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REVISITING CLOSED ASYMPTOTIC COUPLES

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Abstract Every discrete definable subset of a closed asymptotic couple with ordered scalar field $\boldsymbol{k}$ is shown to be contained in a finite-dimensional $\boldsymbol{k}$-linear subspace of that couple. It follows that the differential-valued field $\mathbb{T}$ of transseries induces more structure on its value group than what is definable in its asymptotic couple equipped with its scalar multiplication by real numbers, where this asymptotic couple is construed as a two-sorted structure with $\mathbb{R}$ as the underlying set for the second sort.

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Introduction

The field of Laurent series with real coefficients comes with a natural derivation but is too small to be closed under integration and exponentiation. These defects are cured by passing to a certain canonical extension, the ordered differential field $\mathbb{T}$ of transseries. Transseries are formal series in an indeterminate $x > \mathbb{R}$, such as

$$\begin{aligned} & -3\,\mathrm{e}^{\mathrm{e}^x} + \mathrm{e}^{\frac{\mathrm{e}^x}{\log x} + \frac{\mathrm{e}^x}{\log^2 x} + \frac{\mathrm{e}^x}{\log^3 x} + \cdots} - x^{11} + 7 \\ & + \frac{\pi}{x} + \frac{1}{x \log x} + \frac{1}{x \log^2 x} + \frac{1}{x \log^3 x} + \cdots \\ & + \frac{2}{x^2} + \frac{6}{x^3} + \frac{24}{x^4} + \frac{120}{x^5} + \frac{720}{x^6} + \cdots \\ & + \mathrm{e}^{-x} + 2\,\mathrm{e}^{-x^2} + 3\,\mathrm{e}^{-x^3} + 4\,\mathrm{e}^{-x^4} + \cdots, \end{aligned}$$

where $\log^2 x := (\log x)^2$, etc. Transseries, that is, elements of $\mathbb{T}$, are also the logarithmic-exponential series (LE-series, for short) from [4]; we refer to that paper, or to Appendix A of our book [2], for a detailed construction of $\mathbb{T}$.

What we need for now is that $\mathbb{T}$ is a real closed field extension of the field $\mathbb{R}$ of real numbers and that $\mathbb{T}$ comes equipped with a distinguished element $x > \mathbb{R}$, an exponential operation $\exp\colon \mathbb{T} \to \mathbb{T}$ and a distinguished derivation $\partial\colon \mathbb{T} \to \mathbb{T}$. The exponentiation here is an isomorphism of the ordered additive group of $\mathbb{T}$ onto the ordered multiplicative group $\mathbb{T}^{>}$ of positive elements of $\mathbb{T}$; we set $\mathrm{e}^f := \exp(f)$ for $f \in \mathbb{T}$. The derivation ∂ comes from differentiating a transseries termwise with respect to x, and we set $f' := \partial(f)$, $f'' := \partial^2(f)$, and so on, for $f \in \mathbb{T}$; thus, $x' = 1$, and ∂ is compatible with exponentiation: $(\mathrm{e}^f)' = f'\,\mathrm{e}^f$ for $f \in \mathbb{T}$. Moreover, the constant field of $\mathbb{T}$ is $\mathbb{R}$, that is, $\{f \in \mathbb{T} : f' = 0\} = \mathbb{R}$; see again [2] for details. Before stating our new results, we introduce some conventions:

Notations and conventions. *Throughout, m, n range over $\mathbb{N} = \{0, 1, 2, \ldots\}$. Ordered sets, ordered abelian groups, and ordered fields are totally ordered, by convention. Given an ambient ordered set S, a downward closed subset of S, also called a cut in S, is a set $D \subseteq S$ such that for all $a, b \in S$ with $a < b \in D$ we have $a \in D$. For an (additively written) ordered abelian group Γ we set*

$$\Gamma^{\neq} \ := \ \Gamma \setminus \{0\}, \quad \Gamma^{<} \ := \ \{\gamma \in \Gamma : \gamma < 0\}, \qquad \Gamma^{>} \ := \ \{\gamma \in \Gamma : \gamma > 0\}.$$

For any field K we let $K^{\times} = K \setminus \{0\}$ be its multiplicative group. A differential field is a field K of characteristic 0 with a derivation $\partial\colon K \to K$, and we set $a' := \partial(a)$ for $a \in K$, and let $b^{\dagger} := b'/b$ be the logarithmic derivative of $b \in K^{\times}$ when the ambient differential field K with its derivation ∂ is clear from the context; note that then $(ab)^{\dagger} = a^{\dagger} + b^{\dagger}$ for $a, b \in K^{\times}$.

Our book [2] culminated in an elimination theory for the differential field $\mathbb{T}$ of transseries. As a consequence, we found that the induced structure on its constant field $\mathbb{R}$ is just its semialgebraic structure: if $X \subseteq \mathbb{R}^n$ is definable in $\mathbb{T}$, then X is semialgebraic (in the sense of $\mathbb{R}$). (Here and throughout "definable in $\boldsymbol{M}$" means "definable in $\boldsymbol{M}$ with parameters from $\boldsymbol{M}$".)

The story is more complicated for the structure induced by $\mathbb{T}$ on its value group. To explain this, we recall that the natural valuation ring

$$\mathcal{O}_{\mathbb{T}} = \ \big\{f \in \mathbb{T} : |f| \leqslant r \text{ for some real } r > 0\big\}$$

of the real closed field $\mathbb{T}$ is clearly 0-definable in $\mathbb{T}$ *as a differential field*, which is how we construe $\mathbb{T}$ in the rest of this paper. Let $v\colon \mathbb{T}^{\times} \to \Gamma_{\mathbb{T}}$ be the corresponding valuation on the field $\mathbb{T}$. We may consider $\Gamma_{\mathbb{T}}$ as the quotient $\mathbb{T}^{\times}/{\asymp}$ and v as the natural map to this quotient where $\asymp$ is a 0-definable equivalence relation on $\mathbb{T}^{\times}$.

Thus, $\Gamma_{\mathbb{T}}$ is part of $\mathbb{T}^{\mathrm{eq}}$. What is the structure induced by $\mathbb{T}$ on $\Gamma_{\mathbb{T}}$? It includes the structure of $\Gamma_{\mathbb{T}}$ as an ordered (by convention, additively written) abelian group. Moreover, the derivation of $\mathbb{T}$ induces a function $\psi\colon \Gamma_{\mathbb{T}}^{\neq} \to \Gamma_{\mathbb{T}}$ by $\psi(vf) = v(f^{\dagger})$ for $f \in \mathbb{T}^{\times}$ with $vf \neq 0$. The structure $(\Gamma_{\mathbb{T}}, \psi)$ consisting of the ordered abelian group $\Gamma_{\mathbb{T}}$ with the function ψ is the *asymptotic couple* of $\mathbb{T}$, a notion introduced for differential-valued fields – among

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which is $\mathbb{T}$—by Rosenlicht [7]. There is also a natural 0-definable scalar multiplication

$$(r,\gamma)\mapsto r\gamma : \mathbb{R}\times\Gamma_{\mathbb{T}}\to\Gamma_{\mathbb{T}}$$

that makes $\Gamma_{\mathbb{T}}$ into a vector space over $\mathbb{R}$; it is given by $rv(f)=v(f^r)$ for $f\in\mathbb{T}^>$, and the reason it is 0-definable (in $\mathbb{T}^{\mathrm{eq}}$) is that $r\alpha=\beta$ (for $r\in\mathbb{R}$ and $\alpha,\beta\in\Gamma_{\mathbb{T}}$) iff there are $f,g\in\mathbb{T}^\times$ such that $\alpha=vf$, $\beta=vg$ and $rf^\dagger=g^\dagger$. For this reason, we consider the 2-sorted structure $\boldsymbol{\Gamma}_{\mathbb{T}}=((\Gamma_{\mathbb{T}},\psi),\mathbb{R};\mathrm{sc})$ consisting of the asymptotic couple $(\Gamma_{\mathbb{T}},\psi)$, the field $\mathbb{R}$, and the above scalar multiplication

$$\mathrm{sc}\colon \mathbb{R}\times\Gamma_{\mathbb{T}}\to\Gamma_{\mathbb{T}},\quad \mathrm{sc}(r,\gamma)=r\gamma.$$

The basic elementary properties of this 2-sorted structure were determined in [1]. This structure encodes important features of $\mathbb{T}$, and in this paper, we prove a new result about it in § 5:

Theorem 0.1. *Let $\Gamma_{\mathbb{T}}$ be equipped with its order topology, and let $X\subseteq\Gamma_{\mathbb{T}}$ be definable in $\boldsymbol{\Gamma}_{\mathbb{T}}$. Then the following are equivalent*:

(i) *X is contained in a finite-dimensional $\mathbb{R}$-linear subspace of $\Gamma_{\mathbb{T}}$*;

(ii) *X is discrete*;

(iii) *X has an empty interior in $\Gamma_{\mathbb{T}}$.*

We also know from [2, Corollaries 14.3.10, 14.3.11] that for any non-zero differential polynomial $G(Y)\in\mathbb{T}\{Y\}$ the subset $\{vy : y\in\mathbb{T}^\times,\ G(y)=0\}$ of $\Gamma_{\mathbb{T}}$ is discrete. The set of zeros of

$$G(Y):=Y^2Y'Y^{(3)}-Y^2(Y^{(2)})^2-Y(Y')^2Y^{(2)}+(Y')^4$$

in $\mathbb{T}$ is

$$\{a\,\mathrm{e}^{b\,\mathrm{e}^{cx}} : a,b,c\in\mathbb{R}\}\cup\{a\,\mathrm{e}^{bx} : a,b\in\mathbb{R}\}.$$

For this G the set $\{vy : y\in\mathbb{T}^\times,\ G(y)=0\}$ is not contained in a finite-dimensional $\mathbb{R}$-linear subspace of $\Gamma_{\mathbb{T}}$ and thus *not* definable in the 2-sorted structure $\boldsymbol{\Gamma}_{\mathbb{T}}$ by the theorem above. We treat this example in more detail at the end of § 1.

The authors of [1] had speculated that the subsets of $\Gamma_{\mathbb{T}}$ definable in $\mathbb{T}^{\mathrm{eq}}$ might be just those that are definable in the 2-sorted structure $\boldsymbol{\Gamma}_{\mathbb{T}}$. The above is a counter example but leaves open the possibility that $\Gamma_{\mathbb{T}}$ is *stably embedded* in $\mathbb{T}^{\mathrm{eq}}$. In this connection, we note that for all intents and purposes, we can replace the 2-sorted structure $\boldsymbol{\Gamma}_{\mathbb{T}}$ by the 1-sorted structure $(\Gamma_{\mathbb{T}};\psi,\mathbb{R}1,\mathrm{sc})$ consisting of the asymptotic couple $(\Gamma_{\mathbb{T}};\psi)$ expanded by the set $\mathbb{R}1\subseteq\Gamma_{\mathbb{T}}$, where $1=v(x^{-1})\in\Gamma_{\mathbb{T}}^>$ is the unique fixed point of ψ, and by the function

$$\mathrm{sc}\ :\ (\mathbb{R}1)\times\Gamma_{\mathbb{T}}\to\Gamma_{\mathbb{T}},\quad \mathrm{sc}(r1,\gamma):=r\gamma.$$

Why revisit closed asymptotic couples?

The proof of Theorem 0.1 requires the results of [1], suitably extended. This was our original motive for revisiting the subject of closed asymptotic couples. The theorem itself

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is of interest but is also needed for its application to the induced structure on the value group of $\mathbb{T}$.

The quantifier elimination (QE) for closed asymptotic couples in [1] was expected to help in obtaining a QE for $\mathbb{T}$. The latter is achieved in [2, Chapter 16], but there we needed only a key lemma from [1], not its QE for closed asymptotic couples. That key lemma is [1, Property B], and is given a self-contained proof of five dense pages in [2, § 9.9]. Since then, we found a simpler way to obtain the QE in [1] that does not use the key lemma alluded to but depends on some easier-to-prove new lemmas that have also other applications; see § 2. This new proof of QE, given in § 3, is another reason for revisiting the subject of closed asymptotic couples. (We derive the “key lemma” itself as a routine consequence of the QE for closed asymptotic couples: Proposition 6.3.)

For his study of transexponential pre-H-fields in [6, Chapter 6] and [5], Nigel Pynn-Coates introduced a modified version of “closed asymptotic couple” and adapted accordingly some material from our (unpublished) 2017 version of this paper. Getting the paper published is also more urgent now because in our recent proof that maximal Hardy fields are η_1 we use results from § 4 below.

Finally, this paper gives us an opportunity to enhance and better organize parts of [1], and acknowledge gaps in some proofs there; we intend to close these gaps in a follow-up to the present paper. No familiarity with [1] is needed, but we do assume as background some 20 pages (mainly on asymptotic couples) from [2], namely parts of § 2.4 on ordered abelian groups, Sections 6.5, 9.1 (subsection on asymptotic couples), 9.2 (first four pages), and 9.8. For the reader’s convenience, we also repeat definitions of key notions concerning asymptotic couples and H-fields.

We thank Nigel Pynn-Coates for his careful reading of this paper, and corrections, and the referee for helpful comments.

1. Preliminaries

We only consider asymptotic couples of H-type, calling them *H-couples* for brevity. Thus, an H-couple is a pair (Γ, ψ) consisting of an ordered abelian group Γ with a map $\psi\colon \Gamma^{\neq} \to \Gamma$, such that for all $\alpha, \beta \in \Gamma^{\neq}$,

(AC1) $\alpha+\beta \neq 0 \implies \psi(\alpha+\beta) \geqslant \min(\psi(\alpha), \psi(\beta))$;

(AC2) $\psi(k\alpha)=\psi(\alpha)$ for all $k \in \mathbb{Z}^{\neq}$;

(AC3) $\alpha > 0 \implies \alpha+\psi(\alpha) > \psi(\beta)$;

(HC) $0<\alpha \leqslant \beta \implies \psi(\alpha) \geqslant \psi(\beta)$.

(As an aside, note that (AC2) and (HC) together imply (AC1); had we observed this earlier, it would have shortened some arguments in [2, § 9.8]; the reader can use it to the same effect in § 2 of the present paper.) Let (Γ, ψ) be an H-couple. By (AC1) and (AC2) the function ψ is a valuation on the abelian group Γ; as usual, we extend ψ to $\psi\colon \Gamma \to \Gamma_{\infty} := \Gamma \cup \{\infty\}$ by $\psi(0) := \infty$; we use $\alpha^{\dagger}$ as an alternative notation for $\psi(\alpha)$ and set $\alpha' := \alpha + \alpha^{\dagger}$ for $\alpha \in \Gamma$. Also $\Psi := \psi(\Gamma^{\neq})$. We recall from [2, Corollary 9.2.16] a basic trichotomy for H-couples which says that we are in exactly one of the following three cases:

- (Γ, ψ) has a (necessarily unique) *gap*, that is, an element $\gamma \in \Gamma$ such that $\Psi < \gamma < (\Gamma^{>})'$;
- (Γ, ψ) is *grounded*, that is, Ψ has a largest element;
- (Γ, ψ) has *asymptotic integration*, that is, $\Gamma = (\Gamma^{\neq})'$.

We say that (Γ, ψ) is *closed* if Γ is divisible, $\Psi \subseteq \Gamma$ is downward closed, and (Γ, ψ) has asymptotic integration. We also use the qualifiers having a gap, *grounded*, *having asymptotic integration*, and *closed* for H-couples with extra structure.

An H**-cut** in (Γ, ψ) is a downward closed set $P \subseteq \Gamma$ such that $\Psi \subseteq P < (\Gamma^{>})'$. The set $\Psi^{\downarrow} := \{\alpha \in \Gamma : \alpha \leqslant \beta for some \beta \in \Psi\}$ is an H-cut in (Γ, ψ), and if (Γ, ψ) is grounded or has asymptotic integration, this is the only H-cut in (Γ, ψ). If (Γ, ψ) has a gap β, then $\Psi^{\downarrow} \cup \{\beta\}$ is the only other H-cut in (Γ, ψ).

In particular, if (Γ, ψ) is closed, then Ψ is the only H-cut in (Γ, ψ), but in eliminating quantifiers for closed H-couples in § 3, it is essential to have a predicate for this H-cut in our language.

Where do closed H-couples come from?

We recall from [2, Chapter 10] that an *H-field* is an ordered differential field K with constant field C such that:

(H1) $a' > 0$ for all $a \in K$ with $a > C$;

(H2) $\mathcal{O} = C + o$, where $\mathcal{O}$ is the convex hull of C in the ordered field K, and o is the maximal ideal of the valuation ring $\mathcal{O}$.

Let K be an H-field, and let $\mathcal{O}$ and o be as in (H2). Thus, K is a valued field with valuation ring $\mathcal{O}$. Let $v\colon K^{\times} \to \Gamma$ be the associated valuation. The value group $\Gamma = v(K^{\times})$ is made into an H-couple (Γ, ψ)—the H-couple of K—by $\psi(vf) := v(f^{\dagger})$ for $f \in K^{\times}$ with $vf \neq 0$. We call K *Liouville closed* if it is real closed and for all $a \in K$ there exists $b \in K$ with $a = b'$ and also a $b \in K^{\times}$ such that $a = b^{\dagger}$.

