1. A proof of Dupin's theorem with some simple illustrations of the method employed.

2. Two methods of obtaining Cayley's condition that a family of surfaces may form one of an orthogonal triad.

3. An extension of Dupin's theorem to the case in which a family of surfaces is cut orthogonally by two other families which intersect at a constant angle, with the condition that a family may be capable of being cut in this manner.

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1. *A proof of Dupin's theorem with some simple illustrations of the method employed.*

Before plunging into Dupin's theorem, I think it well to speak of certain infinitesimal rotations which play a part in the proof. By an infinitesimal angle of the first order is meant an angle subtended at the centre of a circle of finite radius by an arc whose length is an infinitesimal of the first order. If we neglect infinitesimals of the second order, equal infinitesimal rotations of the first order about axes which meet and are separated by a small angle of the first order are identical. For instance, if AB and BC be elements of a curve of continuous curvature, an infinitesimal rotation about AB may, if we prefer it, be regarded as taking place about BC; and again, if OA, OB, OC be a set of rectangular axes, small rotations about OA, OB, OC may be regarded as taking place in any order. For if P be a point on a sphere of finite radius, and PQ, PR be the displacements of P due to equal infinitesimal rotations of the first order about two diameters separated by a small angle of the first order, the angle QPR is the angle of separation of the axes, and it follows that QR is an infinitesimal of the second order. Further, if the radius of the sphere is an infinitesimal of the first order, QR is of the third order of small quantities.

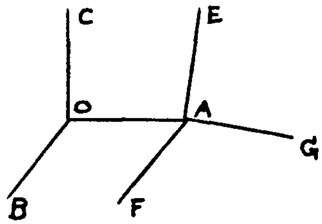

Fig. 1.

Let OA, AG be elements of a curve on a surface; OB and AF elements on the surface at right angles to OA and AG respectively, and OC and AE the normals at O and A. Denote the components along OA, OB, OC of the rotation* which will bring the axes OA, OB, OC into positions parallel to the corresponding axes at A by θ_{A1}, θ_{A2}, θ_{A3}. Then neglecting small quantities of the second order AE and OC will meet, provided $\theta_{A1}=0$. Hence OA is an element of a line of curvature on the surface if $\theta_{A1}=0$, and conversely.

It is well known that if the intersection of two orthogonal surfaces be a line of curvature on one, it is also a line of curvature on the other; but it is here so obvious that it deserves notice. The normals OB and OC and an element OA of the line of intersection form a set of axes at right angles, and the condition $\theta_{A1}=0$ that OA may be an element of a line of curvature on one surface is also the condition that OA may be an element of a line of curvature on the other.

The allied theorem, that if two surfaces cut at a constant angle and if their intersection is a line of curvature on one it is also a line of curvature on the other, can be proved with as little

* It is important that I should be quite clear as to the convention I adopt with regard to positive and negative rotations. I do not care whether the axes of coordinates be right or left handed: I only stipulate that a positive rotation through a right angle about the axis of x shall bring the axi of y into the position formerly occupied by the axis of z, and so on with cyclic interchanges of the letters in the order $x \rightarrow y \rightarrow z \rightarrow x$. When I use axes O(A, B, C) or O(1, 2, 3) I suppose them drawn so that they can be made to coincide with O(x, y, z).

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difficulty. Let OA be an element of the line of intersection, and OB, OC the normals to the surfaces at O: draw $O\gamma$, $O\beta$ at right angles to OB and OC in the plane BOC. Then since the angle BOC is constant the lines $O(ABC\beta\gamma)$ at O, regarded as a rigid system, can be brought into positions parallel to the corresponding lines at A by a small rotation θ. Or, in other words, the rotation for the axes $O(AB\gamma)$ is the same as that for the axes $O(A\beta C)$, and if the component along OA is zero in one case, it is zero in the other also. Thus $\theta_{A1}=0$ is the condition that OA may be an element of a line of curvature on either surface.

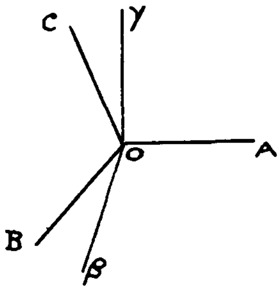

Fig. 2.

We are now in a position to attack Dupin's theorem, that if three families of surfaces cut orthogonally the line of intersection of any member of one family with any member of another is a line of curvature on each. Let OA, OB, OC be elements of the normals to the surfaces at O, or, what is the same thing, elements of the lines of intersection of the three surfaces which pass through O The other lines in the figure are elements of the lines of intersection at A, B, or C, as the case may be.

