathcal{A}$ can also be expressed in terms of generalised hypergeometric functions:

$$\psi(s)={}_1F_1\biggl(\frac{5}{6};\frac{4}{6};\frac{s^2}{6}\biggr)-\frac{4s}{3\sqrt{2\pi}}\,{}_2F_2\biggl(\frac{6}{6},\frac{8}{6};\frac{7}{6},\frac{9}{6};\frac{s^2}{6}\biggr).$$

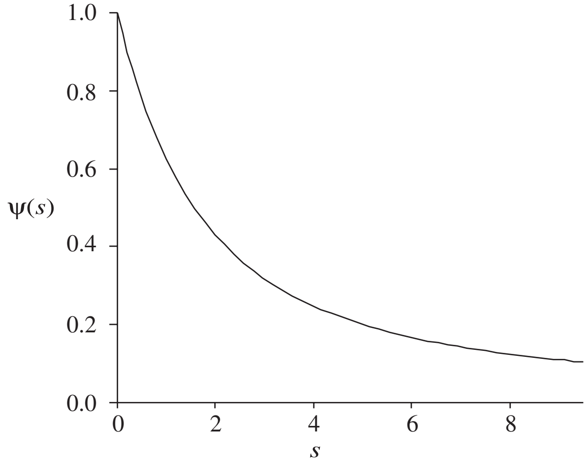

FIGURE 1: The Laplace transform of $\mathcal{A}$.

3. Tail asymptotics

Tauberian theorems by Davies [1] and Kasahara [7] (see [4, Theorem 4.5] for a convenient version) show that the moment asymptotics (2.1) imply the estimate $\ln \mathrm{P}(\mathcal{A} > x) \sim -3x^2/2$ for the tail of the distribution function. Thus, the following corollary is obtained.

Corollary 3.1. *$\mathcal{A}$ has the tail estimate*

$$\mathrm{P}(\mathcal{A} > x) = \exp\biggl\{-\frac{3x^2}{2} + o(x^2)\biggr\}, \qquad x \to \infty.$$

(This result can also be proved by large deviation theory; cf. similar results in [2].)

It seems difficult to obtain more precise tail asymptotics from the moment asymptotics, but it is natural to make a conjecture.

Conjecture 3.1. *$\mathcal{A}$ has a density function $f_{\mathcal{A}}(x)$ satisfying*

$$f_{\mathcal{A}}(x) \sim \frac{2 \cdot 3^{1/6}}{\Gamma(2/3)} x^{1/3} \exp\biggl\{-\frac{3x^2}{2}\biggr\}, \qquad x \to \infty.$$

In fact, if $\mathcal{A}$ has a density with $f_{\mathcal{A}}(x) \sim ax^b \exp\{-cx^d\}$ for some constants $a, b, c,$ and d, then it is the only possible choice that yields the moment asymptotics (2.1); cf. [5].

Conjecture 3.1 may be compared with similar results for several Brownian areas in [5]; see also [3]. Note that in these results for Brownian areas, the exponent of x is always an integer (0, 1, or 2). It is therefore of little surprise that here the exponent seems to be $\frac{1}{3}$, corresponding to the power $n^{1/6}$ in (2.1).

4. Preliminaries on point processes

Let $\mathfrak{S}$ be a measurable space. (In this paper, $\mathfrak{S}$ is either an interval of the real line or the product of two such intervals.) Although a point process Ξ will be regarded as a random set $\{\xi_i\} \subset \mathfrak{S}$, it is technically convenient to formally define it as an integer-valued random

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measure $\sum_i \delta_{\xi_i}$. Hence, $\Xi(A)$ denotes the number of points ξ_i that belong to a (measurable) subset $A \subseteq \mathfrak{S}$. Also, $x \in \Xi$ is equivalent to $\Xi(\{x\}) > 0$. For further details, see, e.g. [6].

A *Poisson process with intensity* $\mathrm{d}\mu$, where $\mathrm{d}\mu$ is a measure on $\mathfrak{S}$, is a point process Ξ such that $\Xi(A)$ has a Poisson distribution with mean $\mu(A)$ for every measurable $A \subseteq \mathfrak{S}$ and $\Xi(A_1), \dots, \Xi(A_k)$ are independent for every family $A_1, \dots, A_k$ of disjoint measurable sets. Lemma 4.1, below, is a standard formula for Laplace functionals; see, e.g. [6, Lemma 12.2(i)].

