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INTERNAL q-HOMOLOGY OF CROSSED MODULES

MANUEL LADRA[1] AND ANA M. VIEITES[2]

[1]*Departamento de Álgebra, Universidad de Santiago de Compostela, E-15782, Spain* (ladra@usc.es)

[2]*Departamento de Matemática Aplicada I, Universidad de Vigo, E-36280, Spain* (avieites@dma.uvigo.es)

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Dedicated to the memory of Professor Alfredo R. Grandjeán

Abstract For q a non-negative integer, we introduce the internal q-homology of crossed modules and we obtain in the case $q = 0$ the homology of crossed modules. In the particular case of considering a group as a crossed module we obtain that its internal q-homology is the homology of the group with coefficients in the ring of the integers modulo q.

The second internal q-homology of crossed modules coincides with the invariant introduced by Grandjeán and López, that is, the kernel of the universal q-central extension. Finally, we relate the internal q-homology of a crossed module to the homology of its classifying space with coefficients in the ring of the integers modulo q.

Keywords: crossed module; classifying space; q-homology; q-commutator; q-centre

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1. Introduction

Crossed modules, introduced in [**18**], provide a simultaneous generalization of the concepts of normal subgroups and modules over a group. Norrie [**15**] has shown that the category of crossed modules has many formal properties analogous to those of groups and there she introduced concepts such as commutator and centre. A first approach to the homology of crossed modules in low dimensions was done in [**7**, **12**]. In [**3**], Carrasco, Cegarra and Grandjeán introduced an 'internal' homology theory of crossed modules in the spirit of [**1**] and Grandjeán and López [**9**] introduce, for q a non-negative integer, an invariant H_2^q for a crossed module that generalizes the second homology group with coefficients in $\mathbb{Z}/q\mathbb{Z}$ and that coincides with the kernel of the universal q-central extension. Baues [**2**] and Ellis [**6**] introduced the homology of crossed modules via classifying space with coefficients in a π_1-module, where π_1 is the first homotopy group of the crossed module.

In this paper, we introduce an internal q-homology theory for a crossed module, H_n^q, obtaining on the one hand, for $q = 0$, the homology theory given in [**3**], and on the other, for $n = 2$, the invariant H_2^q introduced in [**9**], and we study its relationship with the homology of groups with coefficients in $\mathbb{Z}/q\mathbb{Z}$ and the homology of the classifying space of a crossed module with coefficients in $\mathbb{Z}/q\mathbb{Z}$.

The article is divided into five sections. In § 2, we give some concepts in the theory of crossed modules, with special mention to q-commutator and q-centre. In § 3, we introduce the internal q-homology of a crossed module and its relationship with the homology of groups with coefficients in $\mathbb{Z}/q\mathbb{Z}$. In § 4, we give an exact sequence of five terms associated with an extension of crossed modules from which we deduce an expression for the second internal q-homology that extends Hopf's formula in group homology, and that allows us to identify such second internal q-homology with the invariant introduced in [**9**], which coincides with the kernel of the universal q-central extension of a q-perfect crossed module. We also obtain a version in crossed modules of the 'basic theorem' of Stallings [**16**] relating nilpotent groups to their homology. Finally, in § 5, we relate the internal q-homology of a crossed module with the homology of its classifying space introduced in [**2**, **6**] via a natural long exact sequence. In the case $q = 0$, we obtain the result established in [**10**].

2. Preliminaries

Let us remember [**18**] that a *crossed module* $\boldsymbol{T} = (T, G, \partial)$ consists of a group homomorphism $\partial : T \to G$, the boundary map, together with a group action $g \to {}^g t$ of G on T satisfying $\partial({}^g t) = g\partial(t)g^{-1}$ and ${}^{\partial(t)}t' = tt't^{-1}$, for all $t, t' \in T$ and $g \in G$. If $\boldsymbol{T}' = (T', G', \partial')$ is another crossed module, a *crossed module morphism* $\boldsymbol{f} = (f_1, f_2) : \boldsymbol{T} \to \boldsymbol{T}'$ is a pair of group homomorphisms $f_1 : T \to T'$ and $f_2 : G \to G'$, such that $\partial' f_1 = f_2 \partial$ and, for all $g \in G$, $t \in T$, $f_1({}^g t) = {}^{f_2(g)} f_1(t)$. The corresponding category of crossed modules is denoted by $\mathcal{CM}$.

We can consider a group as a crossed module via the following functors

$$\mathcal{G} \xrightarrow{\quad\iota\quad} \mathcal{CM} \qquad\qquad \mathcal{G} \xrightarrow{\quad\varepsilon\quad} \mathcal{CM},$$

where $\mathcal{G}$ is the category of groups and $\iota(G) = (1, G, i)$, $\varepsilon(G) = (G, G, \mathrm{id}_G)$. If we also consider the functors

$$\mathcal{CM} \xrightarrow{\quad\kappa\quad} \mathcal{G} \qquad\qquad \mathcal{CM} \xrightarrow{\quad\zeta\quad} \mathcal{G},$$

given by $\kappa(T, G, \partial) = G$ and $\zeta(T, G, \partial) = T$, then we get the adjoint pairs $\iota \dashv \kappa$, $\kappa \dashv \varepsilon$ and $\varepsilon \dashv \zeta$ [**3**].

A *crossed submodule* $\boldsymbol{N} = (N, R, \partial)$ of a crossed module $\boldsymbol{T}$ is given by the subgroups R of G and N of T, if these inclusions define a crossed module morphism. If R is a normal subgroup of G and for all elements $g \in G$, $n \in N$, $r \in R$ and $t \in T$ we have ${}^g n \in N$ and ${}^r t t^{-1} \in N$, then $\boldsymbol{N}$ is a *normal crossed submodule of* $\boldsymbol{T}$. In this case we can consider the triple $(T/N, G/R, \bar{\partial})$, where $\bar{\partial}$ is induced by ∂, and the new action is given by ${}^{gR}(tN) = ({}^g t)N$. This is the *quotient crossed module of* $\boldsymbol{T}$ *by* $\boldsymbol{N}$.

If $\boldsymbol{S} = (S, H, \partial)$ and $\boldsymbol{N}$ are normal crossed submodules of $\boldsymbol{T}$, the *commutator crossed submodule* of $\boldsymbol{S}$ and $\boldsymbol{N}$ is $([R, S][H, N], [H, R], \partial)$, where $[R, S] = \langle \{ {}^r s s^{-1} \mid s \in S,\ r \in R \} \rangle$

is the displacement subgroup of S relative to R [**15**]. In particular, the commutator crossed submodule of $\boldsymbol{T}$ is $[\boldsymbol{T},\boldsymbol{T}] = ([G,T],[G,G],\partial)$, where $[G,G]$ is the commutator subgroup of G.