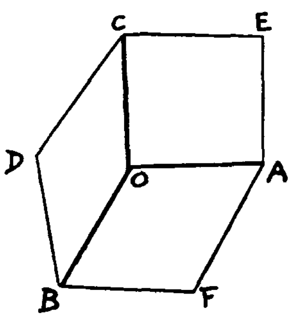

Fig. 3.

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Now the distance of D from the plane BOC is θ_{B2} . OC, and is also $-\theta_{C3}$. OB.

$$\therefore \qquad \frac{\theta_{B2}}{OB}+\frac{\theta_{C3}}{OC}=0,$$

and similarly

$$\frac{\theta_{C3}}{OC}+\frac{\theta_{A1}}{OA}=0,$$

$$\frac{\theta_{A1}}{OA}+\frac{\theta_{B2}}{OB}=0.$$

$$\therefore \qquad \theta_{A1}=0,\ \theta_{B2}=0,\ \theta_{C3}=0,^*$$

i.e. OA, OB, OC are elements of line of curvature of the surfaces on which they lie.

Or we may argue thus:—Suppose θ_{A1} to be positive: then F is on the positive side of the plane AOB, and therefore θ_{B2} is negative. Hence D is on the negative side of BOC and θ_{C3} is positive; and it follows that E is on the positive side of COA, and θ_{A1} is negative. Thus θ_{A1} must be zero.

We can give another simple illustration of the use of the condition for an element of a line of curvature. It is required to show that the lines of curvature on a surface cut at right angles. Let OAFB be an infinitesimal mesh formed by two systems of

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* M. Fouché has anticipated the idea which is the basis of my proof in a paper entitled "Démonstration géométrique du théorème de Dupin," *Nouvelles Annales de Mathématiques*, ser. 4, t. 7 (1907). M. Fouché denotes the geodesic torsions $\frac{\theta_{A1}}{OA}$, etc., by τ, τ_1, τ_2, and having obtained an expression for τ, observes that the geodesic torsion vanishes only along the lines of curvature. He then proves (i) that if two curves cut at right angles on a surface their geodesic torsions at their point of intersection are equal and opposite, (ii) that if two surfaces cut at right angles the geodesic torsion of their line of intersection is the same with respect to either surface. Dupin's theorem then follows in a few lines from the equations $\tau+\tau_1=0$, $\tau_1+\tau_2=0$, $\tau_2+\tau=0$. I was inclined to withdraw my proof after finding M. Fouché's memoir, but I think its retention is justified by my use of the framework of elements, which makes obvious the torsion theorems proved at some length, though charmingly, by M. Fouché. In fact, all proofs of Dupin's theorem, with the exception of Herr Sommerfeld's, the substance of which I am giving in a later note, entail the establishing of equations of the type $\tau+\tau_1=0$, whether the geodesic torsions be actually designated as such, or occur analytically like the [$qr.p$], etc., of Cayley's proof, or as coefficients like the b, b', b'' of Lord Kelvin's.

orthogonal curves on a surface, and OC the normal at O. Then $\frac{\theta_{A1}}{OA} + \frac{\theta_{B2}}{OB} = 0$: hence if $\theta_{A1} = 0$, then $\theta_{B2} = 0$ also, *i.e.* if OA be an element of a line of curvature, OB is also an element of a line of curvature.

As a further example of this method, let l, m, n be the direction cosines of a curve at x, y, z: these direction cosines are functions of x, y, z, and the curves form a doubly infinite system. If they can be cut orthogonally by a family of surfaces $\phi(x, y, z) = \text{constant}$, we have $\frac{\partial\phi}{\partial x} = Rl$, $\frac{\partial\phi}{\partial y} = Rm$, $\frac{\partial\phi}{\partial z} = Rn$, and the elimination of R and ϕ leads immediately to the relation

$$l\left(\frac{\partial n}{\partial y} - \frac{\partial m}{\partial z}\right) + m\left(\frac{\partial l}{\partial z} - \frac{\partial n}{\partial x}\right) + n\left(\frac{\partial m}{\partial x} - \frac{\partial l}{\partial y}\right) = 0\,;$$

but the converse presents difficulties owing to the impossibility of tracing back the steps of the elimination, and I do not remember seeing a simple proof.*