Lemma 4.1. *If* Ξ *is a Poisson process with intensity* $\mathrm{d}\mu$ *on a set* $\mathfrak{S}$ *and* $f\colon \mathfrak{S} \to [0, \infty)$ *is a measurable function, then*

$$\mathrm{E}\biggl(\exp\biggl\{-\sum_{\xi \in \Xi} f(\xi)\biggr\}\biggr) = \exp\biggl\{-\int_{\mathfrak{S}} (1 - \mathrm{e}^{-f(x)})\,\mathrm{d}\mu(x)\biggr\}.$$

Lemma 4.2, below, on the other hand, is more of a digression. The result follows from a standard gamma integral by integration by parts. (The result can also be written as $2\Gamma(\frac{1}{2})\lambda^{1/2}$.)

Lemma 4.2. *If* $\lambda > 0$ *then*

$$\int_0^\infty (1 - \mathrm{e}^{-\lambda x}) x^{-3/2}\,\mathrm{d}x = 2\sqrt{\pi\lambda}.$$

5. Proof of Theorem 2.1

The set $\{t\colon B(t) = 0\}$ is almost surely (a.s.) closed and unbounded, so its complement $\{t\colon B(t) \neq 0\}$ is an infinite union of finite open intervals, denoted by $I_\nu = (g_\nu, d_\nu)$, $\nu = 1, 2, \dots$, in some order. (The intervals cannot be ordered by appearance, since there is a.s. an infinite number of them in, say, [0,1]. Fortunately, the order does not matter.) The restrictions of $B(t)$ to these intervals are called the *excursions* of $B(t)$. Let $\hat{e}_\nu$ be the excursion during I_ν.

The local time $L(t)$ is constant during each excursion. Let τ_ν be the local time during $\hat{e}_\nu$ and let $\ell_\nu := d_\nu - g_\nu$ be the length of $\hat{e}_\nu$. It is well known (see [10, Chapter XII]) that the collection of pairs $\{(\tau_\nu, \ell_\nu)\}_{\nu=1}^\infty$ forms a Poisson process in $[0, \infty) \times (0, \infty)$ with intensity

$$\mathrm{d}\Lambda = (2\pi\ell^3)^{-1/2}\,\mathrm{d}\tau\,\mathrm{d}\ell.$$

Note also that, a.s., if the excursion $\hat{e}_{\nu_1}$ comes before $\hat{e}_{\nu_2}$ then $\tau_{\nu_1} < \tau_{\nu_2}$.

Next, consider a Poisson process $\{T_i\}_{i=1}^\infty$ on $[0, \infty)$ with intensity $\lambda\,\mathrm{d}t$, independent of $\{B(t)\}$. Assume that the points are ordered with $0 < T_1 < T_2 < \cdots$. Then $T_1, T_2 - T_1, \dots$ are i.i.d. $\mathrm{Exp}(\lambda)$ random variables with density function $\lambda\mathrm{e}^{-\lambda t}$. Furthermore, T_1 is independent of $\{B(t)\}$ and, thus, of $\{\mathcal{A}(T)\}$. It follows from (1.2) that $\mathcal{A}(T_1) \overset{\mathrm{D}}{=} T_1^{3/2}\mathcal{A}$ and, consequently,

$$\mathrm{E}(\exp\{-\alpha\mathcal{A}(T_1)\}) = \mathrm{E}(\exp\{-\alpha T_1^{3/2}\mathcal{A}\}) = \mathrm{E}(\psi(\alpha T_1^{3/2})) = \lambda\int_0^\infty \mathrm{e}^{-\lambda s}\psi(\alpha s^{3/2})\,\mathrm{d}s. \quad (5.1)$$

The times T_i are called *marks*, and an excursion is called *marked* if it contains at least one of the marks T_i. The marks $\{T_i\}$ are placed by first constructing {B(t)} and then adding marks according to independent Poisson processes with intensities $\lambda\,\mathrm{d}t$ in each excursion. Thus, given the excursions $\{\hat{e}_\nu\}$, each excursion $\hat{e}_\nu$ is marked with probability $1-\exp\{-\lambda\ell_\nu\}$, independently of the other excursions. The Poisson process $\Xi := \{(\tau_\nu, \ell_\nu)\}$ defined by the excursions can be written as the union $\Xi' \cup \Xi''$, where

$$\Xi' := \{(\tau_\nu, \ell_\nu)\colon \hat{e}_\nu \text{ is unmarked}\}, \qquad \Xi'' := \{(\tau_\nu, \ell_\nu)\colon \hat{e}_\nu \text{ is marked}\}.$$