The *centre* of $\boldsymbol{T}$ is the crossed module $Z(\boldsymbol{T}) = (T^G, Z(G) \cap \mathrm{st}_G(T), \partial)$, where $T^G = \{t \in T \mid {}^g t = t \text{ for all } g \in G\}$, $Z(G)$ is the centre of G and $\mathrm{st}_G(T) = \{g \in G \mid {}^g t = t \text{ for all } t \in T\}$ [**15**]. A crossed module $\boldsymbol{T}$ is *abelian* if $\boldsymbol{T} = Z(\boldsymbol{T})$, in other words, G and T are both abelian groups and G acts trivially on T. We will denote the category of abelian crossed modules by $\mathcal{ACM}$, which is a Birkhoff variety [**14**] of $\mathcal{CM}$, that is, closed under the formation of products, submodules and quotients.

Let q be a non-negative integer.

Definition 2.1. Let $\boldsymbol{N} = (N,R,\partial)$ be a normal crossed submodule of $\boldsymbol{T}$. The *q-commutator crossed submodule* of $\boldsymbol{T}$ and $\boldsymbol{N}$ is $\boldsymbol{T}\#_q\boldsymbol{N} = ([R,T](G\#_q N), G\#_q R, \partial)$, where $G\#_q N = \langle\{{}^g n n^{-1} m^q \mid g \in G,\ n,m \in N\}\rangle$ is a subgroup of N, $[R,T]$ is the displacement subgroup of T relative to R, and $G\#_q R$ is the q-commutator subgroup of G and R [**17**].

The q-commutator crossed submodule $\boldsymbol{T}\#_q\boldsymbol{N}$ is a normal crossed submodule of $\boldsymbol{T}$ and, by normality conditions of $\boldsymbol{N}$, it is contained in $\boldsymbol{N}$ [**9**]. In the particular case $\boldsymbol{N} = \boldsymbol{T}$, we obtain the q-commutator of $\boldsymbol{T}$, $(G\#_q T, G\#_q G, \partial)$ [**5**]. In the case $q = 0$, it is the commutator crossed submodule defined by Norrie [**15**].

Example 2.2. If N is a normal subgroup of a group G, the q-commutator crossed submodule of (N,G,i) is $(G\#_q N, G\#_q G, i)$. So, if G is any group, the q-commutator of $\varepsilon(G)$ and $\iota(G)$ are $(G\#_q G, G\#_q G, \mathrm{id})$ and $(1, G\#_q G, i)$, respectively.

The quotient of a crossed module $\boldsymbol{T}$ by its q-commutator, $\boldsymbol{T}_q = \boldsymbol{T}/(\boldsymbol{T}\#_q\boldsymbol{T})$, is an abelian crossed module; in fact, it is even more, it is an abelian crossed module of exponent q, that is it is an abelian crossed module and $T/G\#_q T$ and $G/G\#_q G$ are both of exponent q and $G/G\#_q G$ acts trivially on $T/G\#_q T$. We can construct a functor from the category of crossed modules to that of abelian crossed modules, called *abelianization modulo q*, $(-)_q : \mathcal{CM} \to \mathcal{ACM}$, in the following way: if $\boldsymbol{T}$ is a crossed module, $(\boldsymbol{T})_q = \boldsymbol{T}_q$, and if $\boldsymbol{f} : \boldsymbol{T} \to \boldsymbol{T}'$ is a morphism of crossed modules, $(\boldsymbol{f})_q : \boldsymbol{T}_q \to \boldsymbol{T}'_q$ is the induced map.

Definition 2.3. Let $\boldsymbol{T}$ be a crossed module. The *q-centre* of $\boldsymbol{T}$, $Z^q(\boldsymbol{T})$, is the crossed module $((T^G)^q, Z^q(G) \cap \mathrm{st}_G(T), \partial)$, where

$$\begin{aligned}(T^G)^q &= \{t \in T \mid t^q = 1 \text{ and } {}^g t = t, \text{ for all } g \in G\},\\ Z^q(G) &= \{g \in Z(G) \mid g^q = 1\},\\ \mathrm{st}_G(T) &= \{g \in G \mid {}^g t = t, \text{ for all } t \in T\}.\end{aligned}$$

The q-centre of $\boldsymbol{T}$ is a normal crossed submodule of $\boldsymbol{T}$ and $\boldsymbol{T}\#_q\boldsymbol{T}$ is the smallest normal crossed submodule $\boldsymbol{N}$ of $\boldsymbol{T}$ such that $\boldsymbol{T}/\boldsymbol{N}$ coincides with its q-centre [**5**]. In the case $q = 0$, it is the centre defined by Norrie [**15**].

Example 2.4. If N is a normal subgroup of a group G, the q-centre of (N,G,i) is $(N \cap Z^q(G), Z^q(G), i)$. So, if G is any group, the q-centre of $\varepsilon(G)$ and $\iota(G)$ are $(Z^q(G), Z^q(G), \mathrm{id})$ and $(1, Z^q(G), i)$, respectively.

Example 2.5. If A is a G-module, the q-centre of $(A, G, 0)$ is $(\mathrm{H}^0(G, A; \mathbb{Z}/q\mathbb{Z}), Z^q(G) \cap \mathrm{st}_G(A), 0)$, where $\mathrm{H}^0(G, A; \mathbb{Z}/q\mathbb{Z})$ is the zeroth mod q cohomology group introduced in [**4**].

3. Internal q-homology and connection with the homology of groups

In [**3**], a left adjoint to the faithful functor to groups, $\mathcal{V} : \mathcal{CM} \to \mathcal{G}$, $(T, G, \partial) \to T \times G$, is given as follows: if H is any group, get the free product group $H * H$ with the injections $u_i : H \to H * H$, $i = 1, 2$, and let $\bar{H} = \mathrm{Ker}(p_2 : H * H \to H)$ be the kernel of the retraction p_2, determined by the conditions $p_2 u_1 = 0$ and $p_2 u_2 = \mathrm{id}_H$. The triple $(\bar{H}, H * H, in)$ is a crossed module with the inclusion as boundary map. The functor $H \to (\bar{H}, H * H, in)$ is left adjoint to $\mathcal{V}$.