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* I have read recently a paper by Herr Sommerfeld entitled "Geometrischer Beweis des Dupin'schen Theorems und seiner Umkehrung" (Jahresbericht der deutschen Matematiker Vereinigung Band VI., Heft 1, 1897). Taking u, v, w to be the direction cosines of a Curvencongruenz, Herr Sommerfeld proves the relation $\Sigma u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = 0$ by applying Stokes' theorem. The converse he establishes in a most simple and beautiful manner by the same means, and with the help of this Hülfssatz obtains Dupin's theorem. The proof of Dupin's theorem is so out of the common and can be put so briefly that I think I may give an account of it here. Let u, v, w be the components of the normal displacement which will transfer points on any surface (say AOB, Fig. 3) of one of the families to the consecutive surface. Then since OA, OB, OC remain at right angles after strain they must be the principal axes of strain at O. The strain can be analysed into three parts, a translation, a displacement normal to the strain quadric, and rotations $\frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$, etc., about the axes of co-ordinates. The first two parts do not alter the directions of the principal axes, and since u, v, w are proportional to the direction cosines of a curve congruence orthogonal to a family of surfaces, $\Sigma u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = 0$; so that the component rotation about OC is zero. Hence OC is an element of a line of curvature of each surface on which it lies.

Suppose then that we are given $\Sigma l\left(\frac{\partial m}{\partial z} - \frac{\partial n}{\partial y}\right) = 0$. This relation is independent of choice of axes: let OA be an element of one of the curves and OB, OC any infinitesimal lengths of the same order as OA forming a rectangular system with OA. Let BF, CE be

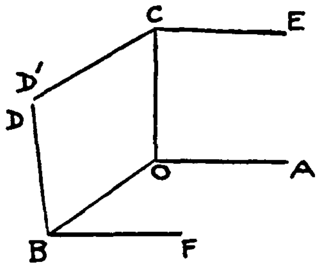

Fig. 4.

elements corresponding to OA of the curves through B and C, and BD, CD′ straight lines whose direction cosines are the same functions of the co-ordinates of B and C as the direction cosines of OC and OB are of the co-ordinates of A. Take OA, OB, OC as axes of x, y, z: then $l=1$, $m=0$, $n=0$, and the given relation becomes

$$\frac{\partial n}{\partial y} = \frac{\partial m}{\partial z}.$$

Now the direction cosines of BF are $1 + \frac{\partial l}{\partial y}\text{OB}$, $0 + \frac{\partial m}{\partial y}\text{OB}$, $0 + \frac{\partial n}{\partial y}\text{OB}$, so that $\frac{\partial n}{\partial y}\text{OB}$ is the cosine of the angle between BF and OC. Thus $\frac{\partial n}{\partial y}\text{OB} = -\theta_{B2}$, and similarly $\frac{\partial m}{\partial z}\text{OC} = \theta_{C3}$.

$$\therefore \quad \frac{\theta_{B2}}{\text{OB}} + \frac{\theta_{C3}}{\text{OC}} = 0,$$

and this is the condition that BD and CD′ may meet, if we neglect infinitesimals of the third order.*

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* It has been shown that if BD and CD′ do meet the condition is satisfied, and it is quite easy to show that the converse holds to the degree of accuracy stated.

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Now if we proceed from B at right angles to BD and BF we can construct a mesh resembling OBDC, and by proceeding in a similar manner from C, D, etc., we can construct a net such that the *l*, *m*, *n* curve at any point is perpendicular to two elements of the net. When the lengths OB, OC, etc., are indefinitely diminished the net becomes a surface which cuts the system of curves orthogonally.

2. *Two methods of obtaining Cayley's condition that a family of surfaces may form one of an orthogonal triad.*

Dupin's theorem certainly imposes a restriction on the manner in which three families of orthogonal surfaces intersect one another, but it is natural to suppose that any family of surfaces can be cut at right angles by two other orthogonal families, provided they follow its lines of curvature. Chasles* fell into this error in his paper, "Sur l'attraction d'une couche ellipsoïdale" in the year 1837, and was corrected by Bouquet† in 1846 in a "Note sur les surfaces orthogonales" in Liouville's Journal—"Peut-être M. Chasles n'a-t-il point accordé une attention suffisante à ce théorème, qui, dans son travail, n'était qu'un accessoire ; car la conclusion à laquelle il a été conduit par un premier aperçu ne me paraît point rigoureuse." Bouquet was thus the first to point out that a family of surfaces may be incapable of being cut at right angles by two other orthogonal families. The subject was pursued by various writers, to whose memoirs references may be found in the Royal Society Index under the heading "Orthogonal Surfaces." Cayley, charmed by an attractive paper by Levy,‡ investigated the general condition that a family of surfaces may be one of an orthogonal triad, and communicated his results in 1872 to the Académie des Sciences in a series of three memoirs, the last of which is concerned with the reduction of the equation by the removal of an irrelevant factor.§ The analysis occupies seventeen or eighteen quarto pages of the Comptes Rendus. Salmon simplified the analysis in his Geometry of Three Dimensions (chap. VIII.), and

* *Journal de l'École Polytechnique*, t. xv.