If K is Liouville closed, its H-couple is closed as is easily verified. We recall from [2] that $\mathbb{T}$ is a Liouville closed H-field.

Ordered vector spaces

Throughout we let $\boldsymbol{k}$, $\boldsymbol{k}_0$, and $\boldsymbol{k}^*$ be ordered fields. Recall that an *ordered vector space over $\boldsymbol{k}$* is an ordered abelian group Γ with a scalar multiplication $\boldsymbol{k} \times \Gamma \to \Gamma$ that makes Γ into a vector space over $\boldsymbol{k}$ such that $c\gamma > 0$ for all $c \in \boldsymbol{k}^{>}$ and $\gamma \in \Gamma^{>}$. Let Γ be an ordered vector space over $\boldsymbol{k}$. Then any $\boldsymbol{k}$-linear subspace of Γ is considered as an ordered vector space over $\boldsymbol{k}$ in the obvious way. We shall need the following easy result about Γ:

Lemma 1.1. *Let Γ_0 be a $\boldsymbol{k}$-linear subspace of Γ. Suppose Γ contains an element ε with $0 < \varepsilon < \Gamma_0^{>}$. Then Γ_0 is closed in Γ with respect to the order topology on Γ.*

Proof. Let $\gamma \in \Gamma \setminus \Gamma_0$. With ε as in the hypothesis, we observe that the interval $(\gamma - \varepsilon, \gamma + \varepsilon)$ can have at most one point in it from Γ_0, and so by decreasing ε we can arrange that $(\gamma - \varepsilon, \gamma + \varepsilon) \cap \Gamma_0 = \emptyset$. $\square$

The ***k*-archimedean class** of $\alpha \in \Gamma$ is

$$[\alpha]_{\boldsymbol{k}} := \big\{\gamma \in \Gamma : |\gamma| \leqslant c|\alpha| \text{ and } |\alpha| \leqslant c|\gamma| \text{ for some } c \in \boldsymbol{k}^{>}\big\}.$$

Let $[\Gamma]_{\boldsymbol{k}}$ be the set of $\boldsymbol{k}$-archimedean classes. Then $[\Gamma]_{\boldsymbol{k}}$ is a partition of Γ, and we linearly order $[\Gamma]_{\boldsymbol{k}}$ by

$$\begin{aligned} [\alpha]_{\boldsymbol{k}} < [\beta]_{\boldsymbol{k}} \quad &:\Longleftrightarrow \quad c|\alpha| < |\beta| \text{ for all } c \in \boldsymbol{k}^{>} \\ &\Longleftrightarrow \quad [\alpha]_{\boldsymbol{k}} \neq [\beta]_{\boldsymbol{k}} \text{ and } |\alpha| < |\beta|. \end{aligned}$$

Thus, $[0]_{\boldsymbol{k}} = \{0\}$ is the smallest $\boldsymbol{k}$-archimedean class. For $\alpha, \beta \in \Gamma$, $c \in \boldsymbol{k}^{\times}$ we have $[c\alpha]_{\boldsymbol{k}} = [\alpha]_{\boldsymbol{k}}$ and $[\alpha + \beta]_{\boldsymbol{k}} \leqslant \max([\alpha]_{\boldsymbol{k}}, [\beta]_{\boldsymbol{k}})$, with equality if $[\alpha]_{\boldsymbol{k}} \neq [\beta]_{\boldsymbol{k}}$.

Lemma 1.2. *Let $\Gamma \neq \{0\}$ be an ordered vector space over $\boldsymbol{k}$ such that $[\Gamma^{\neq}]_{\boldsymbol{k}}$ has no least element. Then every finite-dimensional $\boldsymbol{k}$-linear subspace of Γ is discrete with respect to the order topology on Γ.*

Proof. First note that if $\gamma_1, \ldots, \gamma_n \in \Gamma^{\neq}$ and $[\gamma_1]_{\boldsymbol{k}}, \ldots, [\gamma_n]_{\boldsymbol{k}}$ are distinct, then $\gamma_1, \ldots, \gamma_n$ are $\boldsymbol{k}$-linearly independent. Thus, for a finite-dimensional $\boldsymbol{k}$-linear subspace $\Delta \neq \{0\}$ of Γ we can take $\delta \in \Delta^{\neq}$ such that $[\delta]_{\boldsymbol{k}}$ is minimal in $[\Delta^{\neq}]_{\boldsymbol{k}}$. Then for any $\alpha \in \Delta$ and $\beta \in \Gamma^{\neq}$ with $[\beta]_{\boldsymbol{k}} < [\delta]_{\boldsymbol{k}}$ we have $\alpha + \beta \notin \Delta$. $\square$

Lemma 1.2 takes care of the easy direction (i) $\Rightarrow$ (ii) in Theorem 0.1. The direction (ii) $\Rightarrow$ (iii) is trivial. The harder direction (iii) $\Rightarrow$ (i) uses a generality on expanded vector spaces, to which we now turn.

Let V be a vector space over a field C. We consider the two-sorted structure $(V, C; \mathrm{sc})$ consisting of the abelian group V, the field C, and the scalar multiplication $\mathrm{sc}\colon C \times V \to V$ of the vector space V. Let $X \subseteq V$. Then we have the expansion $\boldsymbol{V} = ((V, X), C; \mathrm{sc})$ of $(V, C; \mathrm{sc})$. Let $\boldsymbol{V}^* = ((V^*, X^*), C^*; \mathrm{sc})$ be an elementary extension of $\boldsymbol{V}$. Let C^*V be the C^*-linear subspace of V^* spanned by V.

Lemma 1.3. *Assume $\boldsymbol{V}^*$ is $|V|^+$-saturated. Then X is contained in a finite-dimensional C-linear subspace of V if and only if $X^* \subseteq C^*V$.*

Proof. If $X \subseteq Cv_1 + \cdots + Cv_n$, $v_1, \ldots, v_n \in V$, then $X^* \subseteq C^*v_1 + \cdots + C^*v_n \subseteq C^*V$. We prove the contrapositive of the other direction, so assume $X \not\subseteq Cv_1 + \cdots + Cv_n$ for all $v_1, \ldots, v_n \in V$. Then $X^* \not\subseteq C^*v_1 + \cdots + C^*v_n$ for all $v_1, \ldots, v_n \in V$, and so by saturation we get an element of X^* that does not lie in C^*V. $\square$

For certain $(V, C; \mathrm{sc})$ this will be applied to sets $X \subseteq V$ that are definable in a suitable expansion of $(V, C; \mathrm{sc})$, with X^* the corresponding set in an elementary extension of that expansion.

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$\boldsymbol{H}$-couples over ordered fields

Ordered vector spaces come into play as follows. Let K be a Liouville closed H-field. It has the (ordered) constant field C, and the H-couple (Γ, ψ). We have a map $(c, \gamma) \mapsto c\gamma\colon C \times \Gamma \to \Gamma$ such that $cvf = vg$ whenever $f, g \in K^{\times}$ and $cf^{\dagger} = g^{\dagger}$. This map makes Γ into an ordered vector space over C, and $\psi(c\gamma) = \psi(\gamma)$ for all $c \in C^{\times}$ and $\gamma \in \Gamma^{\neq}$.

Accordingly, we define an H-couple over $\boldsymbol{k}$ to be an H-couple (Γ, ψ) where the ordered abelian group Γ is also equipped with a map $\boldsymbol{k} \times \Gamma \to \Gamma$ making Γ into an ordered vector space over $\boldsymbol{k}$ such that $\psi(c\gamma) = \psi(\gamma)$ for all $c \in \boldsymbol{k}^{\times}$ and $\gamma \in \Gamma^{\neq}$. Thus, the H-couple of a Liouville closed H-field is naturally an H-couple over its constant field.

Let (Γ, ψ) be an H-couple over $\boldsymbol{k}$. A basic fact is that for distinct $\alpha, \beta \in \Gamma^{\neq}$ we have $[\psi(\alpha) - \psi(\beta)]_{\boldsymbol{k}} < [\alpha - \beta]_{\boldsymbol{k}}$, since for all $c \in \boldsymbol{k}^{>}$, we have $\psi(\alpha) - \psi(\beta) = \psi(c\alpha) - \psi(c\beta) = o(c(\alpha - \beta))$, by [2, 6.5.4(ii)]. Note also that for all $\alpha, \beta \in \Gamma^{\neq}$,

$$[\alpha]_{\boldsymbol{k}} = [\beta]_{\boldsymbol{k}} \implies \psi(\alpha) = \psi(\beta).$$

Hahn spaces

These are the ordered Hahn spaces from [2, § 2.4]: a Hahn space Γ over $\boldsymbol{k}$ is an ordered vector space over $\boldsymbol{k}$ such that for all $\alpha, \beta \in \Gamma^{\neq}$ with $[\alpha]_{\boldsymbol{k}} = [\beta]_{\boldsymbol{k}}$ there exists $c \in \boldsymbol{k}^{\times}$ such that $[\alpha - c\beta]_{\boldsymbol{k}} < [\alpha]_{\boldsymbol{k}}$.

Examples. (1) Any one-dimensional ordered vector space over $\boldsymbol{k}$ is a Hahn space over $\boldsymbol{k}$.

(2) Any $\boldsymbol{k}$-linear subspace of a Hahn space over $\boldsymbol{k}$ is a Hahn space over $\boldsymbol{k}$.

(3) Any ordered vector space over the ordered field $\mathbb{R}$ is a Hahn space over $\mathbb{R}$.

(4) The ordered $\mathbb{Q}$-vector space $\mathbb{Q} + \mathbb{Q}\sqrt{2} \subseteq \mathbb{R}$ is not a Hahn space over $\mathbb{Q}$.

We say that an H-couple (Γ, ψ) over $\boldsymbol{k}$ is of **Hahn type** if for all $\alpha, \beta \in \Gamma^{\neq}$ with $\psi(\alpha) = \psi(\beta)$ there exists a scalar $c \in \boldsymbol{k}$ such that $\psi(\alpha - c\beta) > \psi(\alpha)$; equivalently, Γ is a Hahn space over $\boldsymbol{k}$ and for all $\alpha, \beta \in \Gamma^{\neq}$,

$$\psi(\alpha) = \psi(\beta) \implies [\alpha]_{\boldsymbol{k}} = [\beta]_{\boldsymbol{k}}.$$

Let K be a Liouville closed H-field. We made its H-couple (Γ, ψ) into an H-couple over its constant field C, and as such (Γ, ψ) is of Hahn type.

Details on the example in the introduction

We consider the Liouville closed H-field $\mathbb{T}$ and its element x with $x' = 1$. For $z \in \mathbb{T}$ with $z' \notin \mathbb{R}$ we have

$$\begin{aligned} zz'' = (z')^2 \iff z^{\dagger}{=}(z')^{\dagger} \iff (z'/z)^{\dagger}{=}0 \iff z' = tz \text{ for some } t \in \mathbb{R}^{\times} \\ \iff z = s\,\mathrm{e}^{tx} \text{ for some } s, t \in \mathbb{R}^{\times}. \end{aligned}$$

Considering also the case where $z' \in \mathbb{R}$ we conclude that

$$\big\{z \in \mathbb{T} : zz'' = (z')^2\big\} = \big\{s\,\mathrm{e}^{tx} : s, t \in \mathbb{R}\big\}.$$

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Next, let $y \in \mathbb{T}^\times$ and suppose $z := y^\dagger$ satisfies $zz'' = (z')^2$. Then $y = r\,\mathrm{e}^u$ for some $r \in \mathbb{R}$ and $u \in \mathbb{T}$ with $u' = z$. For $z = s\,\mathrm{e}^{tx}$ with $s,\, t \in \mathbb{R}$ and $u \in \mathbb{T}$, $u' = z$ we get $u \in \mathbb{R}\,\mathrm{e}^{tx} + \mathbb{R}$ if $t \neq 0$, and $u \in \mathbb{R}x + \mathbb{R}$ if $t = 0$. Hence $y = a\,\mathrm{e}^{b\,\mathrm{e}^{cx}}$ or $y = a\,\mathrm{e}^{bx}$ for some $a,\, b,\, c \in \mathbb{R}$. From $zz'' = (z')^2$ we get

$$y^2 y' y^{(3)} - y^2 (y^{(2)})^2 - y(y')^2 y^{(2)} + (y')^4 = 0.$$

In this way, we get for

$$G(Y) := Y^2 Y' Y^{(3)} - Y^2 (Y^{(2)})^2 - Y(Y')^2 Y^{(2)} + (Y')^4$$

that its set of zeros in $\mathbb{T}$ is

$$\big\{a\,\mathrm{e}^{b\,\mathrm{e}^{cx}} : a, b, c \in \mathbb{R}\big\} \cup \big\{a\,\mathrm{e}^{bx} : a, b \in \mathbb{R}\big\}.$$

It is easy to see that for $0 < c < d$ in $\mathbb{R}$ we have $[v(\mathrm{e}^{\mathrm{e}^{cx}})]_{\mathbb{R}} < [v(\mathrm{e}^{\mathrm{e}^{dx}})]_{\mathbb{R}}$, so the set $\big\{vy : y \in \mathbb{T}^\times,\ G(y) = 0\big\}$ is not contained in a finite-dimensional $\mathbb{R}$-linear subspace of $\Gamma_{\mathbb{T}}$.

2. Extensions of $\boldsymbol{H}$-couples

In this section, $(\Gamma,\, \psi)$ *and* $(\Gamma_1,\, \psi_1)$ *are H-couples over* $\boldsymbol{k}$. An **embedding**

$$h\colon\ (\Gamma, \psi) \to (\Gamma_1, \psi_1)$$

is an embedding $h\colon \Gamma \to \Gamma_1$ of ordered vector spaces over $\boldsymbol{k}$ such that

$$h\big(\psi(\gamma)\big) = \psi_1\big(h(\gamma)\big) \text{ for } \gamma \in \Gamma^{\neq}.$$

If $\Gamma \subseteq \Gamma_1$ and the inclusion $\Gamma \hookrightarrow \Gamma_1$ is an embedding $(\Gamma,\, \psi) \to (\Gamma_1,\, \psi_1)$, then we call $(\Gamma_1,\, \psi_1)$ an **extension** of $(\Gamma,\, \psi)$. If $(\Gamma_1,\, \psi_1)$ is of Hahn type and extends $(\Gamma,\, \psi)$, then $(\Gamma,\, \psi)$ is of Hahn type.

Embedding lemmas

The lemmas in this subsection are the analogues for H-couples over $\boldsymbol{k}$ of similar lemmas for H-couples in [2, § 9.8]. The proofs are essentially the same, so we omit them.

Lemma 2.1. *Let β be a gap in $(\Gamma,\, \psi)$. Then there is an H-couple $(\Gamma + \boldsymbol{k}\alpha,\, \psi^\alpha)$ over $\boldsymbol{k}$ that extends $(\Gamma,\, \psi)$ such that:*

(i) *$\alpha > 0$ and $\alpha' = \beta$;*

(ii) *if $i\colon (\Gamma,\, \psi) \to (\Gamma_1,\, \psi_1)$ is an embedding and $\alpha_1 \in \Gamma_1,\ \alpha_1 > 0,\ \alpha_1' = i(\beta)$, then i extends uniquely to an embedding $j\colon \big(\Gamma + \boldsymbol{k}\alpha,\, \psi^\alpha\big) \to (\Gamma_1,\, \psi_1)$ with $j(\alpha) = \alpha_1$.*

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The universal property (ii) determines $(\Gamma+\boldsymbol{k}\alpha,\,\psi^\alpha)$ up to isomorphism over $(\Gamma,\,\psi)$, and $0<c\alpha<\Gamma^>$ for all $c\in\boldsymbol{k}^>$; moreover, for all $\gamma\in\Gamma$ and $c\in\boldsymbol{k}$ with $\gamma+c\alpha\neq 0$,

$$\psi^\alpha(\gamma+c\alpha)=\begin{cases}\psi(\gamma), & \text{if } \gamma\neq 0,\\ \beta-\alpha, & \text{otherwise.}\end{cases}\tag{1}$$

Note also that $[\Gamma+\boldsymbol{k}\alpha]_{\boldsymbol{k}}=[\Gamma]_{\boldsymbol{k}}\cup\big\{[\alpha]_{\boldsymbol{k}}\big\}$, so for $\Psi^\alpha:=\psi^\alpha((\Gamma+\boldsymbol{k}\alpha)^{\neq})$ we have:

$$\Psi^\alpha=\Psi\cup\{\beta-\alpha\},\quad \max\Psi^\alpha=\psi^\alpha(\alpha)=\beta-\alpha.\tag{2}$$

Lemma 2.1 goes through with $\alpha<0$ and $\alpha_1<0$ in place of $\alpha>0$ and $\alpha_1>0$, respectively. In the setting of this modified lemma, we have $\Gamma^<<c\alpha<0$ for all $c\in\boldsymbol{k}^>$, (1) goes through for $\gamma\in\Gamma$ and $c\in\boldsymbol{k}$ with $\gamma+c\alpha\neq 0$, (2) goes through. So we have two ways to remove a gap. Removal of a gap as above leads by (2) to a grounded H-couple over $\boldsymbol{k}$, and this is the situation we consider next.