By composition with the usual forgetful functor $\mathcal{G} \to \mathcal{S}et$ we get the underlying set functor $\mathcal{U} : \mathcal{CM} \to \mathcal{S}et$, $(T, G, \partial) \to T \times G$. Since the forgetful functor $\mathcal{G} \to \mathcal{S}et$ has the free group functor, $X \to F(X)$, as a left adjoint, the functor $\mathcal{U} : \mathcal{CM} \to \mathcal{S}et$ has a left adjoint $\mathcal{F} : \mathcal{S}et \to \mathcal{CM}$ given by $X \to \mathcal{F}(X) = (\overline{F(X)}, F(X) * F(X), in)$. In [**3**] it is also shown that $\mathcal{U}$ is tripleable, hence, for any set X, the free crossed module on X, $\mathcal{F}(X)$, is projective with respect to regular epimorphisms (f_1 and f_2 are surjective morphisms), and every crossed module $\boldsymbol{T}$ admits a projective presentation by means of the free crossed module on its underlying set and the co-unit of the adjunction.

Let $\mathcal{U}$, $\mathcal{F} : \mathcal{CM} \to \mathcal{S}et$ be the tripleable pair of adjoint functors defined before and let $u : \mathrm{id}_{\mathcal{S}et} \to \mathcal{UF}$ and $v : \mathcal{FU} \to \mathrm{id}_{\mathcal{CM}}$ denote the adjunction transformations. We will denote by $(\mathbb{G}, w, v)$ the induced co-triple on the category of crossed modules; thus, $\mathbb{G} = \mathcal{FU} : \mathcal{CM} \to \mathcal{CM}$, the co-multiplication is $w = \mathcal{F}u\mathcal{U} : \mathbb{G} \to \mathbb{G}^2$, and the co-unit is $v : \mathbb{G} \to \mathrm{id}_{\mathcal{CM}}$. Then, any crossed module $\boldsymbol{T}$ has a standard free simplicial resolution $\mathbb{G}_\bullet \boldsymbol{T} \twoheadrightarrow \boldsymbol{T}$. Namely, $\mathbb{G}_\bullet \boldsymbol{T}$ is the simplicial crossed module defined by $\mathbb{G}_n \boldsymbol{T} = \mathbb{G}^{n+1}(\boldsymbol{T})$, $n \geqslant 0$, with face and degeneracy operators

$$d_i = \mathbb{G}^{n-i} v \mathbb{G}^i(\boldsymbol{T}) : \mathbb{G}_n \boldsymbol{T} \to \mathbb{G}_{n-1} \boldsymbol{T}, \qquad 0 \leqslant i \leqslant n,$$
$$s_i = \mathbb{G}^{n-i-1} w \mathbb{G}^i(\boldsymbol{T}) : \mathbb{G}_{n-1} \boldsymbol{T} \to \mathbb{G}_n \boldsymbol{T}, \quad 0 \leqslant i \leqslant n-1.$$

Applying the functor, $(-)_q : \mathcal{CM} \to \mathcal{ACM}$, we obtain an augmented simplicial complex of abelian crossed modules:

$$(\mathbb{G}_\bullet \boldsymbol{T})_q \twoheadrightarrow \boldsymbol{T}_q\ .$$

If we take the alternating sum of the abelianized modulo q face operators, we get the chain complex of abelian crossed modules

$$((\mathbb{G}_\bullet \boldsymbol{T})_q, \delta) = \ \cdots\ (\mathbb{G}_n \boldsymbol{T})_q \xrightarrow{\delta_n} (\mathbb{G}_{n-1} \boldsymbol{T})_q \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_1} (\mathbb{G}_0 \boldsymbol{T})_q \longrightarrow 0.$$

We define the *nth internal q-homology crossed module* by

$$\mathrm{H}_n^q(\boldsymbol{T}) = \mathrm{H}_{n-1}((\mathbb{G}_\bullet \boldsymbol{T})_q, \delta), \quad n \geqslant 1.$$

The following properties of the internal q-homology are a special case of well-known basic results of co-triple cohomology [**1**] in a general setting.

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Proposition 3.1.

(i) $\mathrm{H}_n^q(-) : \mathcal{CM} \to \mathcal{ACM}$ *is a functor for all* $n \geqslant 1$.

(ii) $\mathrm{H}_1^q(\boldsymbol{T}) = \boldsymbol{T}_q$ *for any crossed module* $\boldsymbol{T}$.

(iii) *If* $\boldsymbol{T}$ *is a projective crossed module, then* $\mathrm{H}_n^q(\boldsymbol{T}) = 0$ *for all* $n \geqslant 2$.

Example 3.2. $\mathrm{H}_1^q(N, G, i) = (N/G\#_q N, \mathrm{H}_1^q(G), i)$, where N is a normal subgroup of G.

Example 3.3. If A is a G-module, then $(A, G, 0)$ is a crossed module with $\mathrm{H}_1^q(A, G, 0) = (q - \mathrm{H}_0(G, A), \mathrm{H}_1^q(G), 0)$, because $\mathbb{Z}/q\mathbb{Z} \otimes_G A \cong A/(G\#_q A)$, and where $q - \mathrm{H}_n(G, A) = \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}/q\mathbb{Z}, A)$ are the q-homology groups introduced in [**11**].

In the following theorem we show the connection between our internal q-homology of crossed modules and the homology of groups with coefficients in $\mathbb{Z}/q\mathbb{Z}$, $\mathrm{H}_n^q(G) = \mathrm{H}_n(G, \mathbb{Z}/q\mathbb{Z})$.