† *Liouville*, t. xi.

‡ *Journal de l'École Polytechnique*, t. xxvi.

§ *Comptes Rendus*, t. lxxv.

expressed the equation in a determinantal form, to which he was led by the condition that three quadric cones with a common vertex may be cut by a plane in three pairs of straight lines in involution. In 1887 Mr A. R. Johnson determined the rate at which the principal tangents rotate when the point at which they are drawn travels along an orthogonal trajectory of the family, and deduced Cayley's equation by taking this rate as zero.* My object is to obtain these results by other methods, partly with a view to introducing a degree of simplicity into somewhat laborious analysis, and partly in order to discuss the case in which a family of surfaces can be cut orthogonally by two others which intersect each other at any constant angle.

The following seems to me the most direct rather than the simplest method of attack: it is, however, necessary to my discussion of the more general case, and has at least the merit of obtaining Cayley's final form as easily as the first.

Let OA, OB be elements of lines of curvature on the surface $\phi(x, y, z) = r$, and let the normals at O, A, B meet the consecutive surface, $\phi(x, y, z) = r + dr$, at C, E, D. Then if CE, CD are elements of lines of curvature on the surface $r + dr$, for all points O on the surface r, we can proceed from the surface $r + dr$, along its normals, to the next surface of the family, and so construct a framework which constitutes ultimately three families of ortho-

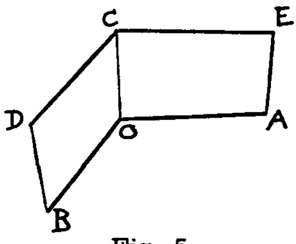

Fig. 5.

gonal surfaces. Hence the sufficient condition that the family r may be one of an orthogonal triad is that the elements CD, CE on the consecutive surface may be elements of lines of curvature. That this condition is necessary as well is immediately obvious from Dupin's theorem. So far I have followed Cayley and Salmon.

* *Quarterly Journal*, vol. 22.

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Let the differential coefficients of ϕ of the first order be denoted by L, M, N, and those of the second order by a, b, c, f, g, h. Let x, y, z be the co-ordinates of O, and $x+d_1x, y+d_1y, z+d_1z$ and $x+d_2x, y+d_2y, z+d_2z$ those of A and B.

Then since OB is at right angles to AE

$$d_2x\left\{\frac{L}{R}+d_1\left(\frac{L}{R}\right)\right\}+d_2y\left\{\frac{M}{R}+d_1\left(\frac{M}{R}\right)\right\}+d_2z\left\{\frac{N}{R}+d_1\left(\frac{N}{R}\right)\right\}=0,$$

or
$$d_2xd_1L+d_2yd_1M+d_2zd_1N=0,$$

where $R^2\equiv L^2+M^2+N^2$, and $d_1\equiv d_1x\frac{\partial}{\partial x}+d_1y\frac{\partial}{\partial y}+d_1z\frac{\partial}{\partial z}$.

Hence, remembering that $\frac{\partial L}{\partial x}=a$ etc., we get

$$d_2x(ad_1x+hd_1y+gd_1z)+d_2y(hd_1x+bd_1y+fd_1z)$$
$$+d_2z(gd_1x+fd_1y+cd_1z)=0.$$

This conjugate* relation with $\Sigma Ld_1x=0$, $\Sigma Ld_2x=0$, and the orthogonal condition

$$d_1xd_2x+d_1yd_2y+d_1zd_2z=0,$$

is sufficient to determine the directions of the lines of curvature at O.

Now the co-ordinates of C may be taken to be $x+L\rho$, $y+M\rho$, $z+N\rho$, and since C lies on the surface $r+dr$

$$\phi(x+L\rho,\ y+M\rho,\ z+N\rho)=r+dr$$
$$L\rho\frac{\partial\phi}{\partial x}+M\rho\frac{\partial\phi}{\partial y}+N\rho\frac{\partial\phi}{\partial z}=dr,$$
$$\therefore\quad R^2\rho=dr.$$

Hence $R^2\rho$ is constant as we proceed along the surface r, and consequently

$$d_1\rho=-\frac{d_1(R^2)}{R^2}\cdot\rho\ \text{ and }\ d_2\rho=-\frac{d_2(R^2)}{R^2}\cdot\rho.$$

We regard ρ as an infinitesimal of the same order as d_1x, etc., so that $d_1\rho$, $d_2\rho$ are of the second order.