Lemma 2.2. *Assume that Ψ has a largest element β. Then there exists an H-couple $(\Gamma+\boldsymbol{k}\alpha,\,\psi^\alpha)$ over $\boldsymbol{k}$ that extends $(\Gamma,\,\psi)$ with $\alpha\neq 0$, $\alpha'=\beta$, such that for any embedding $i\colon(\Gamma,\,\psi)\to(\Gamma_1,\,\psi_1)$ and any $\alpha_1\in\Gamma_1^{\neq}$ with $\alpha_1'=i(\beta)$ there is a unique extension of i to an embedding $j\colon(\Gamma+\boldsymbol{k}\alpha,\,\psi^\alpha)\to(\Gamma_1,\,\psi_1)$ with $j(\alpha)=\alpha_1$.*

Let $(\Gamma+\boldsymbol{k}\alpha,\,\psi^\alpha)$ be as in Lemma 2.2. Then $\Gamma^<<c\alpha<0$ for all $c\in\boldsymbol{k}^>$, $[\Gamma+\boldsymbol{k}\alpha]_{\boldsymbol{k}}=[\Gamma]_{\boldsymbol{k}}\cup\big\{[\alpha]_{\boldsymbol{k}}\big\}$, so (2) holds for $\Psi^\alpha:=\psi^\alpha((\Gamma+\boldsymbol{k}\alpha)^{\neq})$. Thus, our new Ψ-set Ψ^α still has a maximum, but this maximum is larger than the maximum β of the original Ψ-set Ψ. By iterating this construction indefinitely and taking a union, we obtain an H-couple over $\boldsymbol{k}$ with asymptotic integration.

Once we have an H-couple over $\boldsymbol{k}$ with asymptotic integration, we can create an extension with a gap as follows:

Lemma 2.3. *Suppose that $(\Gamma,\,\psi)$ has asymptotic integration. Then there is an H-couple $(\Gamma+\boldsymbol{k}\beta,\,\psi_\beta)$ over $\boldsymbol{k}$ extending $(\Gamma,\,\psi)$ such that*:

(i) $\Psi<\beta<(\Gamma^>)'$;

(ii) *for any $(\Gamma_1,\,\psi_1)$ extending $(\Gamma,\,\psi)$ and $\beta_1\in\Gamma_1$ with $\Psi<\beta_1<(\Gamma^>)'$ there is a unique embedding $(\Gamma+\boldsymbol{k}\beta,\,\psi_\beta)\to(\Gamma_1,\,\psi_1)$ of H-couples over $\boldsymbol{k}$ that is the identity on Γ and sends β to β_1.*

Let $(\Gamma,\,\psi)$ and $(\Gamma+\boldsymbol{k}\beta,\,\psi_\beta)$ be as in Lemma 2.3. If $(\Gamma+\boldsymbol{k}\alpha,\,\psi_\alpha)$ is also an H-couple over $\boldsymbol{k}$ extending $(\Gamma,\,\psi)$ with $\Psi<\alpha<(\Gamma^>)'$, then by (ii) we have an isomorphism $(\Gamma+\boldsymbol{k}\beta,\,\psi_\beta)\to(\Gamma+\boldsymbol{k}\alpha,\,\psi_\alpha)$ of H-couples over $\boldsymbol{k}$ that is the identity on Γ and sends β to α. In this sense, $(\Gamma+\boldsymbol{k}\beta,\,\psi_\beta)$ is unique up to isomorphism over $(\Gamma,\,\psi)$. The construction of $(\Gamma+\boldsymbol{k}\beta,\,\psi_\beta)$ gives the following extra information, with Ψ_β the set of values of ψ_β on $(\Gamma+\boldsymbol{k}\beta)^{\neq}$:

Corollary 2.4. *The set Γ is dense in the ordered abelian group $\Gamma+\boldsymbol{k}\beta$, so $[\Gamma]_{\boldsymbol{k}}=[\Gamma+\boldsymbol{k}\beta]_{\boldsymbol{k}}$, $\Psi_\beta=\Psi$ and β is a gap in $(\Gamma+\boldsymbol{k}\beta,\,\psi_\beta)$.*

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Recall that a cut in an ordered set S is just a downward closed subset of S, and that an element a of an ordered set extending S is said to realize a cut D in S if $D < a < S \setminus D$ (so $a \notin S$).

Lemma 2.5. *Let D be a cut in $[\Gamma^{\neq}]_{\boldsymbol{k}}$ and let $\beta \in \Gamma$ be such that $\beta < (\Gamma^{>})'$, $\gamma^{\dagger} \leqslant \beta$ for all $\gamma \in \Gamma^{\neq}$ with $[\gamma]_{\boldsymbol{k}} > D$, and $\beta \leqslant \delta^{\dagger}$ for all $\delta \in \Gamma^{\neq}$ with $[\delta]_{\boldsymbol{k}} \in D$. Then there exists an H-couple $(\Gamma \oplus \boldsymbol{k}\alpha, \psi^{\alpha})$ over $\boldsymbol{k}$ that extends (Γ, ψ), with $\alpha > 0$, such that:*

(i) *$[\alpha]_{\boldsymbol{k}}$ realizes the cut D in $[\Gamma^{\neq}]_{\boldsymbol{k}}$, and $\psi^{\alpha}(\alpha) = \beta$;*

(ii) *for any embedding $i\colon (\Gamma, \psi) \to (\Gamma_1, \psi_1)$ and $\alpha_1 \in \Gamma_1^{>}$ such that $[\alpha_1]_{\boldsymbol{k}}$ realizes the cut $\{[i(\delta)]_{\boldsymbol{k}} : [\delta]_{\boldsymbol{k}} \in D\}$ in $[i(\Gamma^{\neq})]_{\boldsymbol{k}}$ and $\psi_1(\alpha_1) = i(\beta)$, i extends uniquely to an embedding $j\colon (\Gamma \oplus \boldsymbol{k}\alpha, \psi^{\alpha}) \to (\Gamma_1, \psi_1)$ with $j(\alpha) = \alpha_1$.*

Moreover, $[\Gamma \oplus \boldsymbol{k}\alpha]_{\boldsymbol{k}} = [\Gamma]_{\boldsymbol{k}} \cup \{[\alpha]_{\boldsymbol{k}}\}$ and $\Psi^{\alpha} := \psi^{\alpha}((\Gamma \oplus \boldsymbol{k}\alpha)^{\neq}) = \Psi \cup \{\beta\}$. If (Γ, ψ) is grounded, then so is $(\Gamma \oplus \boldsymbol{k}\alpha, \psi^{\alpha})$. If (Γ, ψ) has asymptotic integration, then so does $(\Gamma \oplus \boldsymbol{k}\alpha, \psi^{\alpha})$. If $\beta \in \Psi^{\downarrow}$, then a gap in (Γ, ψ) remains a gap in $(\Gamma \oplus \boldsymbol{k}\alpha, \psi^{\alpha})$.

Proof. By a straightforward analogue of [2, Lemma 2.4.5] we extend Γ to an ordered vector space $\Gamma^{\alpha} = \Gamma \oplus \boldsymbol{k}\alpha$ over $\boldsymbol{k}$ with $\alpha > 0$ such that $[\alpha]_{\boldsymbol{k}}$ realizes the cut D in $[\Gamma^{\neq}]_{\boldsymbol{k}}$. Then $[\Gamma \oplus \boldsymbol{k}\alpha]_{\boldsymbol{k}} = [\Gamma]_{\boldsymbol{k}} \cup \{[\alpha]_{\boldsymbol{k}}\}$. We extend ψ to $\psi^{\alpha}\colon (\Gamma^{\alpha})^{\neq} \to \Gamma$ by

$$\psi^{\alpha}(\gamma + c\alpha) := \min\{\psi(\gamma), \beta\} \quad \text{for } \gamma \in \Gamma, c \in \boldsymbol{k}^{\times}.$$

Apart from some obvious changes, we now follow the proof of [2, Lemma 9.8.7]. This gives the desired results, except for the last Claim of the lemma. To prove that claim, let $\beta \in \Psi^{\downarrow}$, let $\gamma \in \Gamma$ be a gap in (Γ, ψ), and assume towards a contradiction that γ is not a gap in $(\Gamma^{\alpha}, \psi^{\alpha})$. Then $\gamma > \Psi^{\alpha}$, so $\gamma = (\delta + c\alpha)'$ with $\delta \in \Gamma$, $c \in \boldsymbol{k}^{\times}$ and $0 < \delta + c\alpha < \Gamma^{>}$. Then $[\delta + c\alpha]_{\boldsymbol{k}} \notin [\Gamma]_{\boldsymbol{k}}$, so $[\delta + c\alpha]_{\boldsymbol{k}} = [\alpha]_{\boldsymbol{k}}$. As Ψ has no largest element, we get $\Psi < (\delta + c\alpha)^{\dagger} = \alpha^{\dagger} = \beta$, a contradiction. □

The case of Hahn type

In Lemma 2.1 (and in its variant with $\alpha < 0$), in Lemma 2.2, and in Lemma 2.5 for $\beta \notin \Psi$, we have:

$$\text{if } (\Gamma, \psi) \text{ is of Hahn type, then so is } (\Gamma + \boldsymbol{k}\alpha, \psi^{\alpha}).$$

Suppose (Γ, ψ) and $(\Gamma + \boldsymbol{k}\beta, \psi_{\beta})$ are as in Lemma 2.3, and (Γ, ψ) is of Hahn type. We claim that then $(\Gamma + \boldsymbol{k}\beta, \psi_{\beta})$ is also of Hahn type. To prove this claim, recall from Corollary 2.4 that Γ is dense in $\Gamma + \boldsymbol{k}\beta$. It follows easily that for non-zero $\alpha_1, \alpha_2 \in \Gamma + \boldsymbol{k}\beta$ with $\psi_{\beta}(\alpha_1) = \psi_{\beta}(\alpha_2)$ we have $[\alpha_1]_{\boldsymbol{k}} = [\alpha_2]_{\boldsymbol{k}}$. It remains to show that $\Gamma + \boldsymbol{k}\beta$ is a Hahn space over $\boldsymbol{k}$. So let $\alpha_1, \alpha_2 \in \Gamma + \boldsymbol{k}\beta$ be non-zero with $[\alpha_1]_{\boldsymbol{k}} = [\alpha_2]_{\boldsymbol{k}}$. By density again, and the fact that $[\Gamma]_{\boldsymbol{k}} = [\Gamma + \boldsymbol{k}\beta]_{\boldsymbol{k}}$ has no least element $> [0]_{\boldsymbol{k}}$, we have $\gamma_1, \gamma_2 \in \Gamma$ such that $[\alpha_1 - \gamma_1]_{\boldsymbol{k}} < [\alpha_1]_{\boldsymbol{k}}$ and $[\alpha_2 - \gamma_2]_{\boldsymbol{k}} < [\alpha_2]_{\boldsymbol{k}}$. Take $c \in \boldsymbol{k}^{\times}$ such that $[\gamma_1 - c\gamma_2]_{\boldsymbol{k}} < [\gamma_1]_{\boldsymbol{k}}$. It follows easily that then $[\alpha_1 - c\alpha_2]_{\boldsymbol{k}} < [\alpha_1]_{\boldsymbol{k}}$.

New extension lemmas

The three next lemmas will enable in the next section a simpler proof of QE for closed H-couples than in [1]: in that paper, we needed "properties (A) and (B)" with long and tedious proofs, and here we avoid this.

Lemma 2.6. *Suppose (Γ_1, ψ_1) extends (Γ, ψ). Let $\beta \in \Gamma_1 \setminus \Gamma$ and $\alpha_0 \in \Gamma$ be such that $(\beta - \alpha_0)^\dagger \notin \Gamma$. Then $(\beta - \alpha_0)^\dagger = \max\{(\beta - \alpha)^\dagger : \alpha \in \Gamma\}$. If in addition $\Gamma^<$ is cofinal in $\Gamma_1^<$, then $(\beta - \alpha_0)^\dagger \leqslant$ some element of Ψ.*

Proof. Suppose $\alpha \in \Gamma$ and $(\beta - \alpha)^\dagger > (\beta - \alpha_0)^\dagger$. Then $\alpha - \alpha_0 = (\beta - \alpha_0) - (\beta - \alpha)$ gives $(\beta - \alpha_0)^\dagger = (\alpha - \alpha_0)^\dagger \in \Gamma$, a contradiction. Assume $|\beta - \alpha_0| \geqslant |\gamma|$, $\gamma \in \Gamma^{\neq}$. Then $(\beta - \alpha_0)^\dagger \leqslant \gamma^\dagger \in \Psi$. □

Lemma 2.7. *Suppose (Γ, ψ) is closed and (Γ_1, ψ_1) and (Γ_*, ψ_*) are H-couples over $\boldsymbol{k}$ extending (Γ, ψ). Let $\beta \in \Gamma_1 \setminus \Gamma$ and $\beta_* \in \Gamma_* \setminus \Gamma$ realize the same cut in Γ, and suppose that $\beta^\dagger \notin \Gamma$ and $\Gamma^<$ are cofinal in $(\Gamma + \boldsymbol{k}\beta^\dagger)^<$. Then $\beta_*^\dagger \notin \Gamma$, and $\beta^\dagger$ and $\beta_*^\dagger$ realize the same cut in Γ.*

Proof. Let $\alpha \in \Gamma^{\neq}$. We claim:

$$\beta^\dagger < \alpha^\dagger \;\Rightarrow\; \beta_*^\dagger < \alpha^\dagger, \qquad \beta^\dagger > \alpha^\dagger \;\Rightarrow\; \beta_*^\dagger > \alpha^\dagger.$$

To prove the first implication, assume $\beta^\dagger < \alpha^\dagger$. Then $|\beta| > |\alpha|$, so $|\beta_*| > |\alpha|$, and thus $\beta_*^\dagger \leqslant \alpha^\dagger$. Since (Γ, ψ) is closed and $\Gamma^<$ is cofinal in $(\Gamma + \boldsymbol{k}\beta^\dagger)^<$, we can replace in this argument α by some $\gamma \in \Gamma^{\neq}$ with $\beta^\dagger < \gamma^\dagger < \alpha^\dagger$, to get $\beta_*^\dagger \leqslant \gamma^\dagger < \alpha^\dagger$, and thus $\beta_*^\dagger < \alpha^\dagger$ as claimed. The second implication follows in the same way.

If $\beta^\dagger < \gamma^\dagger$ for some $\gamma \in \Gamma^{\neq}$, then (Γ, ψ) being closed gives the desired conclusion. If $\beta^\dagger > \Psi$, then we use instead $\Psi < \beta^\dagger < (\Gamma^>)'$ and $\Psi < \beta_*^\dagger < (\Gamma^>)'$. □

Lemma 2.8. *Suppose (Γ_1, ψ_1) extends (Γ, ψ). Let $\beta \in \Gamma_1 \setminus \Gamma$ and $\alpha_0, \alpha_1 \in \Gamma$ be such that $\beta_0^\dagger \notin \Gamma$ for $\beta_0 := \beta - \alpha_0$ and $\beta_1^\dagger \notin \Psi$ for $\beta_1 := \beta_0^\dagger - \alpha_1$. Assume also that $|\beta_0| \geqslant |\alpha|$ for some $\alpha \in \Gamma^{\neq}$. Then $\beta_0^\dagger < \beta_1^\dagger$.*

Proof. From $|\beta_0| \geqslant |\alpha|$ with $\alpha \in \Gamma^{\neq}$ we get $\beta_0^\dagger \leqslant \alpha^\dagger$. Also, $\beta_0^\dagger - \alpha^\dagger \notin \Gamma$ and $[\beta_0^\dagger - \alpha_1]_{\boldsymbol{k}} \notin [\Gamma]_{\boldsymbol{k}}$, hence $[\beta_0^\dagger - \alpha^\dagger]_{\boldsymbol{k}} \geqslant [\beta_0^\dagger - \alpha_1]_{\boldsymbol{k}}$. In view of [2, 6.5.4(i)], this gives

$$\beta_0^\dagger = \min(\beta_0^\dagger, \alpha^\dagger) < (\beta_0^\dagger - \alpha^\dagger)^\dagger \;\leqslant\; (\beta_0^\dagger - \alpha_1)^\dagger = \beta_1^\dagger.$$

□

3. Eliminating quantifiers for closed H-couples

Eliminating quantifiers for closed H-couples requires a predicate for their Ψ-set, and in this connection, we need to study the substructures of the thus expanded H-couples. Accordingly, we define an *H-triple over $\boldsymbol{k}$* to be a triple (Γ, ψ, P) where (Γ, ψ) is an H-couple over $\boldsymbol{k}$ and $P \subseteq \Gamma$ is an H-cut in (Γ, ψ).

Lemma 3.1. *Let* (Γ, ψ, P) *be an* H*-triple over* $\boldsymbol{k}$, *and let* $\beta \in P \setminus \Psi$. *Then* (Γ, ψ, P) *can be extended to an* H*-triple* $(\Gamma \oplus \boldsymbol{k}\alpha, \psi^\alpha, P^\alpha)$ *over* $\boldsymbol{k}$ *such that*:

(i) $\alpha > 0$ *and* $\psi^\alpha(\alpha) = \beta$;

(ii) *given any embedding* $i\colon (\Gamma, \psi, P) \to (\Gamma^*, \psi^*, P^*)$ *and any element* $\alpha^* > 0$ *in* Γ^* *with* $\psi^*(\alpha^*) = i(\beta)$, *there is a unique extension of* i *to an embedding* $j\colon (\Gamma \oplus \boldsymbol{k}\alpha, \psi^\alpha, P^\alpha) \to (\Gamma^*, \psi^*, P^*)$ *with* $j(\alpha) = \alpha^*$.