Theorem 3.4. *Let* $\boldsymbol{T} = (T, G, \partial)$ *be a crossed module. Then*

(i) $\mathrm{H}_n^q(\iota(G)) \cong \iota\,\mathrm{H}_n^q(G)$ *and* $\mathrm{H}_n^q(\varepsilon(G)) \cong \varepsilon\,\mathrm{H}_n^q(G)$;

(ii) $\kappa\,\mathrm{H}_n^q(\boldsymbol{T}) \cong \mathrm{H}_n^q(G)$;

(iii) *if* ∂ *is injective, there exists a natural long exact sequence of abelian groups*

$$\cdots\ \mathrm{H}_{n+1}^q(G) \xrightarrow{\bar{p}_r} \mathrm{H}_{n+1}^q(G/\partial T) \longrightarrow \zeta\,\mathrm{H}_n^q(\boldsymbol{T}) \xrightarrow{\bar{\partial}} \mathrm{H}_n^q(G)\ \cdots$$

$$\mathrm{H}_3^q(G/\partial T) \longrightarrow \zeta\,\mathrm{H}_2^q(\boldsymbol{T}) \longrightarrow \mathrm{H}_2^q(G) \longrightarrow \mathrm{H}_2^q(G/\partial T)$$

$$\longrightarrow \frac{T}{G\#_q T} \longrightarrow \mathrm{H}_1^q(G) \longrightarrow \mathrm{H}_1^q(G/\partial T) \longrightarrow 0\,.$$

Proof. (i) As was shown in [**1**], the Eilenberg–MacLane homology groups can be computed through free simplicial resolutions. Hence, if G is a group and $F_\bullet \twoheadrightarrow G$ is a free simplicial resolution of it, and since $\mathbb{Z}/q\mathbb{Z} \otimes_G IG \cong G/(G\#_q G)$, there is a natural isomorphism $\mathrm{H}_{n+1}(G, \mathbb{Z}/q\mathbb{Z}) \cong \mathrm{H}_{n+1}((F_\bullet)_q, \delta)$, $n \geqslant 0$, where $((F_\bullet)_q, \delta)$ is the chain complex obtained by applying the functor $\mathcal{G} \to \mathcal{A}b$, $G \rightsquigarrow G/G\#_q G$ and taking alternating sums of the induced face operators. By [**3**, Proposition 5] the augmented simplicial crossed modules $\iota F_\bullet \twoheadrightarrow \iota G$ and $\varepsilon F_\bullet \twoheadrightarrow \varepsilon G$ are projective simplicial resolutions. Since ι and ε preserve kernels and co-kernels and commute with the functor $(-)_q$, we have

$$\mathrm{H}_n^q(\iota(G)) = \mathrm{H}_{n-1}((\iota F_\bullet)_q) \cong \mathrm{H}_{n-1}(\iota(F_\bullet)_q) \cong \iota\,\mathrm{H}_{n-1}((F_\bullet)_q) \cong \iota\,\mathrm{H}_n^q(G)$$

and

$$\mathrm{H}_n^q(\varepsilon(G)) = \mathrm{H}_{n-1}((\varepsilon F_\bullet)_q) \cong \mathrm{H}_{n-1}(\varepsilon(F_\bullet)_q) \cong \varepsilon\,\mathrm{H}_{n-1}((F_\bullet)_q) \cong \varepsilon\,\mathrm{H}_n^q(G)$$

for all $n \geqslant 1$.

(ii) Since $\kappa \circ (-)_q = (-)_q \circ \kappa$ and for every crossed module $\boldsymbol{T} = (T, G, \partial)$, the simplicial group $\kappa \mathbb{G}_\bullet \boldsymbol{T}$ is a free simplicial resolution of the group G [**3**, Remark 9], we have

$$\kappa \operatorname{H}_n^q(\boldsymbol{T}) = \kappa \operatorname{H}_{n-1}((\mathbb{G}_\bullet \boldsymbol{T})_q) \cong \operatorname{H}_{n-1}((\kappa \mathbb{G}_\bullet \boldsymbol{T})_q) = \operatorname{H}_n^q(G).$$

(iii) We have a short exact sequence of free groups

$$\zeta \mathcal{F}(X) \rightarrowtail \kappa \mathcal{F}(X) \twoheadrightarrow \kappa \mathcal{F}(X)/\zeta \mathcal{F}(X) ,$$

where $\mathcal{F}(X) = (\overline{F(X)}, F(X) * F(X), in)$ is the free crossed module over the set X. Therefore, from the standard simplicial resolution of $\boldsymbol{T} = (T, G, \partial)$ we have a short exact sequence of free simplicial groups:

$$\zeta \mathbb{G}_\bullet \boldsymbol{T} \rightarrowtail \kappa \mathbb{G}_\bullet \boldsymbol{T} \twoheadrightarrow \kappa \mathbb{G}_\bullet \boldsymbol{T}/\zeta \mathbb{G}_\bullet \boldsymbol{T} .$$

By [**3**, Remark 9], $\zeta \mathbb{G}_\bullet \boldsymbol{T}$, $\kappa \mathbb{G}_\bullet \boldsymbol{T}$ are free simplicial resolutions of T and G, respectively. So,

$$\begin{gathered} \pi_n(\zeta \mathbb{G}_\bullet \boldsymbol{T}) = \pi_n(\kappa \mathbb{G}_\bullet \boldsymbol{T}) = 0, \quad n > 0, \\ \pi_0(\zeta \mathbb{G}_\bullet \boldsymbol{T}) = T, \qquad \pi_0(\kappa \mathbb{G}_\bullet \boldsymbol{T}) = G \end{gathered}$$

and so the homotopy exact sequence associated with the above extension of simplicial groups reduces to

$$0 \to \pi_0(\zeta \mathbb{G}_\bullet \boldsymbol{T}) \to \pi_0(\kappa \mathbb{G}_\bullet \boldsymbol{T}) \to \pi_0(\kappa \mathbb{G}_\bullet \boldsymbol{T}/\zeta \mathbb{G}_\bullet \boldsymbol{T}) \to 0.$$

Therefore, the morphism $\pi_0(\zeta \mathbb{G}_\bullet \boldsymbol{T}) \to \pi_0(\kappa \mathbb{G}_\bullet \boldsymbol{T})$ coincides with ∂, because ∂ is injective, and so $\pi_0(\kappa \mathbb{G}_\bullet \boldsymbol{T}/\zeta \mathbb{G}_\bullet \boldsymbol{T}) = G/\partial(T)$. Since $\pi_n(\kappa \mathbb{G}_\bullet \boldsymbol{T}/\zeta \mathbb{G}_\bullet \boldsymbol{T}) = 0$ for $n > 0$ the simplicial group $\kappa \mathbb{G}_\bullet \boldsymbol{T}/\zeta \mathbb{G}_\bullet \boldsymbol{T}$ is a free simplicial resolution of $G/\partial(T)$.