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* It may also be written

$d_1x(ad_2x+hd_2y+gd_2z)+\ldots+\ldots=0$, or $d_1xd_2L+d_1yd_2M+d_1zd_2N=0$.

Let x' denote $x+L\rho$, and let $x'+d_1x'$, etc., be the co-ordinates of E.

Then $$d_1x' = d_1x + \rho d_1L + Ld_1\rho,$$
with five similar equations, and the elements d_1s', d_2s' are at right angles if

$$(d_1x+\rho d_1L+Ld_1\rho)(d_2x+\rho d_2L+Ld_2\rho)+\dots\dots\dots+\dots\dots\dots=0,$$

or $$(d_1xd_2L+d_1yd_2M+d_1zd_2N)+(d_2xd_1L+d_2yd_1M+d_2zd_1N)=0,$$
a condition which is satisfied by virtue of the conjugate relation.

Thus the sufficient and only condition that CE, CD may be elements of lines of curvature is that the conjugate relation may hold when we put $d_1x+\rho d_1L+Ld_1\rho$ for d_1x and so on, and substitute for a, etc., the values that these quantities assume at the point C.

Now at C the value of a is $a+\rho\delta a$, where

$$\delta \equiv L\frac{\partial}{\partial x}+M\frac{\partial}{\partial y}+N\frac{\partial}{\partial z}\text{*},$$

and the condition required is therefore

$$\rho\{\delta a d_1xd_2x+\delta b d_1yd_2y+\delta c d_1zd_2z+\delta f(d_1yd_2z+d_2yd_1z)+\delta g(d_1zd_2x+d_2zd_1x)$$
$$+\delta h(d_1xd_2y+d_2xd_1y)\}$$
$$+2\rho(d_1Ld_2L+d_1Md_2M+d_1Nd_2N)+d_1\rho(Ld_2L+Md_2M+Nd_2N)$$
$$+d_2\rho(Ld_1L+Md_1M+Nd_1N)=0,$$

where

$$d_1\rho=-\frac{d_1(R^2)}{R^2}\cdot\rho=-\frac{2\rho}{R^2}\{L(ad_1x+hd_1y+gd_1z)+M(hd_1x+bd_1y+fd_1z)$$
$$+N(gd_1x+fd_1y+cd_1z)\}$$
$$=-\frac{2\rho}{R^2}\{d_1x\delta L+d_1y\delta M+d_1z\delta N\},$$

and similarly for $d_2\rho$.

After substituting for $d_1\rho$ and $d_2\rho$, and dividing through by ρ, we get

$$\delta a d_1xd_2x+\delta b d_1yd_2y+\delta c d_1zd_2z+\delta f(d_1yd_2z+d_2yd_1z)+\delta g(d_1zd_2x+d_2zd_1x)$$
$$+\delta h(d_1xd_2y+d_2xd_1y)$$
$$+2(d_1Ld_2L+d_1Md_2M+d_1Nd_2N)$$
$$-\frac{4}{R^2}(d_1x\delta L+d_1y\delta M+d_1z\delta N)(d_2x\delta L+d_2y\delta M+d_2z\delta N)=0.$$

* Cayley's use of δ for $L\frac{\partial}{\partial x}+M\frac{\partial}{\partial y}+N\frac{\partial}{\partial z}$ is a little misleading, as the symbol is usually associated with an increment rather than a rate of change along the normal. Salmon, however, retained the notation, and an alteration would create confusion.

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We can effect a reduction by observing that

$$(L^2+M^2+N^2)(d_1Ld_2L+d_1Md_2M+d_1Nd_2N)$$
$$=(d_1x\delta L+d_1y\delta M+d_1z\delta N)(d_2x\delta L+d_2y\delta M+d_2z\delta N);$$

for the right hand side is the same as

$$(Ld_1L+Md_1M+Nd_1N)(Ld_2L+Md_2M+Nd_2N),$$

and the alleged identity is true if

$$(M^2+N^2)d_1Ld_2L+(L^2+N^2)d_1Md_2M+(L^2+M^2)d_1Nd_2N$$
$$=Ld_1L(Md_2M+Nd_2N)+Md_1M(Ld_2L+Nd_2N)$$
$$+Nd_1N(Ld_2L+Md_2M),$$

i.e. if

$$(Md_1N-Nd_1M)(Md_2N-Nd_2M)+(Nd_1L-Ld_1N)(Nd_2L-Ld_2N)$$
$$+(Ld_1M-Md_1L)(Ld_2M-Md_2L)=0.$$