If (Γ, ψ) *is of Hahn type, then so is* $(\Gamma \oplus \boldsymbol{k}\alpha, \psi^\alpha)$.

Proof. Distinguishing various cases this follows from Lemma 2.5, especially the claims beginning with "Moreover". Use also "The case of Hahn type". □

An H*-closure* of an H-triple (Γ, ψ, P) over $\boldsymbol{k}$ is defined to be a closed H-triple (Γ^c, ψ^c, P^c) over $\boldsymbol{k}$ that extends (Γ, ψ, P) such that any embedding

$$(\Gamma, \psi, P) \;\to\; (\Gamma^*, \psi^*, P^*)$$

into a closed H-triple (Γ^*, ψ^*, P^*) over $\boldsymbol{k}$ extends to an embedding

$$(\Gamma^c, \psi^c, P^c) \;\to\; (\Gamma^*, \psi^*, P^*).$$

Corollary 3.2. *Every* H*-triple over* $\boldsymbol{k}$ *has an* H*-closure. Every* H*-triple over* $\boldsymbol{k}$ *of Hahn type has an* H*-closure that is of Hahn type.*

Proof. This is a straightforward consequence of Lemmas 2.1, 2.2, and 3.1, using for the second statement also the remarks in "The case of Hahn type". □

We consider H-triples as $\mathcal{L}_{\boldsymbol{k}}$-structures where $\mathcal{L}_{\boldsymbol{k}}$ is the natural language of ordered vector spaces over $\boldsymbol{k}$, augmented by a constant symbol ∞, a unary function symbol ψ, and a unary relation symbol P. The underlying set of an H-triple (Γ, ψ, P), when construed as an $\mathcal{L}_{\boldsymbol{k}}$-structure, is Γ_∞ rather than Γ, and the symbols of $\mathcal{L}_{\boldsymbol{k}}$ are interpreted in (Γ, ψ, P) as usual, with ∞ serving as a default value:

$$\psi(0) = \psi(\infty) = \gamma + \infty = \infty + \gamma = \infty + \infty = -\infty = c\infty = \infty$$

for $\gamma \in \Gamma$ and $c \in \boldsymbol{k}$. Also $0^\dagger := \infty$ for the zero element $0 \in \Gamma$, so $\Gamma^\dagger = \Psi \cup \{\infty\}$.

Theorem 3.3. *The* $\mathcal{L}_{\boldsymbol{k}}$*-theory of closed* H*-triples over* $\boldsymbol{k}$ *has* QE.

The proof of QE

Towards Theorem 3.3 we consider an H-triple (Γ, ψ, P) over $\boldsymbol{k}$ and closed H-triples (Γ_1, ψ_1, P_1) and (Γ_*, ψ_*, P_*) over $\boldsymbol{k}$ that extend (Γ, ψ, P), and such that (Γ_*, ψ_*, P_*) is $|\Gamma|^+$-saturated. For $\gamma \in \Gamma_1$ we let $(\Gamma\langle\gamma\rangle, \psi_\gamma)$ be the H-couple over $\boldsymbol{k}$ generated by $\Gamma \cup \{\gamma\}$ in (Γ_1, ψ_1), and set $P_\gamma := P_1 \cap \Gamma\langle\gamma\rangle$.

Let $\beta \in \Gamma_1 \setminus \Gamma$. Theorem 3.3 follows if we can show that under these assumptions $(\Gamma\langle\beta\rangle, \psi_\beta, P_\beta)$ can be embedded over Γ into (Γ_*, ψ_*, P_*). We first do this in a situation that may seem rather special:

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Lemma 3.4. *Suppose* (Γ, ψ) *has asymptotic integration and* $(\Gamma + \boldsymbol{k}\beta)^\dagger = \Gamma^\dagger$. *Then* $(\Gamma\langle\beta\rangle, \psi_\beta, P_\beta)$ *can be embedded over* Γ *into* (Γ_*, ψ_*, P_*).

Proof. From $(\Gamma + \boldsymbol{k}\beta)^\dagger = \Gamma^\dagger$ we get $\Gamma\langle\beta\rangle = \Gamma + \boldsymbol{k}\beta$. We have six cases:

Case 1: $(\Gamma^>)^\dagger < \eta < (\Gamma^>)'$ and $\eta \in P_1$ for some $\eta \in \Gamma + \boldsymbol{k}\beta$. Fix such η. Then Γ is dense in $\Gamma + \boldsymbol{k}\eta = \Gamma + \boldsymbol{k}\beta$, by Corollary 2.4, so $[\Gamma + \boldsymbol{k}\beta]_{\boldsymbol{k}} = [\Gamma]_{\boldsymbol{k}}$. Moreover, there is no $\eta_1 \neq \eta$ in $\Gamma + \boldsymbol{k}\beta$ with $(\Gamma^>)^\dagger < \eta_1 < (\Gamma^>)'$. By saturation, we can take $\eta_* \in \Gamma_*$ such that $(\Gamma^>)^\dagger < \eta_* < (\Gamma^>)'$ and $\eta_* \in P_*$. Then [2, 2.4.16] yields an embedding $i\colon \Gamma + \boldsymbol{k}\beta \to \Gamma_*$ of ordered vector spaces over $\boldsymbol{k}$ that is the identity on Γ with $i(\eta) = \eta_*$. This i embeds $(\Gamma\langle\beta\rangle, \psi_\beta, P_\beta)$ into (Γ_*, ψ_*, P_*).

Case 2: $(\Gamma^>)^\dagger < \eta < (\Gamma^>)'$ and $\eta \notin P_1$ for some $\eta \in \Gamma + \boldsymbol{k}\beta$. Fixing such η, we repeat the argument of Case 1, except that now $\eta_* \notin P_*$ instead of $\eta_* \in P_*$.

Case 3: $[\Gamma + \boldsymbol{k}\beta]_{\boldsymbol{k}} = [\Gamma]_{\boldsymbol{k}}$, but there is no $\eta \in \Gamma + \boldsymbol{k}\beta$ with $(\Gamma^>)^\dagger < \eta < (\Gamma^>)'$. Then P_β is the only H-cut of $\Gamma\langle\beta\rangle$. Saturation yields $\beta_* \in \Gamma_*$ realizing the same cut in Γ as β. Then [2, 2.4.16] yields an embedding $i\colon \Gamma + \boldsymbol{k}\beta \to \Gamma_*$ of ordered vector spaces over $\boldsymbol{k}$ that is the identity on Γ with $i(\beta) = \beta_*$. For $\gamma \in \Gamma + \boldsymbol{k}\beta$ we have $[i(\gamma)]_{\boldsymbol{k}} = [\gamma]_{\boldsymbol{k}} \in [\Gamma]_{\boldsymbol{k}}$, so $i(\gamma)^\dagger = \gamma^\dagger \in \Gamma^\dagger$. Thus, i embeds $(\Gamma\langle\beta\rangle, \psi_\beta, P_\beta)$ into (Γ_*, ψ_*, P_*).

Assume next that we are not in Case 1, or Case 2, or Case 3. Then $[\Gamma + \boldsymbol{k}\beta]_{\boldsymbol{k}} \neq [\Gamma]_{\boldsymbol{k}}$. Take $\gamma \in \Gamma\langle\beta\rangle \setminus \Gamma$ such that $\gamma > 0$ and $[\gamma]_{\boldsymbol{k}} \notin [\Gamma]_{\boldsymbol{k}}$, so $[\Gamma\langle\beta\rangle]_{\boldsymbol{k}} = [\Gamma]_{\boldsymbol{k}} \cup \big\{[\gamma]_{\boldsymbol{k}}\big\}$. We are not in Case 1 or Case 2, so P_β is the only H-cut of $(\Gamma\langle\beta\rangle, \psi_\beta)$. Let D be the cut in Γ realized by γ and $E := \Gamma \setminus D$, so $D < \gamma < E$. Then D has no largest element, and so $D \cap \Gamma^> \neq \emptyset$: if $d = \max D$, then we have $0 < \gamma - d < \Gamma^>$, and thus $(\Gamma^>)^\dagger < (\gamma - d)^\dagger < (\Gamma^>)'$, contradicting that we are not in Case 1. Likewise, E has no least element. Here are the remaining cases:

Case 4: $\gamma^\dagger \in (D^{>0})^\dagger \cap E^\dagger$. Saturation yields $\gamma_* \in \Gamma_*$ realizing the same cut in Γ as γ. Then $\gamma_*^\dagger = \gamma^\dagger \in (D^{>0})^\dagger$, and [2, 2.4.16] yields an embedding $i\colon \Gamma + \boldsymbol{k}\beta \to \Gamma_*$ of ordered vector spaces over $\boldsymbol{k}$ that is the identity on Γ with $i(\gamma) = \gamma_*$; this i embeds $(\Gamma\langle\beta\rangle, \psi_\beta, P_\beta)$ into (Γ_*, ψ_*, P_*).

Case 5: $\gamma^\dagger \in (D^{>0})^\dagger > E^\dagger$. Then saturation yields a $\gamma_* \in \Gamma_*$ realizing the same cut in Γ as γ, with $\gamma_*^\dagger = \gamma^\dagger$. By [2, 2.4.16] this yields an embedding $i\colon \Gamma + \boldsymbol{k}\beta \to \Gamma_*$ of ordered vector spaces over $\boldsymbol{k}$ that is the identity on Γ with $i(\gamma) = \gamma_*$, and so as before i embeds $(\Gamma\langle\beta\rangle, \psi_\beta, P_\beta)$ into (Γ_*, ψ_*, P_*).

Case 6: $\gamma^\dagger \in E^\dagger < (D^{>0})^\dagger$. This is handled just like Case 5. □

Note that Cases 4, 5, 6 in the proof above do not occur if (Γ_1, ψ_1) is of Hahn type.

In view of Corollary 3.2 and Lemma 3.4, Theorem 3.3 reduces to:

Lemma 3.5. *Suppose* (Γ, ψ) *is closed and* $(\Gamma + \boldsymbol{k}\gamma)^\dagger \neq \Gamma^\dagger$ *for all* $\gamma \in \Gamma_1 \setminus \Gamma$. *Then* $(\Gamma\langle\beta\rangle, \psi_\beta, P_\beta)$ *embeds into* (Γ_*, ψ_*, P_*) *over* Γ.

Proof. If $\gamma \in \Gamma_1 \setminus \Gamma$ and $\Psi < \gamma < (\Gamma^>)'$, then $(\Gamma + \boldsymbol{k}\gamma)^\dagger = \Gamma^\dagger$, contradicting our assumption. Hence there is no such γ. It follows that $\Gamma^<$ is cofinal in $\Gamma_1^<$.

Take $\alpha_0 \in \Gamma$ such that $(\beta - \alpha_0)^\dagger \notin \Gamma^\dagger$. Since (Γ, ψ) is closed, this means $(\beta - \alpha_0)^\dagger \notin \Gamma$. Next take $\alpha_1 \in \Gamma$ with $((\beta - \alpha_0)^\dagger - \alpha_1)^\dagger \notin \Gamma^\dagger$. Continuing this way, we obtain sequences $\alpha_0, \alpha_1, \alpha_2, \ldots$ in Γ and $\beta_0, \beta_1, \beta_2, \ldots$ in $\Gamma\langle\beta\rangle \setminus \Gamma$ with

$$\beta_0 = \beta - \alpha_0, \qquad \beta_{n+1} = \beta_n^\dagger - \alpha_{n+1} \text{ for all } n,$$

such that $\beta_n^\dagger \notin \Gamma$ for all n. By Lemma 2.8 we have $\beta_0^\dagger < \beta_1^\dagger < \beta_2^\dagger < \cdots$. It follows that $[\beta_0]_{\boldsymbol{k}} > [\beta_1]_{\boldsymbol{k}} > [\beta_2]_{\boldsymbol{k}} > \cdots$, with $[\beta_n]_{\boldsymbol{k}} \notin [\Gamma]_{\boldsymbol{k}}$ for all n. In particular, the family (β_n) is $\boldsymbol{k}$-linearly independent over Γ, and

$$\Gamma\langle\beta\rangle = \Gamma \oplus \boldsymbol{k}\beta_0 \oplus \boldsymbol{k}\beta_1 \oplus \boldsymbol{k}\beta_2 \oplus \cdots .$$

By saturation we can take $\beta_* \in \Gamma_* \setminus \Gamma$ realizing the same cut in Γ as β. This gives an embedding $e_0 \colon \Gamma \oplus \boldsymbol{k}\beta \to \Gamma_*$ of ordered vector spaces over $\boldsymbol{k}$ that is the identity on Γ and sends β to β_*. We define recursively $\beta_{*n} \in (\Gamma_*)_\infty$ by

$$\beta_{*0} := \beta_* - \alpha_0, \qquad \beta_{*(n+1)} := \beta_{*n}^\dagger - \alpha_{n+1}.$$

Assume inductively that $\beta_{*0}, \ldots, \beta_{*n} \in \Gamma_*$ and that we have an embedding

$$e_n \ : \ \Gamma + \boldsymbol{k}\beta_0 + \cdots + \boldsymbol{k}\beta_n \to \Gamma_*$$

of ordered vector spaces over $\boldsymbol{k}$ that is the identity on Γ and sends β_i to β_{*i} for $i = 0, \ldots, n$. Then β_n and β_{*n} realize the same cut in Γ, and so $\beta_{*n}^\dagger \notin \Gamma$, and $\beta_n^\dagger$ and $\beta_{*n}^\dagger$ realize the same cut in Γ by Lemma 2.7. Hence β_{n+1} and $\beta_{*(n+1)} \in \Gamma_* \setminus \Gamma$ realize the same cut in Γ. Moreover, $\beta_{*n}^\dagger < \beta_{*(n+1)}^\dagger$ by Lemma 2.8. We have

$$[\Gamma + \boldsymbol{k}\beta_0 + \cdots + \boldsymbol{k}\beta_n]_{\boldsymbol{k}} = [\Gamma]_{\boldsymbol{k}} \cup \big\{[\beta_0]_{\boldsymbol{k}}, \ldots, [\beta_n]_{\boldsymbol{k}}\big\}, \quad [\beta_0]_{\boldsymbol{k}} > \cdots > [\beta_n]_{\boldsymbol{k}} > [\beta_{n+1}]_{\boldsymbol{k}}.$$

Let D be the cut realized by $[\beta_{n+1}]_{\boldsymbol{k}}$ in $[\Gamma + \boldsymbol{k}\beta_0 + \cdots + \boldsymbol{k}\beta_n]_{\boldsymbol{k}}$. Then the above together with $[\beta_{*n}]_{\boldsymbol{k}} > [\beta_{*(n+1)}]_{\boldsymbol{k}}$ shows that $[\beta_{*(n+1)}]_{\boldsymbol{k}}$ realizes the e_n-image of the cut D in $[e_n(\Gamma + \boldsymbol{k}\beta_0 + \cdots + \boldsymbol{k}\beta_n)]_{\boldsymbol{k}}$. Hence e_n extends to an embedding

$$e_{n+1} \ : \ \Gamma + \boldsymbol{k}\beta_1 + \cdots + \boldsymbol{k}\beta_n + \boldsymbol{k}\beta_{n+1} \to \Gamma_*$$

of ordered vector spaces over $\boldsymbol{k}$ that is the identity on Γ and sends β_{n+1} to $\beta_{*(n+1)}$. This leads to a map $e \colon \Gamma\langle\beta\rangle \to \Gamma_*$ that extends each e_n, and is, therefore, an embedding of H-couples over $\boldsymbol{k}$. Since P_β is the only H-cut in $\Gamma\langle\beta\rangle$, e embeds $(\Gamma\langle\beta\rangle, \psi_\beta, P_\beta)$ into (Γ_*, ψ_*, P_*) over Γ. $\square$

This concludes the proof of Theorem 3.3.

Let $T_{\boldsymbol{k}}$ be the $\mathcal{L}_{\boldsymbol{k}}$-theory of closed H-triples over $\boldsymbol{k}$. Let $T_{\boldsymbol{k}}^{>}$ be the $\mathcal{L}_{\boldsymbol{k}}$-theory whose models are the closed H-triples (Γ, ψ, P) over $\boldsymbol{k}$ with $0 \in P$, equivalently $\Psi \cap \Gamma^{>} \neq \emptyset$. Let $T_{\boldsymbol{k}}^{<}$ be the $\mathcal{L}_{\boldsymbol{k}}$-theory whose models are the closed H-triples (Γ, ψ, P) over $\boldsymbol{k}$ with $0 \notin P$, equivalently $\Psi \subseteq \Gamma^{<}$.