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By the general independence properties of Poisson processes, Ξ' and Ξ'' are *independent* Poisson processes with intensities

$$\mathrm{d}\Lambda' := \mathrm{e}^{-\lambda\ell}\,\mathrm{d}\Lambda = (2\pi)^{-1/2}\ell^{-3/2}\mathrm{e}^{-\lambda\ell}\,\mathrm{d}\tau\,\mathrm{d}\ell \tag{5.2}$$

and

$$\mathrm{d}\Lambda'' = (1-\mathrm{e}^{-\lambda\ell})\,\mathrm{d}\Lambda = (2\pi)^{-1/2}\ell^{-3/2}(1-\mathrm{e}^{-\lambda\ell})\,\mathrm{d}\tau\,\mathrm{d}\ell,$$

respectively. In particular, if the lengths are ignored, the local times of the marked excursions form a Poisson process $\tilde{\Xi}$ on $(0,\infty)$ with intensity

$$\int_{\ell=0}^{\infty}(1-\mathrm{e}^{-\lambda\ell})\,\mathrm{d}\Lambda = \tilde{\lambda}\,\mathrm{d}\tau,$$

where, using Lemma 4.2,

$$\tilde{\lambda} = \int_0^{\infty}(2\pi)^{-1/2}\ell^{-3/2}(1-\mathrm{e}^{-\lambda\ell})\,\mathrm{d}\ell = \sqrt{2\lambda}. \tag{5.3}$$

Owing to the fact that $B(T_1)\neq 0$ a.s., there exists a unique excursion $\hat{e}_{\nu^*}$ that contains the first mark T_1, i.e. $T_1\in I_{\nu^*}$. Let $\zeta := L(T_1) = \tau_{\nu^*}$ be the local time at T_1 (and, thus, during $\hat{e}_{\nu^*}$). Since $\hat{e}_{\nu^*}$ is the first marked excursion, its local time ζ is the first of the points in the Poisson process $\tilde{\Xi}$ and, hence,

$$\zeta \sim \operatorname{Exp}(\sqrt{2\lambda}). \tag{5.4}$$

The restriction of $B(t)$ to the interval $[0,T_1]$ consists of all excursions $\hat{e}_\nu$ with local time $\tau_\nu < \tau_{\nu^*} = \zeta$ and the part of $\hat{e}_{\nu^*}$ on (g_{ν^*},T_1), plus the set

$$[0,T_1]\setminus\bigcup_\nu I_\nu = \{t\le T_1 \colon B(t)=0\},$$

which, a.s., has measure 0 and, thus, may be ignored. Consequently, since $L(t)=\tau_\nu$ on I_ν,

$$\begin{aligned}
\mathcal{A}(T_1) &:= \int_0^{T_1} L(t)\,\mathrm{d}t \\
&= \sum_{\nu\colon \tau_\nu<\tau_{\nu^*}}\int_{I_\nu} L(t)\,\mathrm{d}t + \int_{g_{\nu^*}}^{T_1} L(t)\,\mathrm{d}t \\
&= \sum_{\nu\colon \tau_\nu<\zeta}\tau_\nu\ell_\nu + \zeta(T_1-g_{\nu^*}) \\
&=: \mathcal{A}' + \mathcal{A}''.
\end{aligned}$$

The sum defined as $\mathcal{A}' = \sum_{\nu\colon \tau_\nu<\zeta}\tau_\nu\ell_\nu$ only contains terms for unmarked excursions $\hat{e}_\nu$. Thus,

$$\mathcal{A}' = \sum_{(\tau_\nu,\ell_\nu)\in\Xi'\colon \tau_\nu<\zeta}\tau_\nu\ell_\nu.$$

Recall that ζ is determined by Ξ'' (as the smallest τ with $(\tau,\ell)\in\Xi''$ for some ℓ) and that Ξ' and Ξ'' are independent. Hence, Ξ' and ζ are independent. It follows from Lemma 4.1, with $\mathfrak{S} = (0,\zeta)\times(0,\infty)$ and $f((\tau,\ell)) = \alpha\tau\ell$, that

$$\mathrm{E}(\mathrm{e}^{-\alpha\mathcal{A}'}\mid\zeta) = \exp\biggl\{-\int_{\tau=0}^{\zeta}\int_{\ell=0}^{\infty}(1-\mathrm{e}^{-\alpha\tau\ell})\,\mathrm{d}\Lambda'(\tau,\ell)\biggr\}.$$

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By (5.2) and Lemma 4.2,

$$\begin{aligned}\int_{\tau=0}^{\zeta}\int_{\ell=0}^{\infty}(1-\mathrm{e}^{-\alpha\tau\ell})\,\mathrm{d}\Lambda'(\tau,\ell) &= \int_{\tau=0}^{\zeta}\int_{\ell=0}^{\infty}(1-\mathrm{e}^{-\alpha\tau\ell})(2\pi)^{-1/2}\ell^{-3/2}\mathrm{e}^{-\lambda\ell}\,\mathrm{d}\ell\,\mathrm{d}\tau \\ &= (2\pi)^{-1/2}\int_{\tau=0}^{\zeta}\int_{\ell=0}^{\infty}(\mathrm{e}^{-\lambda\ell}-\mathrm{e}^{-(\lambda+\alpha\tau)\ell})\ell^{-3/2}\,\mathrm{d}\ell\,\mathrm{d}\tau \\ &= \int_{\tau=0}^{\zeta}\sqrt{2}(\sqrt{\lambda+\alpha\tau}-\sqrt{\lambda})\,\mathrm{d}\tau \\ &= \frac{2\sqrt{2}}{3\alpha}((\lambda+\alpha\zeta)^{3/2}-\lambda^{3/2})-\sqrt{2\lambda}\zeta,\end{aligned}$$