Now $\quad d_1xd_2L+d_1yd_2M+d_1zd_2N=0,$

and $\quad d_1xL+d_1yM+d_1zN=0.$

$$\therefore \quad \left\| \begin{matrix} L, & M, & N \\ d_2L, & d_2M, & d_2N \end{matrix} \right\| \models d_1x,\ d_1y,\ d_1z,^*$$

and similarly $\left\| \begin{matrix} L, & M, & N \\ d_1L, & d_1M, & d_1N \end{matrix} \right\| \models d_2x,\ d_2y,\ d_2z.$

But we know that $\Sigma d_1xd_2x=0$, and it follows that the equality we are discussing is true.

The condition that d_1s', d_2s' may be elements of lines of curvature may now be written

$$\delta ad_1xd_2x+\delta bd_1yd_2y+\delta cdz_1dz_2+\delta f(d_1yd_2z+d_2yd_1z)+\delta g(d_1zd_2x+d_2zd_1x)$$
$$+\delta h(d_1xd_2y+d_2xd_1y)$$
$$-\frac{2}{R^2}(d_1x\delta L+d_1y\delta M+d_1z\delta N)(d_2x\delta L+d_2y\delta Md_2z\delta N)=0,$$

and it remains to eliminate d_1xd_2x, etc., between this equation and the equations for the lines of curvature.

We have, of course, the conjugate relation

$$ad_1xd_2x+bd_1yd_2y+cd_1zd_2z+f(d_1yd_2z+d_2yd_1z)+g(d_1zd_2x+d_2zd_1x)$$
$$+h(d_1xd_2y+d_2xd_1y)=0,$$

and the orthogonal condition

$$d_1xd_2x+d_1yd_2y+d_1zd_2z=0.$$

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* The notation is sufficiently obvious: the symbol $\models$ is to be read "are proportional to."

Further, from $\Sigma L d_1 x = 0$ and $\Sigma L d_2 x = 0$ we get

$$L d_1 x d_2 x \qquad\qquad + \frac{N}{2}(d_1 z d_2 x + d_2 z d_1 x) + \frac{M}{2}(d_1 x d_2 y + d_2 x d_1 y) = 0,$$

$$M d_1 y d_2 y \qquad + \frac{N}{2}(d_1 y d_2 z + d_2 y d_1 z) \qquad + \frac{L}{2}(d_1 x d_2 y + d_2 x d_1 y) = 0,$$

$$N d_1 z d_2 z + \frac{M}{2}(d_1 y d_2 z + d_2 y d_1 z) + \frac{L}{2}(d_1 z d_2 x + d_2 z d_1 x) \qquad = 0,$$

and the elimination of $d_1 x d_2 x$, etc., from the six equations gives

$$\begin{vmatrix} \delta a - \frac{2(\delta L)^2}{R^2}, & \delta b - \frac{2(\delta M)^2}{R^2}, & \delta c - \frac{2(\delta N)^2}{R^2}, & 2\left(\delta f - \frac{2\delta M \delta N}{R^2}\right), & 2\left(\delta g - \frac{2\delta N \delta L}{R^2}\right), & 2\left(\delta h - \frac{2\delta L \delta M}{R^2}\right) \\ a, & b, & c, & 2f, & 2g, & 2h \\ 1, & 1, & 1, & 0, & 0, & 0 \\ L, & 0, & 0, & 0, & N, & M \\ 0, & M, & 0, & N, & 0, & L \\ 0, & 0, & N, & M, & L, & 0 \end{vmatrix} = 0$$

This is Salmon's determinantal form of Cayley's unreduced equation: if $\mathfrak{A}$*, etc., are the co-factors of the terms of the first row of the determinant, it may be written

$$\mathfrak{A}\left\{\delta a - \frac{2(\delta L)^2}{R^2}\right\} + \ldots + \ldots + 2\mathfrak{F}\left\{\delta f - \frac{2\delta M \delta N}{R^2}\right\} + \ldots + \ldots = 0.$$

After the expansion of the determinant the equation has to be cleared of fractions by multiplying by R^2. Cayley found that the equation in this form contained the factor $L^2 + M^2 + N^2$, and the last of his communications on the subject to the Académie de

* It is easily found by actual expansion that

$$\tfrac{1}{2}\mathfrak{A} = (M^2 + N^2)(Mg - Nh) + L\{(b - c)MN - f(M^2 - N^2)\}.$$

$$\mathfrak{F} = (L^2 + M^2 + N^2)\{(b - c)L + gN - hM\} + (a - b)LM^2$$
$$(c - a)LN^2 + gN(L^2 + M^2 - N^2) - hM(L^2 - M^2 + N^2).$$

The equivalent forms $\tfrac{1}{2}\mathfrak{A} = (L^2 + M^2 + N^2)(Mg - Nh) + L(N\delta M - M\delta N)$

$$\mathfrak{F} = (L^2 + M^2 + N^2)\{(b - c)L - hM + gN)\}$$
$$+ M(M\delta L - L\delta M) + N(L\delta N - N\delta L)$$

are given by Cayley and quoted by Salmon.