Corollary 3.6. *The $\mathcal{L}_{\boldsymbol{k}}$-theory $T_{\boldsymbol{k}}$ has exactly two completions: $T_{\boldsymbol{k}}^{>}$ and $T_{\boldsymbol{k}}^{<}$.*

Proof. We have an H-triple $(\{0\}, \psi_0, \{0\})$ over $\boldsymbol{k}$ that embeds into every model of $T_{\boldsymbol{k}}^{>}$, and an H-triple $(\{0\}, \psi_0, \emptyset)$ over $\boldsymbol{k}$ that embeds into every model of $T_{\boldsymbol{k}}^{<}$. Here ψ_0 is the "empty" function $\emptyset \to \{0\}$. $\square$

Suppose K is a Liouville closed H-field. Then its H-couple (Γ, ψ) is naturally an H-couple over its constant field C. The case $(\Gamma, \psi) \models T_C^{>}$ corresponds to the derivation ∂ of K being small (that is, $\partial f \prec 1$ for all $f \prec 1$ in K), while the case $(\Gamma, \psi) \models T_C^{<}$

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corresponds to this derivation not being small. For example, the usual derivation $\frac{d}{dx}$ of $\mathbb{T}$ is small. The derivation $x^2\frac{d}{dx}$ on $\mathbb{T}$ is not small, but $\mathbb{T}$ with this derivation is still Liouville closed.

4. Simple extensions

Let (Γ, ψ) be an H-couple over $\boldsymbol{k}$ with asymptotic integration, and let (Γ^*, ψ^*) be an H-couple over $\boldsymbol{k}$ that extends (Γ, ψ). For $\gamma \in \Gamma^*$, let $(\Gamma\langle\gamma\rangle, \psi_\gamma)$ denote the H-couple over $\boldsymbol{k}$ generated by $\Gamma \cup \{\gamma\}$ in (Γ^*, ψ^*). Let $\beta \in \Gamma^* \setminus \Gamma$. The following result yields a useful description of the "simple" extension $(\Gamma\langle\beta\rangle, \psi_\beta)$, where i, n range over $\mathbb{N} = \{0, 1, 2, \ldots\}$:

Proposition 4.1. *One of the following occurs*:

(a) $(\Gamma + \boldsymbol{k}\beta)^\dagger = \Gamma^\dagger$;

(b) *there are sequences* (α_i) *in* Γ *and* (β_i) *in* Γ^* *such that* (β_i) *is* $\boldsymbol{k}$*-linearly independent over* Γ, $\beta_0 = \beta - \alpha_0$ *and* $\beta_{i+1} = \beta_i^\dagger - \alpha_{i+1}$ *for all* i, *and such that* $\Gamma\langle\beta\rangle = \Gamma \oplus \bigoplus_{i=0}^\infty \boldsymbol{k}\beta_i$.

$(\mathrm{c})_n$ *there are* $\alpha_0, \ldots, \alpha_n \in \Gamma$, *and non-zero* $\beta_0, \ldots, \beta_n \in \Gamma^*$ *such that* $\beta_0 = \beta - \alpha_0$, $\beta_{i+1} = \beta_i^\dagger - \alpha_{i+1}$ *for* $i < n$, *the vectors* $\beta_0, \ldots, \beta_n, \beta_n^\dagger$ *are* $\boldsymbol{k}$*-linearly independent over* Γ, $(\Gamma + \boldsymbol{k}\beta_n^\dagger)^\dagger = \Gamma^\dagger$, *and* $\Gamma\langle\beta\rangle = \Gamma \oplus \bigoplus_{i=0}^n \boldsymbol{k}\beta_i \oplus \boldsymbol{k}\beta_n^\dagger$.

$(\mathrm{d})_n$ *there are* $\alpha_0, \ldots, \alpha_n \in \Gamma$, *and non-zero* $\beta_0, \ldots, \beta_n \in \Gamma^*$ *such that* $\beta_0 = \beta - \alpha_0$, $\beta_{i+1} = \beta_i^\dagger - \alpha_{i+1}$ *for* $i < n$, *the vectors* $\beta_0, \ldots, \beta_n$ *are* $\boldsymbol{k}$*-linearly independent over* Γ, $\beta_n^\dagger \in \Gamma \setminus \Gamma^\dagger$, *and* $\Gamma\langle\beta\rangle = \Gamma \oplus \bigoplus_{i=0}^n \boldsymbol{k}\beta_i$.

Note that in case (a) we have $\Gamma\langle\beta\rangle = \Gamma \oplus \boldsymbol{k}\beta$, a case described in more detail in Lemma 3.4. The proof below gives extra information about the other cases.

Proof. Suppose we are not in case (a). Then we have $\alpha_0 \in \Gamma$ and $\beta_0 := \beta - \alpha_0$ with $\beta_0^\dagger \notin \Gamma^\dagger$. This is the first step in inductively constructing elements $\alpha_i \in \Gamma$ and $\beta_i \in \Gamma\langle\beta\rangle \setminus \Gamma_0$, either for all i, or for all $i \leqslant n$ for a certain n. Suppose we already have $\alpha_0, \ldots, \alpha_n \in \Gamma$ and $\beta_0, \ldots, \beta_n \in \Gamma\langle\beta\rangle \setminus \Gamma$ with α_0 and β_0 as above, $\beta_{i+1} = \beta_i^\dagger - \alpha_{i+1}$ and $\beta_i^\dagger \notin \Gamma$ for $i < n$, and $\beta_n^\dagger \notin \Gamma^\dagger$. Thus, $[\beta_i]_{\boldsymbol{k}} \notin [\Gamma]_{\boldsymbol{k}}$ for $i \leqslant n$.

Claim 1: $\beta_0^\dagger < \cdots < \beta_n^\dagger$.

Claim 2: there is no $\eta \in \Gamma + \boldsymbol{k}\beta_0 + \cdots + \boldsymbol{k}\beta_n$ with $\Psi < \eta < (\Gamma^>)'$.

To prove Claim 1, assume towards a contradiction that $\beta_i^\dagger \geqslant \beta_{i+1}^\dagger$, $i < n$. Then by Lemma 2.8 we have $0 < |\beta_i| < \Gamma^>$, so $\Psi < \beta_i^\dagger < (\Gamma^>)'$, and thus $[\beta_{i+1}]_{\boldsymbol{k}} \in [\Gamma]_{\boldsymbol{k}}$ by Corollary 2.4, a contradiction. It follows from Claim 1 that $[\beta_0]_{\boldsymbol{k}} > \cdots > [\beta_n]_{\boldsymbol{k}}$ and that $\beta_0, \ldots, \beta_n$ are $\boldsymbol{k}$-linearly independent over Γ. As to Claim 2, suppose towards a contradiction that $\Psi < \gamma + \delta < (\Gamma^>)'$ where $\gamma \in \Gamma$, $\delta \in \boldsymbol{k}\beta_0 + \cdots + \boldsymbol{k}\beta_n$. Then $\delta \neq 0$, and so $[\delta]_{\boldsymbol{k}} \notin [\Gamma]_{\boldsymbol{k}}$. With $D := \Psi - \gamma$ and $E := (\Gamma^>)' - \gamma$, we have $D < \delta < E$. On the other hand, for every $\varepsilon \in \Gamma^>$ there are $d \in D$ and $e \in E$ with $e - d < \varepsilon$, so Γ is dense in $\Gamma + \boldsymbol{k}\delta$ by [2, 2.4.17], contradicting $[\delta]_{\boldsymbol{k}} \notin [\Gamma]_{\boldsymbol{k}}$. This concludes the proof of Claim 2.

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If $(\beta_n^\dagger - \alpha_{n+1})^\dagger \notin \Gamma^\dagger$ for some $\alpha_{n+1} \in \Gamma$ (so $\beta_n^\dagger \notin \Gamma$), then we take such an α_{n+1} and set $\beta_{n+1} := \beta_n^\dagger - \alpha_{n+1}$. If there is no such α_{n+1}, then the construction breaks off, with α_n and β_n as the last vectors.

Suppose the construction goes on indefinitely. Then it yields infinite sequences (α_i) and (β_i) as in case (b), in particular, $\Gamma\langle\beta\rangle = \Gamma \oplus \bigoplus_{i=0}^{\infty} \boldsymbol{k}\beta_i$,

$$\Psi_\beta \;:=\; \psi^*\big(\Gamma\langle\beta\rangle^{\neq}\big) = \Psi \cup \big\{\beta_i^\dagger : i \in \mathbb{N}\big\},$$

and $(\Gamma\langle\beta\rangle, \psi_\beta)$ has asymptotic integration by Claim 2.

Next, assume that the construction stops with α_n and β_n as the last vectors. Thus, $(\Gamma + \boldsymbol{k}\beta_n^\dagger)^\dagger = \Gamma^\dagger$. We have two cases:

Case 1: $\beta_n^\dagger \notin \Gamma$. Then $\alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n$ are as in case $(\mathrm{c})_n$. Here is why. Set $\Delta := \Gamma + \boldsymbol{k}\beta_n^\dagger$, so $\Delta^\dagger = \Gamma^\dagger$. From $\beta_i^\dagger \notin \Delta^\dagger$ for all $i \leqslant n$ and Claim 1 we obtain that $\beta_0, \ldots, \beta_n$ are $\boldsymbol{k}$-linearly independent over Δ, with

$$(\Delta + \boldsymbol{k}\beta_0 + \cdots + \boldsymbol{k}\beta_n)^\dagger \;\subseteq\; \Delta + \boldsymbol{k}\beta_0 + \cdots + \boldsymbol{k}\beta_n,$$

and $\beta \in \Delta + \boldsymbol{k}\beta_0$, which proves the assertion.

Case 2: $\beta_n^\dagger \in \Gamma$. Then $\alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n$ are as in case $(\mathrm{d})_n$. Here is why. From $\beta_i^\dagger \notin \Gamma^\dagger$ for all $i \leqslant n$ and Claim 1 we obtain that $\beta_0, \ldots, \beta_n$ are $\boldsymbol{k}$-linearly independent over Γ, with

$$(\Gamma + \boldsymbol{k}\beta_0 + \cdots + \boldsymbol{k}\beta_n)^\dagger \;\subseteq\; \Gamma + \boldsymbol{k}\beta_0 + \cdots + \boldsymbol{k}\beta_n,$$

and $\beta \in \Gamma + \boldsymbol{k}\beta_0$, which proves the assertion. $\square$

In case $(\mathrm{d})_n$ we have $\beta_n^\dagger \in \Gamma \setminus \Gamma^\dagger$, and this cannot happen if (Γ, ψ) is closed. The proof of Proposition 4.1 yields some further results that are needed later:

Lemma 4.2. *Let* (α_i) *and* (β_i) *be as in* (b). *Then*:

(i) $\beta_i^\dagger \notin \Gamma$ *for all* i, *and thus* $[\beta_i]_{\boldsymbol{k}} \notin [\Gamma]_{\boldsymbol{k}}$ *for all* i;

(ii) $(\beta_i^\dagger)$ *is strictly increasing, and thus* $([\beta_i]_{\boldsymbol{k}})$ *is strictly decreasing*;

(iii) $[\Gamma\langle\beta\rangle]_{\boldsymbol{k}} = [\Gamma]_{\boldsymbol{k}} \cup \big\{[\beta_i]_{\boldsymbol{k}} : i \in \mathbb{N}\big\}$, *and thus* $\Psi_\beta = \Psi \cup \big\{\beta_i^\dagger : i \in \mathbb{N}\big\}$;

(iv) *there is no* $\eta \in \Gamma\langle\beta\rangle$ *with* $\Psi < \eta < (\Gamma^>)'$;

(v) $(\Gamma\langle\beta\rangle, \psi_\beta)$ *has asymptotic integration*;

(vi) $\Gamma^<$ *is cofinal in* $\Gamma\langle\beta\rangle^<$.

If (Γ, ψ) *is closed and* $\gamma \in \Gamma^* \setminus \Gamma$ *realizes the same cut in* Γ *as* β, *then we have an isomorphism* $(\Gamma\langle\beta\rangle, \psi_\beta) \to (\Gamma\langle\gamma\rangle, \psi_\gamma)$ *of* H*-couples over* $\boldsymbol{k}$ *that is the identity on* Γ *and sends* β *to* γ. *If* (Γ, ψ) *is of Hahn type, then so is* $(\Gamma\langle\beta\rangle, \psi_\beta)$.

Proof. As to (i), this follows from the $\boldsymbol{k}$-linear independence of (β_i) over Γ and from $\beta_i^\dagger = \beta_{i+1} + \alpha_{i+1}$. Hence the sequences (α_i), and (β_i) conform to the construction in the proof of Proposition 4.1, and so other parts of that proof yield (ii)–(vi). The next statement follows as in the proof of Lemma 3.5 using Lemma 2.7 and (iv).

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Suppose that (Γ, ψ) is of Hahn type. We show that then $\Gamma\langle\beta\rangle$ is a Hahn space; the additional argument required for showing that $(\Gamma\langle\beta\rangle, \psi_\beta)$ is of Hahn type is similar and left to the reader. So let $\delta_1, \delta_2 \in \Gamma\langle\beta\rangle^{\neq}$ satisfy $[\delta_1]_{\boldsymbol{k}} = [\delta_2]_{\boldsymbol{k}}$; we have to find $c \in \boldsymbol{k}$ such that $[\delta_1 - c\delta_2]_{\boldsymbol{k}} < [\delta_1]_{\boldsymbol{k}}$. Now

$$\delta_1 = \gamma_1 + \sum_i c_{i1}\beta_i, \quad \delta_2 = \gamma_2 + \sum_i c_{i2}\beta_i, \quad \gamma_1, \gamma_2 \in \Gamma,$$

with all $c_{i1}, c_{i2} \in \boldsymbol{k}$, and $c_{i1} = c_{i2} = 0$ for all but finitely many i. Consider first the case $[\delta_1]_{\boldsymbol{k}} \in [\Gamma]_{\boldsymbol{k}}$. Then $[\gamma_1]_{\boldsymbol{k}} > [\beta_i]_{\boldsymbol{k}}$ for all i with $c_{i1} \neq 0$, by (i), (ii), (iii), and so $\delta_1 = \gamma_1 + \alpha_1$ with $[\alpha_1]_{\boldsymbol{k}} < [\gamma_1]_{\boldsymbol{k}} = [\delta_1]_{\boldsymbol{k}}$, and likewise $\delta_2 = \gamma_2 + \alpha_2$ with $[\alpha_2]_{\boldsymbol{k}} < [\gamma_2]_{\boldsymbol{k}} = [\delta_2]_{\boldsymbol{k}}$. Take $c \in \boldsymbol{k}$ such that $[\gamma_1 - c\gamma_2]_{\boldsymbol{k}} < [\gamma_1]_{\boldsymbol{k}}$. Then $\delta_1 - c\delta_2 = \gamma_1 - c\gamma_2 + \alpha_1 - c\alpha_2$, so $[\delta_1 - c\delta_2]_{\boldsymbol{k}} < [\gamma_1]_{\boldsymbol{k}} = [\delta_1]_{\boldsymbol{k}}$. Next, suppose $[\delta_1]_{\boldsymbol{k}} \notin [\Gamma]_{\boldsymbol{k}}$. Then $c_{i1} \neq 0$ for some i; let j be the least such i. Then $[\gamma_1]_{\boldsymbol{k}} < [\beta_j]_{\boldsymbol{k}}$ and $[\delta_1]_{\boldsymbol{k}} = [\beta_j]_{\boldsymbol{k}}$ by (ii). Now j is also the least i with $c_{i2} \neq 0$, in view of $[\delta_1]_{\boldsymbol{k}} = [\delta_2]_{\boldsymbol{k}}$. Then $[\delta_1 - c\delta_2]_{\boldsymbol{k}} < [\delta_1]_{\boldsymbol{k}}$ for $c \in \boldsymbol{k}$ with $c_{j1} = cc_{j2}$. $\square$

Lemma 4.3. *Let $\alpha_0, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be as in (c)$_n$, and set $\Delta := \Gamma + \boldsymbol{k}\beta_n^\dagger$, so $\Delta^\dagger = \Gamma^\dagger$ and $\Gamma\langle\beta\rangle = \Delta \oplus \boldsymbol{k}\beta_0 \oplus \cdots \oplus \boldsymbol{k}\beta_n$. Then:*

(i) *$\Gamma^<$ is cofinal in $\Delta^<$;*

(ii) *$\beta_0^\dagger, \ldots, \beta_n^\dagger \notin \Gamma$,, and thus $[\beta_0]_{\boldsymbol{k}}, \ldots, [\beta_n]_{\boldsymbol{k}} \notin [\Delta]_{\boldsymbol{k}}$;*

(iii) *$\beta_0^\dagger < \cdots < \beta_n^\dagger$, and thus $[\beta_0]_{\boldsymbol{k}} > \cdots > [\beta_n]_{\boldsymbol{k}}$;*

(iv) *$\Psi_\beta = \Psi \cup \{\beta_0^\dagger, \ldots, \beta_n^\dagger\}$ and $[\Gamma\langle\beta\rangle]_{\boldsymbol{k}} = [\Delta]_{\boldsymbol{k}} \cup \big\{[\beta_0]_{\boldsymbol{k}}, \ldots, [\beta_n]_{\boldsymbol{k}}\big\}$;*

(v) *there is no $\gamma \in \Delta + \boldsymbol{k}\beta_0 + \cdots + \boldsymbol{k}\beta_{n-1}$ with $0 < \gamma < \Gamma^>$;*