Since $\mathcal{F}(X)$ has as abelianization modulo q,

$$\mathcal{F}(X)_q = \Bigl(\bigoplus_X \mathbb{Z}/q\mathbb{Z}, \Bigl(\bigoplus_X \mathbb{Z}/q\mathbb{Z} \Bigr) \oplus \Bigl(\bigoplus_X \mathbb{Z}/q\mathbb{Z} \Bigr), i_1 \Bigr),$$

we have a short exact sequence of abelian groups,

$$\zeta(\mathcal{F}(X)_q) \rightarrowtail \kappa(\mathcal{F}(X)_q) \twoheadrightarrow (\kappa \mathcal{F}(X)/\zeta \mathcal{F}(X))_q ,$$

and a short exact sequence of abelian groups chain complexes,

$$\zeta((\mathbb{G}_\bullet \boldsymbol{T})_q) \rightarrowtail \kappa((\mathbb{G}_\bullet \boldsymbol{T})_q) \twoheadrightarrow (\kappa \mathbb{G}_\bullet \boldsymbol{T}/\zeta \mathbb{G}_\bullet \boldsymbol{T})_q ,$$

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whose associated long exact homology sequence yields (iii), because

$$\zeta\,\mathrm{H}_n^q(\boldsymbol{T}) = \mathrm{H}_{n-1}(\zeta(\mathbb{G}_\bullet \boldsymbol{T})_q), \qquad \zeta\,\mathrm{H}_1^q(\boldsymbol{T}) = \frac{T}{G\#_q T},$$
$$\mathrm{H}_n^q(G) = \mathrm{H}_{n-1}((\kappa\mathbb{G}_\bullet \boldsymbol{T})_q), \qquad \mathrm{H}_n^q(G/\partial(T)) = \mathrm{H}_{n-1}((\kappa\mathbb{G}_\bullet \boldsymbol{T}/\zeta\mathbb{G}_\bullet \boldsymbol{T})_q).$$

□

Example 3.5. Let G be a group and let

$$R \overset{i}{\rightarrowtail} F \twoheadrightarrow G$$

be a free presentation of G. By the theorem above, we get the following exact sequence for the crossed module (R, F, i):

$$\mathrm{H}_{n+1}^q(F) \longrightarrow \mathrm{H}_{n+1}^q(G) \longrightarrow \zeta\,\mathrm{H}_n^q(R, F, i) \longrightarrow \mathrm{H}_n^q(F)\ .$$

Since F is free, $\mathrm{H}_n^q(F) = 0$, for $n \geqslant 2$. Then $\mathrm{H}_{n+1}^q(G) \cong \zeta\,\mathrm{H}_n^q(R, F, i)$, for $n \geqslant 2$.

4. Five-term exact sequence and basic theorem

Lemma 4.1. *Let*

$$e : \boldsymbol{N} \overset{\boldsymbol{i}}{\rightarrowtail} \boldsymbol{T} \overset{\boldsymbol{p}}{\twoheadrightarrow} \boldsymbol{M}$$

be an extension of crossed modules. If $\boldsymbol{p}$ *admits a section, then the sequence*

$$\boldsymbol{0} \longrightarrow \frac{\boldsymbol{N}}{\boldsymbol{T}\#_q\boldsymbol{N}} \longrightarrow \mathrm{H}_1^q(\boldsymbol{T}) \longrightarrow \mathrm{H}_1^q(\boldsymbol{M}) \longrightarrow \boldsymbol{0}$$

is a split short exact sequence of abelian crossed modules.

Proof. If

$$e : 1 \longrightarrow \boldsymbol{N} \overset{\boldsymbol{i}}{\longrightarrow} \boldsymbol{T} \overset{\boldsymbol{p}}{\longrightarrow} \boldsymbol{M} \longrightarrow 1$$

is an extension of $\boldsymbol{N}$ by $\boldsymbol{M}$, since $(\boldsymbol{N}\#_q\boldsymbol{N}) \subseteq (\boldsymbol{T}\#_q\boldsymbol{N}) \subseteq (\boldsymbol{T}\#_q\boldsymbol{T})$ and $\boldsymbol{p}$ clearly restricts to a regular epimorphism $\boldsymbol{p} : \boldsymbol{T}\#_q\boldsymbol{T} \to \boldsymbol{M}\#_q\boldsymbol{M}$, we get the following exact sequence of abelian crossed modules:

$$\frac{\boldsymbol{N}}{\boldsymbol{T}\#_q\boldsymbol{N}} \overset{\bar{\boldsymbol{i}}}{\longrightarrow} \boldsymbol{T}_q \overset{\bar{\boldsymbol{p}}}{\longrightarrow} \boldsymbol{M}_q \longrightarrow 0\ .$$

If $\boldsymbol{p}$ admits a section, that is, if there exists a crossed module morphism $\boldsymbol{s} : \boldsymbol{M} \to \boldsymbol{T}$ such that $\boldsymbol{ps} = \mathrm{id}_{\boldsymbol{M}}$, then

$$0 \longrightarrow \frac{\boldsymbol{N}}{\boldsymbol{T}\#_q\boldsymbol{N}} \overset{\bar{\boldsymbol{i}}}{\longrightarrow} \boldsymbol{T}_q \overset{\bar{\boldsymbol{p}}}{\longrightarrow} \boldsymbol{M}_q \longrightarrow 0$$

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is a split short exact sequence of abelian crossed modules since we can prove that there exists a crossed module morphism

$$\bar{\boldsymbol{r}} = (\bar{r}_1, \bar{r}_2) : \boldsymbol{T}_q \to \frac{\boldsymbol{N}}{\boldsymbol{T}\#_q\boldsymbol{N}}$$

that verifies $\bar{\boldsymbol{r}}\bar{\boldsymbol{i}} = \mathrm{id}$ and it is given by

$$\bar{r}_1(t(G\#_qT)) = \overline{ts_1p_1(t)^{-1}} \in \frac{N}{[R,T](G\#_qN)}, \quad t \in T,$$
$$\bar{r}_2(g(G\#_qG)) = \overline{gs_2p_2(g)^{-1}} \in \frac{R}{G\#_qR}, \qquad g \in G,$$

where the overbar denotes the equivalence class in each case. In fact, $\bar{\boldsymbol{r}}$ is a morphism of crossed modules.