Sciences was concerned with its removal. The ultimate form is easily obained by this method, for the first of our six equations could have been written

$$\delta a d_1x d_1y + \ldots + \ldots + \delta f(d_1y d_2z + d_2y d_1z) + \ldots + \ldots$$
$$- 2(d_1\mathrm{L} d_2\mathrm{L} + d_1\mathrm{M} d_2\mathrm{M} + d_1\mathrm{N} d_2\mathrm{N}) = 0,$$

or $$\delta a d_1x d_1y + \ldots + \ldots + \delta f(d_1y d_2z + d_2y d_1z) + \ldots + \ldots$$
$$- 2\{(ad_1x + hd_1y + gd_1z)(ad_2x + hd_2y + gd_2z)$$
$$+ (hd_1x + bd_1y + fd_1z)(hd_2x + bd_2y + fd_2z)$$
$$+ (gd_1x + fd_1y + cd_1z)(gd_2x + fd_2y + cd_2z)\} = 0.$$

The other equations are the same as before, and the elimination gives us for the first row of the determinant

$$\delta a - 2(a^2 + h^2 + g^2),\ \delta b - 2(h^2 + b^2 + f^2),\ \delta c - 2(g^2 + f^2 + c^2),$$
$$2\{\delta f - 2(hg + bf + fc)\}, 2\{\delta g - 2(ag + hf + gc)\}, 2\{\delta h - 2(ah + hb + gf)\}.$$

By adding $2(f^2 + g^2 + h^2) - 2(bc + ca + ab)$ times the third row and $2(a + b + c)$ times the second row to the first row, we get the final form

$$\mathfrak{A}(\delta a - 2\mathrm{A}) + \mathfrak{B}(\delta b - 2\mathrm{B}) + \mathfrak{C}(\delta c - 2\mathrm{C}) + 2\mathfrak{F}(\delta f - 2\mathrm{F}) + 2\mathfrak{G}(\delta g - 2\mathrm{G})$$
$$+ 2\mathfrak{H}(\delta h - 2\mathrm{H}) = 0,$$

where A, etc., are the co-factors of a, etc., in the determinant whose rows are a, h, g ; h, b, f ; g, f, c.

The second method of obtaining Cayley's equation I present on account of its simplicity. It resembles Mr Johnson's in that it determines the rate at which the principal tangents at a point on any member of the family $\phi(x, y, z) = r$ rotate about the normal when the point describes an orthogonal trajectory of the family.

Let λ_1, μ_1, ν_1 and λ_2, μ_2, ν_2 be the direction cosines of OA and OB, the principal tangents at O.

Then since

$$\lambda_2, \mu_2, \nu_2 = \left\| \begin{matrix} \mathrm{L}, & \mathrm{M}, & \mathrm{N}, \\ \lambda_1, & \mu_1, & \nu_1, \end{matrix} \right\|$$

the conjugate relation may be written

$$\begin{vmatrix} a\lambda_1 + h\mu_1 + g\nu_1, & h\lambda_1 + b\mu_1 + f\nu_1, & g\lambda_1 + f\mu_1 + c\nu_1 \\ \lambda_1 & \mu_1 & \nu_1 \\ \mathrm{L} & \mathrm{M} & \mathrm{N} \end{vmatrix} = 0,$$

and, of course, λ_1, μ_1, ν_1 may be replaced by λ_2, μ_2, ν_2.

Thus the direction cosines of either principal tangent satisfy equations of the type

$$a\lambda + h\mu + g\nu = K\lambda + \chi L$$

$$h\lambda + b\lambda + f\nu = K\mu + \chi M$$

$$g\lambda + f\mu + c\nu = K\nu + \chi N,$$

in which the same suffixes must be attached to K and χ as to λ, μ, ν.