(vi) *if $|\beta_n| \geqslant \alpha$ for some $\alpha \in \Gamma^>$, then $\Gamma^<$ is cofinal in $\Gamma\langle\beta\rangle^<$ and so a gap in (Δ, ψ_Δ), if any, remains a gap in $(\Gamma\langle\beta\rangle, \psi_\beta)$;*

(vii) *if $|\beta_n| < \Gamma^>$, then $(\Gamma\langle\beta\rangle, \psi_\beta)$ is grounded with $\max \Psi_\beta = \beta_n^\dagger$;*

(viii) *if (Δ, ψ_Δ) has no gap, then there is no $\eta \in \Gamma\langle\beta\rangle$ with $\Psi < \eta < (\Gamma^>)'$, and so $\Gamma^<$ is cofinal in $\Gamma\langle\beta\rangle^<$ and $(\Gamma\langle\beta\rangle, \psi_\beta)$ has asymptotic integration.*

Proof. As to (i), if $\delta \in \Delta$ and $\Gamma^< < \delta < 0$, then $\Psi < \delta^\dagger$, contradicting $\Delta^\dagger = \Gamma^\dagger$. Item (ii) follows from the $\boldsymbol{k}$-linear independence of $\beta_0, \ldots, \beta_n, \beta_n^\dagger$ over Γ and from $\beta_i^\dagger = \beta_{i+1} + \alpha_{i+1}$ for $i < n$. Next, we obtain (iii) from Claim 1 in the proof of Proposition 4.1, and then (iv) follows easily. As to (v), by (ii) and (iii) we have

$$[\Delta + \boldsymbol{k}\beta_0 + \cdots + \boldsymbol{k}\beta_{n-1}]_{\boldsymbol{k}} = [\Delta]_{\boldsymbol{k}} \cup \big\{[\beta_0]_{\boldsymbol{k}}, \ldots, [\beta_{n-1}]_{\boldsymbol{k}}\big\}.$$

Thus, assuming towards a contradiction that (v) is false gives $\gamma \in \Delta \cup \{\beta_0, \ldots, \beta_{n-1}\}$ with $0 < |\gamma| < \Gamma^>$. Then $\Psi < \gamma^\dagger < (\Gamma^>)'$, and so $\gamma \notin \Delta$. Hence $\gamma = \beta_i$ with $i < n$, and so $\gamma^\dagger \in \Gamma + \boldsymbol{k}\beta_0 + \cdots + \boldsymbol{k}\beta_n$, contradicting Claim 2 in the proof of Proposition 4.1 with $\gamma^\dagger$ in the role of η. By similar arguments, if $0 < \gamma < \Gamma^>$ for some $\gamma \in \Gamma\langle\beta\rangle$, then $0 < |\beta_n| < \Gamma^>$. This gives (vi). For (vii), assume $|\beta_n| < \Gamma^>$. Then (i), (iv), (v) give $[\beta_n]_{\boldsymbol{k}} = \min[\Gamma\langle\beta\rangle^{\neq}]_{\boldsymbol{k}}$, and thus $\max \Psi_\beta = \beta_n^\dagger$.

As to (viii), note first that $\Psi = \Psi_\Delta$. Assume (Δ, ψ_Δ) has no gap. Then (Δ, ψ_Δ) has asymptotic integration. Hence by Claim 2 in the proof of Proposition 4.1, applied to Δ instead of Γ, there is no $\eta \in \Gamma\langle\beta\rangle$ with $\Psi < \eta < (\Gamma^>)'$. □

Lemma 4.4. *Let $\alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n$ be as in (d)$_n$. Then:*

(i) $\beta_0^\dagger, \ldots, \beta_{n-1}^\dagger \notin \Gamma$, $\beta_n^\dagger \notin \Psi$, *and thus* $[\beta_0]_{\boldsymbol{k}}, \ldots, [\beta_n]_{\boldsymbol{k}} \notin [\Gamma]_{\boldsymbol{k}}$;

(ii) $\beta_0^\dagger < \cdots < \beta_n^\dagger$, *and thus* $[\beta_0]_{\boldsymbol{k}} > \cdots > [\beta_n]_{\boldsymbol{k}}$;

(iii) $\Psi_\beta = \Psi \cup \{\beta_0^\dagger, \ldots, \beta_n^\dagger\}$ *and* $[\Gamma\langle\beta\rangle]_{\boldsymbol{k}} = [\Gamma]_{\boldsymbol{k}} \cup \big\{[\beta_0]_{\boldsymbol{k}}, \ldots, [\beta_n]_{\boldsymbol{k}}\big\}$;

(iv) *there is no* $\eta \in \Gamma\langle\beta\rangle$ *with* $\Psi < \eta < (\Gamma^>)'$;

(v) $\Gamma^<$ *is cofinal in* $\Gamma\langle\beta\rangle^<$, *and* $(\Gamma\langle\beta\rangle, \psi_\beta)$ *has asymptotic integration.*

Proof. The first part of (i) follows from the recursion satisfied by $\beta_0, \ldots, \beta_n$, the $\boldsymbol{k}$-linear independence of $\beta_0, \ldots, \beta_n$ over Γ, and $\beta_n^\dagger \notin \Psi$. Claim 1 in the proof of Proposition 4.1 gives (ii), which together with (i) yields (iii). Claim 2 in that proof gives (iv), which has (v) as an easy consequence. □

The next result is crucial in the proof of Theorem 0.1 in § 5. Here (Γ^*, ψ^*) is equipped with an H-cut P^*, and we set $P := P^* \cap \Gamma = \Psi^\downarrow$, and $P_\gamma := P^* \cap \Gamma\langle\gamma\rangle$ for $\gamma \in \Gamma^*$, so we have the H-triples (Γ, ψ, P), $(\Gamma\langle\gamma\rangle, \psi_\gamma, P_\gamma) \subseteq (\Gamma^*, \psi^*, P^*)$ over $\boldsymbol{k}$.

Lemma 4.5. *Assume (Γ^*, ψ^*) is closed, of Hahn type, and $\Gamma^<$ is not cofinal in $(\Gamma^*)^<$. Then for some $\delta \in (\Gamma^*)^>$, all $\gamma \in \Gamma^*$ with $|\beta - \gamma| < \delta$ yield an isomorphism $(\Gamma\langle\beta\rangle, \psi_\beta, P_\beta) \to (\Gamma\langle\gamma\rangle, \psi_\gamma, P_\gamma)$ over Γ sending β to γ.*

Proof. Suppose we are in Case (a) of Proposition 4.1. There are three subcases:

Subcase 1: $(\Gamma^>)^\dagger < \eta < (\Gamma^>)'$ and $\eta \in P^*$ for some $\eta \in \Gamma + \boldsymbol{k}\beta$. Fix such η and recall from Case 1 in the proof of Lemma 3.4 that Γ is dense in $\Gamma + \boldsymbol{k}\eta = \Gamma + \boldsymbol{k}\beta$. Thus, if $\varepsilon \in \Gamma^*$ and $0 < \varepsilon < \Gamma^>$, then $(\Gamma^>)^\dagger < \eta - \varepsilon < \eta$. Moreover, P^* has no largest element, so we can take $\varepsilon \in (\Gamma^*)^>$ so small that for all $\zeta \in \Gamma^*$ with $|\eta - \zeta| < \varepsilon$ we have $(\Gamma^>)^\dagger < \zeta < (\Gamma^>)'$ and $\zeta \in P^*$; in particular, such ζ realizes the same cut in Γ as η. Take $\alpha \in \Gamma$ and $c \in \boldsymbol{k}^\times$ with $\beta = \alpha + c\eta$. Then for ζ as above and $\gamma := \alpha + c\zeta$ the condition $|\eta - \zeta| < \varepsilon$ amounts to $|\beta - \gamma| < \delta := |c|\varepsilon$, with an isomorphism $(\Gamma\langle\beta\rangle, \psi_\beta, P_\beta) \to (\Gamma\langle\gamma\rangle, \psi_\gamma, P_\gamma)$ over Γ sending β to γ.

Subcase 2: $(\Gamma^>)^\dagger < \eta < (\Gamma^>)'$ and $\eta \notin P^*$ for some $\eta \in \Gamma + \boldsymbol{k}\beta$. This can be treated in the same way as Subcase 1.

Subcase 3: there is no $\eta \in \Gamma + \boldsymbol{k}\beta$ with $(\Gamma^>)^\dagger < \eta < (\Gamma^>)'$. Take $\delta \in \Gamma^*$ such that $0 < \delta < \Gamma^>$. Then all $\gamma \in \Gamma^*$ with $|\gamma - \beta| < \delta$ realize the same cut in Γ as β: otherwise we would have $\alpha \in \Gamma$ with $0 < |\alpha - \beta| < \Gamma^>$, so $(\Gamma^>)^\dagger < (\alpha - \beta)^\dagger < (\Gamma^>)'$, a contradiction. Now (Γ^*, ψ^*) is of Hahn type, so $[\Gamma + \boldsymbol{k}\beta]_{\boldsymbol{k}} = [\Gamma]_{\boldsymbol{k}}$. As in Case 3 in the proof of Lemma 3.4 this yields for any such γ an isomorphism $(\Gamma\langle\beta\rangle, \psi_\beta, P_\beta) \to (\Gamma\langle\gamma\rangle, \psi_\gamma, P_\gamma)$ over Γ sending β to γ.

Assume we are in Case (b) of Proposition 4.1, and let (α_i) and (β_i) be as in that case. Let $\varepsilon \in \Gamma^*$ be such that $[\varepsilon]_{\boldsymbol{k}} < [\beta_0]_{\boldsymbol{k}}$. Then $\beta_0 + \varepsilon = (\beta + \varepsilon) - \alpha_0$, $[\beta_0 + \varepsilon]_{\boldsymbol{k}} = [\beta_0]_{\boldsymbol{k}}$, and

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thus $(\beta_0+\varepsilon)^\dagger=\beta_0^\dagger$. It follows that with $\beta+\varepsilon$ instead of β we are also in case (b), with associated sequences (α_i) and $(\beta_{i,\varepsilon})$, with $\beta_{0,\varepsilon}:=\beta_0+\varepsilon$ and $\beta_{i,\varepsilon}:=\beta_i$ for $i\geqslant 1$. As noted in the proof of Lemma 4.2, the sequences (α_i), (β_i) conform to the construction in the proof of Proposition 4.1, and so the latter proof yields an isomorphism $(\Gamma\langle\beta\rangle,\psi_\beta,P_\beta)\to(\Gamma\langle\beta+\varepsilon\rangle,\psi_{\beta+\varepsilon},P_{\beta+\varepsilon})$ over Γ that sends β_i to $\beta_{i,\varepsilon}$ for each i, and thus β to $\beta+\varepsilon$.

Next, assume we are in Case $(\mathrm{c})_n$ of Proposition 4.1, and let $\alpha_0,\ldots,\alpha_n,\beta_0,\ldots,\beta_n$ be as in that case. As before, let $\varepsilon\in\Gamma^*$ be such that $[\varepsilon]_{\boldsymbol{k}}<[\beta_0]_{\boldsymbol{k}}$. Then $\beta_0+\varepsilon=(\beta+\varepsilon)-\alpha_0$, $[\beta_0+\varepsilon]_{\boldsymbol{k}}=[\beta_0]_{\boldsymbol{k}}$, so $(\beta_0+\varepsilon)^\dagger=\beta_0^\dagger$. Hence with $\beta+\varepsilon$ instead of β we are again in case $(\mathrm{c})_n$, with associated sequences $\alpha_0,\ldots,\alpha_n$ and $\beta_{0,\varepsilon},\ldots,\beta_{n,\varepsilon}$, with $\beta_{0,\varepsilon}:=\beta_0+\varepsilon$ and $\beta_{i,\varepsilon}:=\beta_i$ for $1\leqslant i\leqslant n$. Note also that β and $\beta+\varepsilon$ give rise to the same $\Delta=\Gamma+\boldsymbol{k}\beta_n^\dagger=\Gamma+\boldsymbol{k}\beta_{n,\varepsilon}^\dagger$. It now follows from Lemma 4.3 that we have an isomorphism $(\Gamma\langle\beta\rangle,\psi_\beta)\to(\Gamma\langle\beta+\varepsilon\rangle,\psi_{\beta+\varepsilon})$ of H-couples over $\boldsymbol{k}$ that is the identity on Δ and sends β_i to $\beta_{i,\varepsilon}$ for each $i\leqslant n$, and thus, β to $\beta+\varepsilon$. Since β and $\beta+\varepsilon$ yield the same Δ, it follows easily from (vi), (vii), (viii) of Lemma 4.3 that this isomorphism maps P_β onto $P_{\beta+\varepsilon}$.

Finally, assume we are in Case $(\mathrm{d})_n$ of Proposition 4.1, and let $\alpha_0,\ldots,\alpha_n,\beta_0,\ldots,\beta_n$ be as in that case. Let $\varepsilon\in\Gamma^*$ be such that $[\varepsilon]_{\boldsymbol{k}}<[\beta_0]_{\boldsymbol{k}}$. Then $\beta_0+\varepsilon=(\beta+\varepsilon)-\alpha_0$, $[\beta_0+\varepsilon]_{\boldsymbol{k}}=[\beta_0]_{\boldsymbol{k}}$, so $(\beta_0+\varepsilon)^\dagger=\beta_0^\dagger$. Hence with $\beta+\varepsilon$ instead of β we are again in case $(\mathrm{d})_n$, with associated sequences $\alpha_0,\ldots,\alpha_n$ and $\beta_{0,\varepsilon},\ldots,\beta_{n,\varepsilon}$, with $\beta_{0,\varepsilon}:=\beta_0+\varepsilon$ and $\beta_{i,\varepsilon}:=\beta_i$ for $1\leqslant i\leqslant n$. Then Lemma 4.4 yields an isomorphism $(\Gamma\langle\beta\rangle,\psi_\beta,P_\beta)\to(\Gamma\langle\beta+\varepsilon\rangle,\psi_{\beta+\varepsilon},P_{\beta+\varepsilon})$ of H-triples over $\boldsymbol{k}$ that is the identity on Γ and sends β_i to $\beta_{i,\varepsilon}$ for each $i\leqslant n$, and thus β to $\beta+\varepsilon$. $\square$

5. Closed *H*-couples of Hahn type

So far we have treated H-couples over $\boldsymbol{k}$ as one-sorted structures, by keeping $\boldsymbol{k}$ fixed and having for each scalar c a separate unary function symbol that is interpreted as scalar multiplication by c. We now go to the setting where an H-couple over $\boldsymbol{k}$ is viewed as a 2-sorted structure with $\boldsymbol{k}$ as a second sort, and thus with "Hahn type" as a first-order condition. Extending an H-couple may now involve extending $\boldsymbol{k}$, so we begin with a subsection on the process of scalar extension for Hahn spaces. We remind the reader that the ordered scalar field $\boldsymbol{k}$ is not necessarily real closed.

Scalar extension

Let Γ be a Hahn space over $\boldsymbol{k}$, and let $\boldsymbol{k}^*$ be an ordered field extension of $\boldsymbol{k}$. Then we have the vector space $\Gamma_{\boldsymbol{k}^*}:=\boldsymbol{k}^*\otimes_{\boldsymbol{k}}\Gamma$ over $\boldsymbol{k}^*$. We have the $\boldsymbol{k}$-linear embedding $\gamma\mapsto 1\otimes\gamma\colon\Gamma\to\Gamma_{\boldsymbol{k}^*}$ via which we identify Γ with a $\boldsymbol{k}$-linear subspace of $\Gamma_{\boldsymbol{k}^*}$. We make $\Gamma_{\boldsymbol{k}^*}$ into a Hahn space over $\boldsymbol{k}^*$ as follows: for any $\gamma\in\Gamma^{\neq}_{\boldsymbol{k}^*}$ we have $\gamma=c_1\gamma_1+\cdots+c_m\gamma_m$ with $m\geqslant 1$, $c_1,\ldots,c_m\in(\boldsymbol{k}^*)^\times$, $\gamma_1\ldots,\gamma_m\in\Gamma^>$, $[\gamma_1]_{\boldsymbol{k}}>\cdots>[\gamma_m]_{\boldsymbol{k}}$; then $\gamma>0$ iff $c_1>0$. This makes Γ into an ordered $\boldsymbol{k}$-linear subspace of $\Gamma_{\boldsymbol{k}^*}$, and we have an order-preserving bijection $[\gamma]_{\boldsymbol{k}}\to[\gamma]_{\boldsymbol{k}^*}\colon[\Gamma]_{\boldsymbol{k}}\to[\Gamma_{\boldsymbol{k}^*}]_{\boldsymbol{k}^*}$.