(i) $\bar{r}_1$ is a group morphism:

$$\begin{aligned}\bar{r}_1(\bar{t})\bar{r}_1(\bar{t}_1) &= \overline{ts_1p_1(t)^{-1}t_1s_1p_1(t_1)^{-1}} \\ &= \overline{tt_1(s_1p_1(t)s_1p_1(t_1))^{-1}\,{}^{\partial s_1p_1(t)}(s_1p_1(t_1)t_1^{-1})(s_1p_1(t_1)t_1^{-1})^{-1}} \\ &= \overline{tt_1s_1p_1(tt_1)^{-1}} = \bar{r}_1(\overline{tt_1}).\end{aligned}$$

(ii) $\bar{r}_2$ is a group morphism:

$$\begin{aligned}\bar{r}_2(\bar{g})\bar{r}_2(\bar{g}_1) &= \overline{gs_2p_2(g)^{-1}g_1s_2p_2(g_1)^{-1}} \\ &= \overline{gg_1s_2p_2(g_1)^{-1}s_2p_2(g)^{-1}[s_2p_2(g), s_2p_2(g_1)g_1^{-1}]} \\ &= \overline{gg_1s_2p_2(gg_1)^{-1}} = \bar{r}_2(\bar{g}\bar{g}_1).\end{aligned}$$

(iii) It is enough to prove that $\bar{r}_1(\overline{{}^gt}) = \bar{r}_1(\bar{t})$, since $\boldsymbol{N}/\boldsymbol{T}\#_q\boldsymbol{N}$ is abelian. It is easy to see that

$$x = {}^gts_1p_1({}^gt)^{-1}(ts_1p_1(t)^{-1})^{-1} \in [R,T](G\#_qN)$$

because

$$x = {}^{gs_2p_2(g)^{-1}}({}^{s_2p_2(g)}t)({}^{s_2p_2(g)}t)^{-1\,s_2p_2(g)}(ts_1p_1(t)^{-1})(ts_1p_1(t)^{-1})^{-1}$$

with

$${}^{gs_2p_2(g)^{-1}}({}^{s_2p_2(g)}t)({}^{s_2p_2(g)}t)^{-1} \in [R,T]$$

and

$${}^{s_2p_2(g)}(ts_1p_1(t)^{-1})(ts_1p_1(t)^{-1})^{-1} \in G\#_qN.$$

$\square$

Theorem 4.2. *Let*

$$e : \boldsymbol{N} \rightarrowtail^{\boldsymbol{i}} \boldsymbol{T} \overset{\boldsymbol{p}}{\twoheadrightarrow} \boldsymbol{M}$$

be an extension of crossed modules. There exists a natural exact sequence of abelian crossed modules:

$$\mathrm{H}_2^q(\boldsymbol{T}) \longrightarrow \mathrm{H}_2^q(\boldsymbol{M}) \xrightarrow{\theta_\star(e)} \frac{\boldsymbol{N}}{\boldsymbol{T}\#_q\boldsymbol{N}} \longrightarrow \mathrm{H}_1^q(\boldsymbol{T}) \xrightarrow{\boldsymbol{p}_\star} \mathrm{H}_1^q(\boldsymbol{M}) \longrightarrow 0\,.$$

The proof of this theorem is analogous to the one in [**3**, Theorem 12] using the previous lemma.

Example 4.3. When $q = 0$, we get the five-term exact sequences which appear in the papers [**3**, **8**].

Example 4.4. If we consider crossed modules of the form either $\iota(G) = (1, G, i)$ or $\varepsilon(G) = (G, G, \mathrm{id}_G)$, we get the generalization of the five-term exact sequence

$$\mathrm{H}_2^q(G) \longrightarrow \mathrm{H}_2^q(M) \longrightarrow \frac{N}{G\#_q N} \longrightarrow \mathrm{H}_1^q(G) \longrightarrow \mathrm{H}_1^q(M) \longrightarrow 0\,,$$

which appears in [**17**].

Theorem 4.2 will help us to give a Hopf-type formula for the second internal q-homology crossed module, $\mathrm{H}_2^q(\boldsymbol{M})$, in terms of an arbitrary projective presentation of the crossed module $\boldsymbol{M}$.

Corollary 4.5. *If*

$$\boldsymbol{N} \rightarrowtail \boldsymbol{T} \overset{\pi}{\twoheadrightarrow} \boldsymbol{M}$$

is a projective presentation of $\boldsymbol{M}$*, then there exists a natural isomorphism of abelian crossed modules:*

$$\begin{aligned}\mathrm{H}_2^q(\boldsymbol{M}) &\cong \frac{\boldsymbol{N} \cap (\boldsymbol{T}\#_q\boldsymbol{T})}{\boldsymbol{T}\#_q\boldsymbol{N}} \\ &= \left(\frac{N \cap (G\#_q T)}{[R,T](G\#_q N)}, \frac{R \cap (G\#_q G)}{G\#_q R}, \bar{\partial}\right).\end{aligned}$$

Proof. It follows from the five-term exact sequence applied to the projective presentation of $\boldsymbol{M}$, and since $\mathrm{H}_2^q(\boldsymbol{T}) = \boldsymbol{0}$, that

$$\mathrm{Ker}\left(\frac{\boldsymbol{N}}{(\boldsymbol{T}\#_q\boldsymbol{N})} \to \boldsymbol{T}_q\right) = \frac{\boldsymbol{N} \cap (\boldsymbol{T}\#_q\boldsymbol{T})}{\boldsymbol{T}\#_q\boldsymbol{N}} \cong \mathrm{H}_2^q(\boldsymbol{M}).$$

□

Example 4.6. If $q = 0$, then we get the Hopf formula for crossed modules, which appears in [**3**, **8**].

Example 4.7. When the crossed modules are either $\iota(G) = (1, G, i)$ or $\varepsilon(G) = (G, G, \mathrm{id}_G)$, we get the Hopf formula generalization [**17**].

By Corollary 4.5 the formula for $\mathrm{H}_2^q(\boldsymbol{T})$ coincides with the invariant introduced in [**9**], which is identified with the kernel of the universal q-central extension of a q-perfect crossed module.

Example 4.8. If A is a G-module, then $\mathrm{H}_2^q(A, G, 0) = (q - \mathrm{H}_1(G, A), \mathrm{H}_2^q(G), 0)$ [**9**].

Definition 4.9. Let $\boldsymbol{T}$ be a crossed module. The *lower* (q) *central series* $\{\boldsymbol{T}_n^q\}$ of $\boldsymbol{T}$ is

$$\begin{aligned} \boldsymbol{T}_1^q &= \boldsymbol{T}, \\ \boldsymbol{T}_{i+1}^q &= \boldsymbol{T} \#_q \boldsymbol{T}_i^q, \quad i = 1, 2, \dots . \end{aligned}$$

The crossed module $\boldsymbol{T}$ is called (q) *nilpotent* if there exists $n \geqslant 1$ such that $\boldsymbol{T}_n^q = \mathbf{1}$. We define $\boldsymbol{T}_\omega^q = \bigcap_{i=1}^\infty \boldsymbol{T}_i^q$.