and it follows that

$$\mathrm{E}(\mathrm{e}^{-\alpha\mathcal{A}'}\mid\zeta)=\exp\biggl\{\sqrt{2\lambda}\zeta-\frac{2\sqrt{2}}{3\alpha}((\lambda+\alpha\zeta)^{3/2}-\lambda^{3/2})\biggr\}. \tag{5.5}$$

Now consider $\mathcal{A}''=\zeta(T_1-g_{\nu^*})$. Note that $T_1-g_{\nu^*}$ is the location (relative to the left endpoint of the excursion) of the first mark in the first marked excursion. Since Ξ is a Poisson process with intensity independent of τ, the location $T_1-g_{\nu^*}$ is independent of the local time ζ of the first marked excursion. Furthermore, the joint distribution of $(\ell_{\nu^*}, T_1-g_{\nu^*})$ has density

$$(\tilde{\lambda})^{-1}\lambda\mathrm{e}^{-\lambda y}(2\pi)^{-1/2}\ell^{-3/2}\,\mathrm{d}\ell\,\mathrm{d}y, \qquad 0<y<\ell<\infty,$$

where the normalisation constant $\tilde{\lambda}$ is given by (5.3). Consequently,

$$\begin{aligned}\mathrm{E}(\mathrm{e}^{-\alpha\mathcal{A}''}\mid\zeta) &= \mathrm{E}(\exp\{-\alpha\zeta(T_1-g_{\nu^*})\}\mid\zeta) \\ &= \int_{y=0}^{\infty}\int_{\ell=y}^{\infty}\mathrm{e}^{-\alpha\zeta y}(\tilde{\lambda})^{-1}\lambda\mathrm{e}^{-\lambda y}(2\pi)^{-1/2}\ell^{-3/2}\,\mathrm{d}\ell\,\mathrm{d}y \\ &= \pi^{-1/2}\lambda^{1/2}\int_{y=0}^{\infty}\mathrm{e}^{-(\lambda+\alpha\zeta)y}y^{-1/2}\,\mathrm{d}y \\ &= \lambda^{1/2}(\lambda+\alpha\zeta)^{-1/2}.\end{aligned} \tag{5.6}$$

Again, since Ξ' and Ξ'' are independent, $\mathcal{A}'$ and $\mathcal{A}''$ are conditionally independent given ζ. Thus, (5.5) and (5.6) yield

$$\begin{aligned}\mathrm{E}(\exp\{-\alpha\mathcal{A}(T_1)\}\mid\zeta) &= \mathrm{E}(\mathrm{e}^{-\alpha\mathcal{A}'}\mid\zeta)\,\mathrm{E}(\mathrm{e}^{-\alpha\mathcal{A}''}\mid\zeta) \\ &= \biggl(\frac{\lambda}{\lambda+\alpha\zeta}\biggr)^{1/2}\exp\biggl\{\sqrt{2\lambda}\zeta-\frac{2\sqrt{2}}{3\alpha}((\lambda+\alpha\zeta)^{3/2}-\lambda^{3/2})\biggr\}.\end{aligned}$$

By (5.4), ζ has the density $\sqrt{2\lambda}\mathrm{e}^{-\sqrt{2\lambda}x}$, $x>0$, and it follows that

$$\mathrm{E}(\exp\{-\alpha\mathcal{A}(T_1)\})=\lambda\sqrt{2}\int_0^{\infty}(\lambda+\alpha x)^{-1/2}\exp\biggl\{-\frac{2\sqrt{2}}{3\alpha}((\lambda+\alpha x)^{3/2}-\lambda^{3/2})\biggr\}\,\mathrm{d}x.$$

Finally, the substitution

$$\frac{2\sqrt{2}}{3\alpha\lambda}((\lambda+\alpha x)^{3/2}-\lambda^{3/2})\mapsto s$$

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provides the slightly simpler formula

$$\mathrm{E}(\exp\{-\alpha\mathcal{A}(T_1)\}) = \lambda\int_0^\infty \left(1+\frac{3\alpha s}{2\sqrt{2\lambda}}\right)^{-2/3} \mathrm{e}^{-\lambda s}\,\mathrm{d}s.$$

The result now follows by a comparison with (5.1).

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