Lemma 5.1. *Assume $[\Gamma^{\neq}]_{\boldsymbol{k}}$ has no least element. Then for every $\gamma^*\in\Gamma_{\boldsymbol{k}^*}\setminus\Gamma$ there is an element $\varepsilon\in\Gamma^>$ such that $|\gamma^*-\gamma|>\varepsilon$ for all $\gamma\in\Gamma$.*

Proof. Let $\gamma^*\in\Gamma_{\boldsymbol{k}^*}\setminus\Gamma$, so $\gamma^*=c_1\gamma_1+\cdots+c_m\gamma_m$ with $m\geqslant 1$, $c_1,\ldots,c_m\in(\boldsymbol{k}^*)^\times$, $\gamma_1\ldots,\gamma_m\in\Gamma^>$, $[\gamma_1]_{\boldsymbol{k}}>\cdots>[\gamma_m]_{\boldsymbol{k}}$. To show that γ^* has the claimed property we can assume $c_1 \notin \boldsymbol{k}$. Take any $\varepsilon \in \Gamma^>$ with $[\varepsilon]_{\boldsymbol{k}} < [\gamma_1]_{\boldsymbol{k}}$, and assume towards a contradiction that $\gamma \in \Gamma$ and $|\gamma^* - \gamma| \leqslant \varepsilon$. Then $[\gamma]_{\boldsymbol{k}^*} = [\gamma^*]_{\boldsymbol{k}^*} = [\gamma_1]_{\boldsymbol{k}^*}$, so $[\gamma]_{\boldsymbol{k}} = [\gamma_1]_{\boldsymbol{k}}$, and hence $[\gamma - c\gamma_1]_{\boldsymbol{k}} < [\gamma_1]_{\boldsymbol{k}}$ with $c \in \boldsymbol{k}$. In view of

$$\gamma^* - \gamma = (c_1 - c)\gamma_1 + c_2\gamma_2 + \cdots + c_m\gamma_m - (\gamma - c\gamma_1)$$

and $c_1 \neq c$, this yields a contradiction. $\square$

We also have the following universal property:

Corollary 5.2. *Any embedding $\Gamma \to \Gamma^*$ of ordered vector spaces over $\boldsymbol{k}$ into an ordered vector space Γ^* over $\boldsymbol{k}^*$ such that the induced map $[\Gamma]_{\boldsymbol{k}} \to [\Gamma^*]_{\boldsymbol{k}^*}$ is injective extends uniquely to an embedding $\Gamma_{\boldsymbol{k}^*} \to \Gamma^*$ of ordered vector spaces over $\boldsymbol{k}^*$.*

Let (Γ, ψ) be an H-couple over $\boldsymbol{k}$ of Hahn type and $\boldsymbol{k}^*$ an ordered field extension of $\boldsymbol{k}$. The H-couple $(\Gamma, \psi)_{\boldsymbol{k}^*} := (\Gamma_{\boldsymbol{k}^*}, \psi_{\boldsymbol{k}^*})$ over $\boldsymbol{k}^*$ is determined by requiring that $\psi_{\boldsymbol{k}^*}$ extends ψ. Note that then $(\Gamma, \psi)_{\boldsymbol{k}^*}$ is also of Hahn type and has the same Ψ-set as (Γ, ψ). The following is close to [1, Lemma 3.7], whose proof uses a form of Hahn's Embedding Theorem. Here we use instead Lemma 5.1.

Lemma 5.3. *If $\gamma \in \Gamma$ is a gap in (Γ, ψ), then γ remains a gap in $(\Gamma, \psi)_{\boldsymbol{k}^*}$. If γ^* is a gap in $(\Gamma, \psi)_{\boldsymbol{k}^*}$, then $\gamma^* \in \Gamma$. Thus, (Γ, ψ) has asymptotic integration if and only if $(\Gamma, \psi)_{\boldsymbol{k}^*}$ has asymptotic integration.*

Proof. Suppose towards a contradiction that $\gamma \in \Gamma$ is a gap in (Γ, ψ), but not in $(\Gamma, \psi)_{\boldsymbol{k}^*}$. Then $\gamma = \alpha'$ with $\alpha \in \Gamma^>_{\boldsymbol{k}^*} \setminus \Gamma$. From $\gamma < (\Gamma^>)'$ we get $0 < \alpha < \Gamma^>$, but this contradicts that by Lemma 5.1 we have $|\alpha| > \varepsilon$ for some $\varepsilon \in \Gamma^>$.

Next, assume γ^* is a gap in $(\Gamma, \psi)_{\boldsymbol{k}^*}$. Then $\Psi < \gamma^* < (\Gamma^>)'$, and for all $\varepsilon \in \Gamma^>$ there are $\alpha \in \Psi$ and $\beta \in (\Gamma^>)'$ (namely $\alpha := \varepsilon^\dagger$ and $\beta := \varepsilon'$) with $\beta - \alpha \leqslant \varepsilon$. In view of Lemma 5.1 this yields $\gamma^* \in \Gamma$. $\square$

Normalized *H*-couples

Let (Γ, ψ) be an H-couple over $\boldsymbol{k}$. By [2, § 9.2], if $\Psi \cap \Gamma^> \neq \emptyset$, then $\psi(\gamma) = \gamma$ for a unique $\gamma \in \Gamma^>$; this unique fixed point of ψ on $\Gamma^>$ is then denoted by 1. Referring to (Γ, ψ) as a *normalized H-couple* means that $\Psi \cap \Gamma^> \neq \emptyset$, and that we consider Γ as equipped with this fixed point 1 as a distinguished element. (The term "normalized" is justified, because for any H-couple over $\boldsymbol{k}$ with underlying ordered vector space $\Gamma \neq \{0\}$ we can arrange $\Psi \cap \Gamma^> \neq \emptyset$ by replacing its function ψ with a suitable "shift" $\alpha + \psi$ where $\alpha \in \Gamma$.) For minor technical reasons, it is convenient to restrict our attention in the remainder of this paper to normalized H-couples; this is hardly a loss of generality, as we saw. Note also that the H-couple of $\mathbb{T}$ is normalized by taking $1 = v(x^{-1})$.

Below we construe a normalized H-couple over $\boldsymbol{k}$ as a 2-sorted structure

$$\boldsymbol{\Gamma} = \big((\Gamma, \psi), \boldsymbol{k}; \mathrm{sc}\big)$$

where (Γ, ψ) is an H-couple as defined in the beginning of § 1, $\boldsymbol{k}$ is an ordered field, and $\mathrm{sc}\colon \boldsymbol{k} \times \Gamma \to \Gamma$ is a scalar multiplication that makes Γ into an ordered vector space over

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$\boldsymbol{k}$ (but we shall write $c\gamma$ instead of $\mathrm{sc}(c,\gamma)$ for $c \in \boldsymbol{k}$ and $\gamma \in \Gamma$), such that $\psi(c\gamma) = \psi(\gamma)$ for $c \in \boldsymbol{k}^{\times}$, $\gamma \in \Gamma$; in addition we assume Γ to be equipped with an element $1 > 0$ such that $\psi(1) = 1$. Such $\boldsymbol{\Gamma}$ is said to be *of Hahn type* if the H-couple (Γ, ψ) over $\boldsymbol{k}$ is of Hahn type as defined in § 1. In the same way, we may consider a normalized H-triple over $\boldsymbol{k}$ as a 2-sorted structure

$$\boldsymbol{\Gamma} = \big((\Gamma, \psi, P), \boldsymbol{k}; \mathrm{sc}\big).$$

The language and theory of normalized H-triples of Hahn type

We construe a normalized H-triple $\boldsymbol{\Gamma} = ((\Gamma, \psi, P), \boldsymbol{k}; \mathrm{sc})$ of Hahn type as an $\mathcal{L}_H$-structure, where $\mathcal{L}_H$ is the two-sorted language with the following non-logical symbols:

(i) P, $<$, 0, 1,∞, $-$, $+$, ψ, interpreted as usual in $\Gamma_\infty := \Gamma \cup \{\infty\}$, the linear ordering on Γ being extended to a linear order on Γ_∞ by $\gamma < \infty$ for $\gamma \in \Gamma$, and with ∞ serving as a default value by setting $-\infty = \infty$, $\gamma + \infty = \infty + \gamma = \infty + \infty = \psi(0) = \psi(\infty) = \infty$ for $\gamma \in \Gamma$;

(ii) $<$, 0, 1, ∞, $-$, $+$, $\cdot$, interpreted as usual in $\boldsymbol{k}_\infty := \boldsymbol{k} \cup \{\infty\}$, the linear ordering on $\boldsymbol{k}$ being extended to a linear order on $\boldsymbol{k}_\infty$ by $c < \infty$ for $c \in \boldsymbol{k}$, and with ∞ serving as a default value by setting $-\infty = \infty$, $c + \infty = \infty + c = \infty + \infty = c\infty = \infty c = \infty\infty = \infty$ for $c \in \boldsymbol{k}$;

(iii) a symbol sc for the map $\boldsymbol{k}_\infty \times \Gamma_\infty \to \Gamma_\infty$ that is the scalar multiplication on $\boldsymbol{k} \times \Gamma$, and taking the value ∞ at all other points of $\boldsymbol{k}_\infty \times \Gamma_\infty$;

(iv) a symbol : for the function $\Gamma^2_\infty \to \boldsymbol{k}_\infty$ that assigns to every $(\alpha, \beta) \in \Gamma^2$ with $[\alpha]_{\boldsymbol{k}} \leqslant [\beta]_{\boldsymbol{k}}$ and $\beta \neq 0$ the unique scalar $\alpha : \beta = c \in \boldsymbol{k}$ such that $[\alpha - c\beta]_{\boldsymbol{k}} < [\beta]_{\boldsymbol{k}}$, and assigns to all other pairs in Γ^2_∞ the value ∞.

The symbols in (i) should be distinguished from those in (ii) even though we use the same written signs for convenience. The two default values ∞ are included to make all primitives totally defined. Note that in (iv) we have $\alpha : \beta = 0$ if $[\alpha]_{\boldsymbol{k}} < [\beta]_{\boldsymbol{k}}$.

Using $a1 : b1 = a/b$ for $a, b \in \boldsymbol{k}$ with $b \neq 0$, we see that a substructure of a normalized H-triple of Hahn type is also a normalized H-triple of Hahn type, with possibly smaller scalar field. Thus, the $\mathcal{L}_H$-theory of normalized H-triples of Hahn type has a universal axiomatization (which would be easy to specify). Let there be given normalized H-triples of Hahn type,

$$\boldsymbol{\Gamma}_0 = \big((\Gamma_0, \psi_0, P_0), \boldsymbol{k}_0; \mathrm{sc}_0\big) \quad \text{and} \quad \boldsymbol{\Gamma} = \big((\Gamma, \psi, P), \boldsymbol{k}; \mathrm{sc}\big).$$

An *embedding* $\boldsymbol{\Gamma}_0 \to \boldsymbol{\Gamma}$ is a pair $i = (i_{\mathrm{v}}, i_{\mathrm{s}})$ whose vector part $i_{\mathrm{v}} \colon \Gamma_0 \to \Gamma$ is an embedding of ordered abelian group and whose scalar part $i_{\mathrm{s}} \colon \boldsymbol{k}_0 \to \boldsymbol{k}$ is an embedding of ordered fields such that $i_{\mathrm{v}}(c\gamma) = i_{\mathrm{s}}(c) i_{\mathrm{v}}(\gamma)$ and $\gamma \in P_0 \Leftrightarrow i_{\mathrm{v}}(\gamma) \in P$ for all $c \in \boldsymbol{k}_0$ and $\gamma \in \Gamma_0$, and $i_{\mathrm{v}}(\psi_0(\gamma)) = \psi(i_{\mathrm{v}}(\gamma))$ for all non-zero $\gamma \in \Gamma_0$ (and so $i_{\mathrm{v}}(1) = 1$ and $i_c(\alpha : \beta) = i_{\mathrm{v}}(\alpha) : i_{\mathrm{v}}(\beta)$ for all $\alpha, \beta \in \Gamma$). If $\boldsymbol{k}_0 = \boldsymbol{k}$, then an embedding $e \colon (\Gamma_0, \psi_0, P_0) \to (\Gamma, \psi, P)$ of H-triples over $\boldsymbol{k}$ in the usual sense yields an embedding $(e, \mathrm{id}_{\boldsymbol{k}}) \colon \boldsymbol{\Gamma}_0 \to \boldsymbol{\Gamma}$ as above.

Quantifier elimination

Let T_H be the $\mathcal{L}_H$-theory of normalized closed H-triples of Hahn type, and recall that the H-couple of $\mathbb{T}$ is naturally a model of T_H. In this subsection, we let $\boldsymbol{\Gamma} = ((\Gamma, \psi, P), \boldsymbol{k}; \mathrm{sc})$ and $\boldsymbol{\Gamma}^* = ((\Gamma^*, \psi^*, P^*), \boldsymbol{k}^*; \mathrm{sc}^*)$ denote normalized closed H-triples of Hahn type, construed as models of T_H. The key embedding result is as follows:

Proposition 5.4. *Assume $\boldsymbol{\Gamma}^*$ is κ-saturated for $\kappa = |\Gamma|^+$. Let $\boldsymbol{\Gamma}_0$ be a substructure of $\boldsymbol{\Gamma}$ with scalar field $\boldsymbol{k}_0$. Let an embedding $i_0 \colon \boldsymbol{\Gamma}_0 \to \boldsymbol{\Gamma}^*$ be given, and an embedding $e\colon \boldsymbol{k} \to \boldsymbol{k}^*$ of ordered fields such that $e|_{\boldsymbol{k}_0} = (i_0)_{\mathrm{s}}$. Then i_0 can be extended to an embedding $i\colon \boldsymbol{\Gamma} \to \boldsymbol{\Gamma}^*$ such that $i_{\mathrm{s}} = e$.*

Proof. By Corollary 5.2 on extending scalars, the remarks following it, and (to handle the P-predicate) Lemma 5.3 we can reduce to the case $\boldsymbol{k}_0 = \boldsymbol{k}$. It remains to appeal to the embedding result established in the proof of Theorem 3.3. $\square$

In what follows, *formula* means $\mathcal{L}_H$*-formula*. Let $x = (x_1, \dots, x_m)$ denote a tuple of distinct scalar variables and $y = (y_1, \dots, y_n)$ a tuple of distinct vector variables.

Corollary 5.5. *Suppose that $\boldsymbol{\Gamma}$ is a substructure of $\boldsymbol{\Gamma}^*$. Then*

$$\boldsymbol{\Gamma} \preccurlyeq \boldsymbol{\Gamma}^* \ (\text{as } \mathcal{L}_H\text{-structures}) \quad \Longleftrightarrow \quad \boldsymbol{k} \preccurlyeq \boldsymbol{k}^* \ (\text{as ordered fields}).$$

Proof. The direction $\Rightarrow$ being trivial, we assume $\boldsymbol{k} \preccurlyeq \boldsymbol{k}^*$ and shall derive $\boldsymbol{\Gamma} \preccurlyeq \boldsymbol{\Gamma}^*$. By induction on formulas $\phi(x, y)$ (with x and y as above) we show that for all $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}^*$ as in the hypothesis of the lemma and all $c \in \boldsymbol{k}^m$ and $\gamma \in \Gamma^n$,

$$\boldsymbol{\Gamma} \models \phi(c, \gamma) \quad \Longleftrightarrow \quad \boldsymbol{\Gamma}^* \models \phi(c, \gamma). \tag{$*$}$$

For the inductive step, let $\phi = \exists z \theta$, where $\theta = \theta(x, y, z)$ is a formula and z is a single variable of the scalar or vector sort. The direction $\Rightarrow$ in (??) holds by the (implicit) inductive assumption. Assume $\boldsymbol{\Gamma}^* \models \phi(c, \gamma)$ where $c \in \boldsymbol{k}^m$ and $\gamma \in \Gamma^n$. Take a κ-saturated elementary extension $\boldsymbol{\Gamma}_1$ of $\boldsymbol{\Gamma}$, where $\kappa = |\Gamma^*|^+$. Let $\boldsymbol{k}_1$ be the scalar field of Γ_1. Then we have an elementary embedding $e\colon \boldsymbol{k}^* \to \boldsymbol{k}_1$ that is the identity on $\boldsymbol{k}$. Proposition 5.4 (with $\boldsymbol{\Gamma}$, $\boldsymbol{\Gamma}^*$, $\boldsymbol{\Gamma}_1$ in the roles of $\boldsymbol{\Gamma}_0$, $\boldsymbol{\Gamma}$, $\boldsymbol{\Gamma}^*$) gives an embedding $i\colon \boldsymbol{\Gamma}^* \to \boldsymbol{\Gamma}_1$ where $i_{\mathrm{s}} = e$ and i_{v} is the identity on Γ. By the (tacit) inductive hypothesis on θ we obtain $\boldsymbol{\Gamma}_1 \models \phi(c, \gamma)$, and thus $\boldsymbol{\Gamma} \models \phi(c, \gamma)$. $\square$

With x, y as above, call a formula $\eta(x, y)$ a *scalar formula* if it has the form $\zeta(s_1(x, y), \dots, s_N(x, y))$ where $\zeta(z_1, \dots, z_N)$ is a formula in the language of ordered rings (as specified in (ii) of the description of $\mathcal{L}_H$), where $z_1, \dots, z_N$ are distinct scalar variables and $s_1(x, y), \dots, s_N(x, y)$ are scalar-valued terms of $\mathcal{L}_H$.

Theorem 5.6. *Every formula $\phi(x, y)$ is T_H-equivalent to a boolean combination of scalar formulas $\eta(x, y)$ and atomic formulas $\alpha(x, y)$.*

As a consequence, extending T_H by axioms that the scalar field is real closed gives outright QE, without requiring scalar formulas.