The following theorem is a generalization of the basic theorem in [**16, 17**] and is given in [**8**] with restricted hypothesis.

Theorem 4.10. *Let* $\boldsymbol{f} : \boldsymbol{T} \to \boldsymbol{X}$ *be a morphism of crossed modules. If* $\boldsymbol{f}$ *induces an isomorphism* $\boldsymbol{f}_q : \boldsymbol{T}_q \to \boldsymbol{X}_q$ *and an epimorphism* $\mathrm{H}_2^q(\boldsymbol{T}) \twoheadrightarrow \mathrm{H}_2^q(\boldsymbol{X})$, *then* $\boldsymbol{f}$ *induces isomorphisms*

$$\boldsymbol{f}_i^q : \frac{\boldsymbol{T}}{\boldsymbol{T}_i^q} \cong \frac{\boldsymbol{X}}{\boldsymbol{X}_i^q}$$

for any $i \geqslant 0$, *and a monomorphism*

$$\boldsymbol{f}_\omega^q : \frac{\boldsymbol{T}}{\boldsymbol{T}_\omega^q} \to \frac{\boldsymbol{X}}{\boldsymbol{X}_\omega^q}.$$

If $\boldsymbol{T}$ *and* $\boldsymbol{X}$ *are both* (q) *nilpotent, then* $\boldsymbol{f}$ *is an isomorphism.*

The proof is parallel to the one in [**8**, Basic Theorem, p. 422].

Example 4.11. If we apply this theorem either to the crossed module $\iota(G) = (1, G, i)$ or to $\varepsilon(G) = (G, G, \mathrm{id}_G)$, we get the corresponding version q of the 'basic' theorem for groups in [**17**].

Example 4.12. When $q = 0$, we get the crossed modules basic theorem which appears in [**3, 8**].

5. Relation between the internal q-homology and the homology of the classifying space

Another description of the category of crossed modules is given by its equivalence with the category of simplicial groups whose Moore complex has length 1 [**13**]. To any given

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crossed module $\boldsymbol{T} = (T, G, \partial)$ we can associate the simplicial group $N_*^{-1}(T, G, \partial)$ as follows: $N_n^{-1}(T, G, \partial) = T^n \rtimes G$, $n \geqslant 0$, where G operates component-wise on T^n,

$$\partial_i(t_1, \dots, t_n, g) = (t_1, \dots, \hat{t}_i, \dots, t_n, g), \quad 1 \leqslant i \leqslant n,$$
$$\partial_0(t_1, \dots, t_n, g) = (t_2 t_1^{-1}, \dots, t_n t_1^{-1}, \partial t_1 g), \qquad s_i(t_1, \dots, t_n, g) = (t_1, \dots, t_i, 1, \dots, t_n, g).$$

Given any crossed module $\boldsymbol{T} = (T, G, \partial)$ its *classifying space* $B\boldsymbol{T} = B(T, G, \partial)$ is defined as the classifying space of the simplicial group $N_*^{-1}(T, G, \partial)$, and therefore $\pi_i B(\boldsymbol{T}) = \pi_{i-1} N_*^{-1}(\boldsymbol{T})$. As a consequence, we have $\pi_1 B(T, G, \partial) \cong G/\partial(T)$, $\pi_2 B(T, G, \partial) \cong \operatorname{Ker} \partial$ and $\pi_n B(T, G, \partial) = 0$, $n > 2$. Clearly, $BG = B(1, G, i)$ for any group G.

In [**2, 6**] the homology of a crossed module $\boldsymbol{T}$ with coefficients in a $\pi_1 B(\boldsymbol{T})$-module A is defined as the homology of the classifying space $B\boldsymbol{T}$ of the crossed module with coefficients in A, $\mathrm{H}_n(\boldsymbol{T}, A) := \mathrm{H}_n(B(\boldsymbol{T}), A)$. The aim of this section is to prove that the internal q-homology of a crossed module $\boldsymbol{T}$ defined in § 3 is related by a long exact sequence to the homology of the classifying space $B(\boldsymbol{T})$ of the crossed module with coefficients in $\mathbb{Z}/q\mathbb{Z}$.

The inclusion

$$(1, \mathrm{id}_G) : (1, G, i) \to (T, G, \partial)$$

yields an injective map of simplicial sets

$$i_{(T,G,\partial)} : B(G) \to B(T, G, \partial)$$

whose co-fibre is denoted by $\beta(T, G, \partial)$.

Proposition 5.1. *If $\boldsymbol{T} = (T, G, \partial)$ is a projective object in the category $\mathcal{CM}$, then $\mathrm{H}_n(\beta(\boldsymbol{T}), \mathbb{Z}/q\mathbb{Z}) = 0$ for any $n > 2$ and $\mathrm{H}_2(\beta(\boldsymbol{T}), \mathbb{Z}/q\mathbb{Z}) \cong T/(G\#_q T)$.*

Proof. If $\boldsymbol{T} = (T, G, \partial)$ is a projective crossed module, then it is a retract of a free crossed module, and so ∂ is injective and T, G and $G/\partial(T)$ are free groups. Therefore, $B(T, G, \partial)$ has the same homotopy type as $K(G/\partial(T), 1)$ and applying the homology exact sequence of the co-fibration to

$$BG \to B(T, G, \partial) \to \beta(T, G, \partial)$$

we have that $\mathrm{H}_n(\beta(T, G, \partial), \mathbb{Z}/q\mathbb{Z}) = 0$ for any $n > 2$ and

$$\begin{aligned}\mathrm{H}_2(\beta(T, G, \partial), \mathbb{Z}/q\mathbb{Z}) &\cong \operatorname{Ker}(\mathrm{H}_1(BG, \mathbb{Z}/q\mathbb{Z}) \to \mathrm{H}_1(B(T, G, \partial), \mathbb{Z}/q\mathbb{Z})) \\ &\cong \operatorname{Ker}(\mathrm{H}_1(G, \mathbb{Z}/q\mathbb{Z}) \to \mathrm{H}_1(G/\partial(T), \mathbb{Z}/q\mathbb{Z})).\end{aligned}$$

On the other hand, the five-term homology exact sequence associated with the extension of groups

$$T \rightarrowtail G \twoheadrightarrow G/\partial(T)$$

reduces to

$$\begin{aligned}0 = \mathrm{H}_2(G/\partial(T), \mathbb{Z}/q\mathbb{Z}) \to \mathrm{H}_0(G/\partial(T), \mathrm{H}_1(T, \mathbb{Z}/q\mathbb{Z})) \to \mathrm{H}_1(G, \mathbb{Z}/q\mathbb{Z}) \\ \to \mathrm{H}_1(G/\partial(T), \mathbb{Z}/q\mathbb{Z}) \to 0.\end{aligned}$$