Proof. Suppose $(c, \gamma) \in \boldsymbol{k}^m \times \Gamma^n$ and $(c^*, \gamma^*) \in (\boldsymbol{k}^*)^m \times (\Gamma^*)^n$ satisfy the same scalar formulas $\eta(x, y)$ and atomic formulas $\alpha(x, y)$ in $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}^*$, respectively. It suffices to derive from this assumption that (c, γ) and (c^*, γ^*) satisfy the same formulas in $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}^*$. We may assume that $\boldsymbol{\Gamma}^*$ is κ-saturated where $\kappa = |\Gamma|^+$. Let $\boldsymbol{\Gamma}_0$ with scalar field $\boldsymbol{k}_0$ be the substructure of $\boldsymbol{\Gamma}$ generated by (c, γ). Since (c, γ) and (c^*, γ^*) realize the same atomic formulas $\alpha(x, y)$, we have an embedding $i_0 \colon \boldsymbol{\Gamma}_0 \to \boldsymbol{\Gamma}^*$ such that $i_0(c) = c^*$ and $i_0(\gamma) = \gamma^*$. They also realize the same scalar formulas $\eta(x, y)$, so we have an elementary embedding $e \colon \boldsymbol{k} \to \boldsymbol{k}^*$ agreeing with $(i_0)_{\mathrm{s}}$ on $\boldsymbol{k}_0$. Proposition 5.4 then yields an embedding $i \colon \boldsymbol{\Gamma} \to \boldsymbol{\Gamma}^*$ extending i_0 with $i_{\mathrm{s}} = e$. Then i is an elementary embedding by Corollary 5.5, so (c, γ) and (c^*, γ^*) do indeed satisfy the same formulas in $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}^*$. $\square$

Discrete definable sets

We are finally ready to prove the theorem announced in the introduction. We state it here in its natural general setting:

Theorem 5.7. *Let $\boldsymbol{\Gamma} = ((\Gamma, \psi, P), \boldsymbol{k}; \mathrm{sc})$ be a normalized closed H-triple of Hahn type and let $X \subseteq \Gamma$ be definable in $\boldsymbol{\Gamma}$. Then the following are equivalent*:

(i) *X is contained in a finite-dimensional $\boldsymbol{k}$-linear subspace of Γ*;

(ii) *X is discrete*;

(iii) *X has empty interior in Γ.*

Proof. The direction (i) $\Rightarrow$ (ii) holds by Lemma 1.2. The direction (ii) $\Rightarrow$ (iii) is obvious. (These two implications do not need X to be definable.)

As to (iii) $\Rightarrow$ (i), assume X has empty interior. Take a formula $\phi(y)$ over $\boldsymbol{\Gamma}$ in a single vector variable y that defines the set X in $\boldsymbol{\Gamma}$. We use Theorem 5.6 to arrange that $\phi(y)$ is a boolean combination of scalar formulas over $\boldsymbol{\Gamma}$ and atomic formulas over $\boldsymbol{\Gamma}$. Take a $|\Gamma|^+$-saturated elementary extension $\boldsymbol{\Gamma}^* = ((\Gamma^*, \psi^*, P^*), \boldsymbol{k}^*; \mathrm{sc}^*)$ of $\boldsymbol{\Gamma}$, and let $X^* \subseteq \Gamma^*$ be defined by $\phi(y)$ in $\boldsymbol{\Gamma}^*$. We identify $\Gamma_{\boldsymbol{k}^*}$ with $\boldsymbol{k}^*\Gamma \subseteq \Gamma^*$ in the usual way. We Claim that $X^* \subseteq \Gamma_{\boldsymbol{k}^*}$. (This gives (i) by Lemma 1.3.) Consider the substructure $\boldsymbol{\Gamma}_{\boldsymbol{k}^*} = ((\Gamma_{\boldsymbol{k}^*}, \psi_{\boldsymbol{k}^*}, P_{\boldsymbol{k}^*}), \boldsymbol{k}^*; \mathrm{sc}^*)$ of $\boldsymbol{\Gamma}^*$; it has asymptotic integration by Lemma 5.3. Let $X_{\boldsymbol{k}^*} \subseteq \Gamma_{\boldsymbol{k}^*}$ be defined in $\boldsymbol{\Gamma}_{\boldsymbol{k}^*}$ by $\phi(y)$. Then $X_{\boldsymbol{k}^*} = X^* \cap \Gamma_{\boldsymbol{k}^*}$, so our claim amounts to $X^* = X_{\boldsymbol{k}^*}$. Suppose towards a contradiction that $\gamma^* \in X^* \setminus X_{\boldsymbol{k}^*}$. In particular, $\gamma^* \in \Gamma^* \setminus \Gamma_{\boldsymbol{k}^*}$. Saturation yields an $\varepsilon \in \Gamma^*$ such that $0 < \varepsilon < c^*\gamma$ for all positive c^* in $\boldsymbol{k}^*$ and all positive $\gamma \in \Gamma$, so $0 < \varepsilon < \Gamma_{\boldsymbol{k}^*}^{>}$, and thus $\Gamma_{\boldsymbol{k}^*}^{>}$ is not coinitial in $(\Gamma^*)^{>}$. Lemma 4.5 then yields a $\delta > 0$ in Γ^* such that all $\gamma \in \Gamma^*$ with $|\gamma - \gamma^*| < \delta$ yield an isomorphism

$$\big(\Gamma_{\boldsymbol{k}^*}\langle\gamma^*\rangle, \psi_{\gamma^*}, P_{\gamma^*}\big) \;\cong\; \big(\Gamma_{\boldsymbol{k}^*}\langle\gamma\rangle, \psi_{\gamma}, P_{\gamma}\big) \;\subseteq\; \big(\Gamma^*, \psi^*, P^*\big)$$

of H-triples over $\boldsymbol{k}^*$ sending γ^* to γ. Hence $s(\gamma^*) = s(\gamma)$ for such γ and any scalar-valued $\mathcal{L}_H$-term $s(y)$ over $\boldsymbol{\Gamma}$, and so $\boldsymbol{\Gamma}^* \models \phi(\gamma)$ for those γ. Thus, the interval $(\gamma^* - \delta, \gamma^* + \delta)$ in Γ^* lies entirely in X^*, contradicting that X^* is discrete in Γ^*. $\square$

6. Further results about closed *H*-couples

We briefly return to the *one-sorted* setting of H-couples (or H-triples) and give two easy applications of Theorem 3.3.

Definable closure

Let $\mathbf{\Gamma}^* = (\Gamma^*, \psi^*, P^*)$ be a closed H-triple over $\boldsymbol{k}$. Then we have the notion of the *definable closure* of a set $\Gamma \subseteq \Gamma^*$ in $\mathbf{\Gamma}^*$, and thus of such a set Γ being *definably closed* in $\mathbf{\Gamma}^*$. If $\Gamma \subseteq \Gamma^*$ is definably closed in $\mathbf{\Gamma}^*$, then Γ is (the underlying set of) a subgroup of Γ^* with $\psi^*(\Gamma^{\neq}) \subseteq \Gamma$, and thus we have an H-triple (Γ, ψ, P) over $\boldsymbol{k}$ with $(\Gamma, \psi, P) \subseteq \mathbf{\Gamma}^*$.

Proposition 6.1. *Let* (Γ, ψ, P) *be an* H*-triple over* $\boldsymbol{k}$ *with* $(\Gamma, \psi, P) \subseteq \mathbf{\Gamma}$. *Then:*

$$\Gamma \text{ is definably closed in } (\Gamma^*, \psi^*, P^*) \quad \Longleftrightarrow \quad (\Gamma, \psi) \text{ has asymptotic integration.}$$

Proof. For $\Rightarrow$, note that for every $\gamma \in \Gamma$ there is a unique $\alpha \in (\Gamma^*)^{\neq}$ with $\gamma = \alpha'$.

For the converse, assume that (Γ, ψ) has asymptotic integration (so $P = \Psi^{\downarrow}$). Iterating the construction of Lemma 3.1, we obtain an increasing continuous chain

$$\big((\Gamma_\lambda, \psi_\lambda, P_\lambda)\big)_{\lambda<\nu} \qquad (\text{with } \nu \text{ an ordinal})$$

of H-triples contained in (Γ^*, ψ^*, P^*) as substructures, with $(\Gamma_0, \psi_0, P_0) = (\Gamma, \psi, P)$, such that every $(\Gamma_\lambda, \psi_\lambda, P_\lambda)$ has asymptotic integration with P_λ being the downward closure of Ψ_0 in Γ_λ, and such that the union

$$(\Gamma^{c}, \psi^{c}, P^{c}) \;:=\; \bigcup_{\lambda<\nu} (\Gamma_\lambda, \psi_\lambda, P_\lambda)$$

is closed. The reference to Lemma 3.1 means that for $\lambda < \lambda + 1 < \nu$ we have $\Gamma_{\lambda+1} = \Gamma_\lambda \oplus \boldsymbol{k}\alpha_\lambda$ with $\alpha_\lambda > 0$ and $\alpha_\lambda^{\dagger} \in P_\lambda \setminus \psi_\lambda(\Gamma_\lambda^{\neq})$. That the chain is continuous means that $(\Gamma_\mu, \psi_\lambda, P_\mu) = \bigcup_{\lambda<\mu}(\Gamma_\lambda, \psi_\lambda, P_\lambda)$ for limit ordinals $\mu < \nu$. Any such $(\Gamma^{c}, \psi^{c}, P^{c})$ is clearly, an H-closure of (Γ, ψ, P), which explains the superscript c. Since $(\Gamma^{c}, \psi^{c}, P^{c}) \preccurlyeq (\Gamma^*, \psi^*, P^*)$, any element of Γ^* that is definable in $\mathbf{\Gamma}^*$ over Γ must lie in Γ^{c}. So let $\gamma^{c} \in \Gamma^{c} \setminus \Gamma$; to show that then γ^{c} is not definable in $\mathbf{\Gamma}^*$ over Γ it suffices by Theorem 3.3 that γ^{c} realizes in $\mathbf{\Gamma}^*$ the same quantifier-free type over Γ as some $\gamma \in \Gamma^{c}$ with $\gamma \neq \gamma^{c}$. Take λ with $\lambda < \lambda + 1 < \nu$ such that $\gamma^{c} \in \Gamma_{\lambda+1} \setminus \Gamma_\lambda$. Then

$$\gamma^{c} = \gamma_\lambda + d\alpha_\lambda \qquad (\gamma_\lambda \in \Gamma_\lambda,\ d \in \boldsymbol{k}^{\times}).$$

Take any $\alpha \neq \alpha_\lambda$ in $\Gamma_{\lambda+1}^{>}$ such that $[\alpha]_{\boldsymbol{k}} = [\alpha_\lambda]_{\boldsymbol{k}}$. Then $\gamma^{c} \neq \gamma := \gamma_\lambda + d\alpha$. Lemma 3.1 gives an automorphism σ of $(\Gamma_{\lambda+1}, \psi_{\lambda+1}, P_{\lambda+1})$ over Γ_λ with $\sigma(\alpha) = \alpha_\lambda$, so $\sigma(\gamma^{c}) = \gamma$. Thus, γ^{c} and γ realize in $\mathbf{\Gamma}^*$ the same quantifier-free type over Γ. $\square$

A closure property of closed *H*-couples

We show here how [1, Properties A and B] and its variant [2, § 9.9] follow from our QE.

Let (Γ, ψ) be an H-couple over $\boldsymbol{k}$. We extend $\psi\colon \Gamma^{\neq} \to \Gamma$ to a function $\psi\colon \Gamma_\infty \to \Gamma_\infty$ by $\psi(0) = \psi(\infty) := \infty$. For $\alpha_1, \ldots, \alpha_n \in \Gamma$, $n \geqslant 1$, we define $\psi_{\alpha_1,\ldots,\alpha_n}\colon \Gamma_\infty \to \Gamma_\infty$ by

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recursion on n:

$$\psi_{\alpha_1}(\gamma) \ := \ \psi(\gamma-\alpha_1), \quad \psi_{\alpha_1,\dots,\alpha_n}(\gamma) \ := \ \psi\big(\psi_{\alpha_1,\dots,\alpha_{n-1}}(\gamma)-\alpha_n\big) \text{ for } n\geqslant 2.$$

Let D be a subset of an ordered abelian group Δ. Call D *bounded* if $D\subseteq[p,\,q]$ for some $p\leqslant q$ in Δ, and otherwise, call D *unbounded*. (These notions and the next one are with respect to the ambient Δ.) A **(convex) component of** D is by definition a non-empty convex subset S of Δ such that $S\subseteq D$ and S is maximal with these properties. The components of D partition the set D: for $d\in D$ the unique component of D containing d is

$$\big\{\gamma\in D^{\leqslant d}:\,[\gamma,d]\subseteq D\big\}\cup\big\{\gamma\in D^{\geqslant d}:\,[d,\gamma]\subseteq D\big\}.$$

Let $n\geqslant 1$, and let α be a sequence $\alpha_1,\,\dots,\,\alpha_n$ from Γ. We set

$$D_\alpha \ := \ \big\{\gamma\in\Gamma:\,\psi_\alpha(\gamma)\neq\infty\big\}.$$

Thus,

$$D_\alpha=\Gamma\setminus\{\alpha_1\} \text{ for } n=1, \text{ and}$$
$$D_\alpha=\big\{\gamma\in D_{\alpha'}:\,\psi_{\alpha'}(\gamma)\neq\alpha_n\big\} \text{ for } n>1 \text{ and } \alpha'=\alpha_1,\dots,\alpha_{n-1}.$$

One checks easily by induction on n that for distinct $\gamma,\,\gamma'\in D_\alpha$,

$$\psi_\alpha(\gamma)-\psi_\alpha(\gamma')=o(\gamma-\gamma').$$

Let $n\geqslant 1$, let $\alpha_1,\,\dots,\,\alpha_n\in\Gamma$, set $\alpha:=(\alpha_1,\,\dots,\,\alpha_n)$, and let $c_1,\,\dots,\,c_n\in\boldsymbol{k}$.

The next lemma is [2, Lemma 9.9.3], generalized from $\boldsymbol{k}=\mathbb{Q}$ to arbitrary $\boldsymbol{k}$, with the same (easy) proof.

Lemma 6.2. *The function*

$$\gamma\mapsto\gamma+c_1\psi_{\alpha_1}(\gamma)+\cdots+c_n\psi_{\alpha_1,\dots,\alpha_n}(\gamma)\ :\ D_\alpha\to\Gamma$$

is strictly increasing. Moreover, this function has the intermediate value property on every component of D_α.

Proposition 6.3. *Suppose $(\Gamma,\,\psi)$ is closed, $(\Gamma^*,\,\psi^*)$ is an H-couple over $\boldsymbol{k}$ extending $(\Gamma,\,\psi)$, and $\gamma\in\Gamma^*$ is such that*

$$\psi^*_{\alpha_1,\dots,\alpha_n}(\gamma)\neq\infty \quad (\textit{so } \psi^*_{\alpha_1,\dots,\alpha_i}(\gamma)\neq\infty \textit{ for } i=1,\dots,n), \quad \textit{and}$$
$$\gamma+c_1\psi^*_{\alpha_1}(\gamma)+\cdots+c_n\psi^*_{\alpha_1,\dots,\alpha_n}(\gamma)\in\Gamma.$$

Then $\gamma\in\Gamma$.

Proof. By extending $(\Gamma^*,\,\psi^*)$ we arrange it to be closed. Then by Theorem 3.3, $(\Gamma,\,\psi,\,\Psi)\preccurlyeq(\Gamma^*,\,\psi^*,\,\Psi^*)$, and so we have $\beta\in\Gamma$ such that $\psi_{\alpha_1,\dots,\alpha_n}(\beta)\neq\infty$ and

$$\beta+c_1\psi_{\alpha_1}(\beta)+\cdots+c_n\psi_{\alpha_1,\dots,\alpha_n}(\beta)=\gamma+c_1\psi^*_{\alpha_1}(\gamma)+\cdots+c_n\psi^*_{\alpha_1,\dots,\alpha_n}(\gamma).$$

It remains to note that then $\beta=\gamma$ by Lemma 6.2. □

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7. Final remarks

In [1], we adopted the 2-sorted setting and "Hahn type" at the outset and only observed in its last section that much went through in a one-sorted setting without Hahn type assumption and just rational scalars. Here we have reversed this order, since our proof of Theorem 0.1 required various facts, such as Lemmas 2.7 and 4.5, about "one-sorted" closed H-couples over an arbitrary ordered scalar field that are not readily available in [1].

There remain several parts in [1] that we have not tried to cover or extend here. These concern the definable closure of an H-couple in an ambient closed H-couple, the uniqueness of H-closures, the well-orderedness of Ψ for finitely generated H-couples, the weak o-minimality of closed H-couples, and the local o-minimality and o-minimality at infinity of models of T_H. We alert the reader that our terminology (and notation) concerning asymptotic couples have evolved since [1], and are now in line with [2], and so comparisons with the material here and in [1] require careful attention to the exact meaning of words.

We do intend to treat some of these topics in a follow-up, since our revisit also uncovered errors in the alleged proofs of weak o-minimality and local o-minimality in [1]. These can be corrected using the present paper, but this is not entirely a routine matter.

Competing interests. The authors declare none.

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