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Since $\mathbb{Z}/q\mathbb{Z} \otimes_{G/\partial(T)} T/[T,T] \cong T/G\#_q T$ and $\mathrm{H}_1(C, \mathbb{Z}/q\mathbb{Z}) \cong C/C\#_q C$ for any group C [**17**], we obtain that $\mathrm{H}_2(\beta(\boldsymbol{T}), \mathbb{Z}/q\mathbb{Z}) \cong T/(G\#_q T)$. □

Theorem 5.2. *For any crossed module* $\boldsymbol{T}$, *there exists a natural isomorphism*

$$\mathrm{H}_{n+1}(\beta(\boldsymbol{T}), \mathbb{Z}/q\mathbb{Z}) \cong \zeta\, \mathrm{H}_n^q(\boldsymbol{T}), \quad n \geqslant 1.$$

Proof. Let $(T, G, \partial)_* = (T_*, G_*, \partial_*)$ be a projective simplicial resolution of (T, G, ∂) in $\mathcal{CM}$. Then by [**10**, Proposition 2] $\beta(T_*, G_*, \partial_*) \to \beta(T, G, \partial)$ is a weak equivalence of bisimplicial sets, where $\beta(T, G, \partial)$ is considered as a bisimplicial set, which is constant in vertical directions. Therefore, we have a spectral sequence of the form

$$E^1_{pr} = \mathrm{H}_r(\beta(T_p, G_p, \partial_p), \mathbb{Z}/q\mathbb{Z}) \implies \mathrm{H}_{p+r}(\beta(T, G, \partial), \mathbb{Z}/q\mathbb{Z}).$$

Since (T_p, G_p, ∂_p) is a projective object in $\mathcal{CM}$, we can apply the above proposition to get

$$E^1_{pr} = 0, \quad \text{if } r \neq 0, 2, \qquad E^1_{p0} \cong \mathbb{Z}/q\mathbb{Z}, \qquad E^1_{p2} \cong T_p/G_p\#_q T_p.$$

Thus, E^1_{p0} is a constant simplicial abelian group, and hence $E^2_{p0} = 0$ for $p > 0$. Therefore, the spectral sequence degenerates and gives the expected isomorphism:

$$\mathrm{H}_{n+2}(\beta(T, G, \partial), \mathbb{Z}/q\mathbb{Z}) \cong \pi_n(T_*/G_*\#_q T_*), \quad n \geqslant 0.$$

□

Corollary 5.3. *For any crossed module* $\boldsymbol{T} = (T, G, \partial)$,

(i) $\mathrm{H}_2(\beta(\boldsymbol{T}), \mathbb{Z}/q\mathbb{Z}) \cong T/(G\#_q T)$;

(ii) *there exists a natural exact sequence*

$$\cdots \longrightarrow \mathrm{H}_{n+1}(B(\boldsymbol{T}), \mathbb{Z}/q\mathbb{Z}) \longrightarrow \zeta\, \mathrm{H}_n^q(\boldsymbol{T}) \longrightarrow \mathrm{H}_n^q(G)$$
$$\longrightarrow \mathrm{H}_n(B(\boldsymbol{T}), \mathbb{Z}/q\mathbb{Z}) \longrightarrow \cdots .$$

Proof. Part (i) follows from Proposition 3.1 and part (ii) from the homology exact sequence of the co-fibration

$$B(G) \to B(T, G, \partial) \to \beta(T, G, \partial).$$

□

When $q = 0$ we get the results of [**10**].

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References

1. M. Barr and J. Beck, Homology and standard constructions, in *Seminar on triples and categorical homology theory*, Lecture Notes in Mathematics, vol. 80, pp. 245–335 (Springer, 1969).
2. H.-J. Baues, *Combinatorial homotopy and 4-dimensional complexes* (De Gruyter, Berlin, 1991).
3. P. Carrasco, A. M. Cegarra and A. R. Grandjeán, (Co)homology of crossed modules, *J. Pure Appl. Alg.* **168**(2,3) (2002), 147–176.
4. D. Conduché, H. Inassaridze and N. Inassaridze, Mod q cohomology and Tate cohomology of groups, *J. Pure Appl. Alg.* **189** (2004), 61–87.
5. J. L. Doncel-Juárez and A. R. Grandjeán, q-perfect crossed modules, *J. Pure Appl. Alg.* **81** (1992), 279–292.
6. G. J. Ellis, Homology of 2-types, *J. Lond. Math. Soc.* **46** (1992), 1–27.
7. N. D. Gilbert, The low-dimensional homology of crossed modules, *Homology Homotopy Appl.* **2** (2000), 41–50.
8. A. R. Grandjeán and M. Ladra, Nilpotent crossed modules and homology, *Commun. Alg.* **26**(2) (1998), 417–424.
9. A. R. Grandjeán and M. P. López, $\mathrm{H}_2^q(T, G, \partial)$ and q-perfect crossed modules, *Appl. Categ. Struct.* **11** (2003), 171–184.
10. A. R. Grandjeán, M. Ladra and T. Pirashvili, CCG-homology of crossed modules via classifying spaces, *J. Alg.* **229** (2000), 660–665.
11. N. Inassaridze, On nonabelian tensor product modulo q of groups, *Commun. Alg.* **29** (2001), 2657–2687.
12. M. Ladra and A. R. Grandjeán, Crossed modules and homology, *J. Pure Appl. Alg.* **95** (1994), 41–55.
13. J. L. Loday, Spaces with finitely many homotopy groups, *J. Pure Appl. Alg.* **24** (1982), 179–202.
14. E. Manes, *Algebraic theories*, Graduate Texts in Mathematics, vol. 26 (Springer, 1976).
15. K. J. Norrie, Actions and automorphisms of crossed modules, *Bull. Soc. Math. France* **118** (1990), 129–146.
16. J. Stallings, Homology and central series of groups, *J. Alg.* **2** (1965), 170–181.
17. U. Stammbach, *Homology in group theory*, Lecture Notes in Mathematics, vol. 359 (Springer, 1973).
18. J. H. C. Whitehead, Combinatorial homotopy, II, *Bull. Am. Math. Soc.* **55** (1949), 453–496.