On multiplying by λ, μ, ν and adding, we have

$$K = (a, b, c, f, g, h)(\lambda, \mu, \nu)^2,*$$

and on multiplying by L, M, N and adding

$$R^2\chi = \lambda\delta L + \mu\delta M + \nu\delta N.$$

The elimination of λ, μ, ν, χ from the K, χ equation and the equation $\Sigma\lambda L = 0$ gives for k the quadratic

$$\begin{vmatrix} a-K, & h, & g, & L \\ h, & b-K, & f, & M \\ g, & f, & c-K, & N \\ L, & M, & N, & O \end{vmatrix} = 0.$$

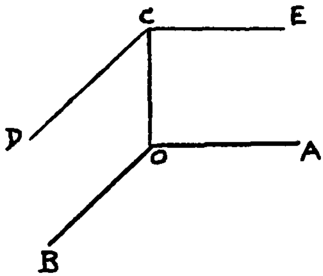

Fig. 6.

Let the normal at O meet the consecutive surface of the family at C, and let OC $= \rho$R, so that the co-ordinates of C are, as before,

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* It is hardly necessary to prove that $\frac{K_1}{R}$, $\frac{K_2}{R}$ are the curvatures of the principal normal sections considered positive when the corresponding centres of curvature are on the side of the tangent plane remote from the region into which the normal $\frac{L}{R}$, $\frac{M}{R}$, $\frac{N}{R}$ is drawn: R is, of course, always positive.

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$x+\rho L$, $y+\rho M$, $z+\rho N$. The direction cosines of CE, the principal tangent at C corresponding to OA, are

$$\lambda_1+\left(\rho L\frac{\partial}{\partial x}+\rho M\frac{\partial}{\partial y}+\rho N\frac{\partial}{\partial z}\right)\lambda_1,\text{ etc., or }\lambda_1+\rho\delta\lambda_1,\text{ etc.}$$

Let θ_{C3} be the component along OC of the rotation which will bring the system O(A, B, C) into a position parallel to that of the corresponding system at C: then

$$\begin{aligned}\theta_{C3}&=\lambda_2(\lambda_1+\rho\delta\lambda_1)+\ldots+\ldots\\&=\rho(\lambda_2\delta\lambda_1+\mu_2\delta\mu_1+\nu_2\delta\nu_1),\end{aligned}$$

and if for $\dfrac{\theta_{C3}}{OC}$ we write $\dfrac{\partial\theta}{\partial\nu}$,

$$\frac{\partial\theta}{\partial\nu}=\frac{\theta_{C3}}{\rho R}=\frac{1}{R}(\lambda_2\delta\lambda_1+\mu_2\delta\mu_1+\nu_2\delta\nu_1)=-\frac{1}{R}(\lambda_1\delta\lambda_2+\mu_1\delta\mu_2+\nu_1\delta\nu_2).$$

Now the K, χ equations give

$$\begin{aligned}&\delta a\lambda_1+\delta h\mu_1+\delta g\nu_1+a\delta\lambda_1+h\delta\mu_1+g\delta\nu_1=K_1\delta\lambda_1+\lambda_1\delta K_1+\chi_1\delta L+L\delta\chi_1,\\&\delta h\lambda_1+\delta b\mu_1+\delta f\nu_1+h\delta\lambda_1+b\delta\mu_1+f\delta\nu_1=K_1\delta\mu_1+\mu_1\delta K_1+\chi_1\delta M+M\delta\chi_1,\\&\delta g\lambda_1+\delta f\mu_1+\delta c\nu_1+g\delta\lambda_1+f\delta\mu_1+c\delta\nu_1=K_1\delta\nu_1+\nu_1\delta K_1+\chi_1\delta N+N\delta\chi_1.\end{aligned}$$

Multiplying by λ_2, μ_2, ν_2 and adding

$$\begin{aligned}\delta a\lambda_1\lambda_2+\ldots+\ldots+\delta f(\mu_1\nu_2+\mu_2\nu_1)+\ldots+\ldots+a\lambda_2\delta\lambda_1+\ldots+\ldots+f(\mu_2\delta\nu_1+\nu_2\delta\mu_1)+\ldots+\ldots\\=K_1(\lambda_2\delta\lambda_1+\mu_2\delta\mu_1+\nu_2\delta\nu_1)+\chi_1(\lambda_2\delta L+\mu_2\delta M+\nu_2\delta N);\end{aligned}$$

and similarly

$$\begin{aligned}\delta a\lambda_1\lambda_2+\ldots+\ldots+\delta f(\mu_1\nu_2+\mu_2\nu_1)+\ldots+\ldots+a\lambda_1\delta\lambda_2+\ldots+\ldots+f(\mu_1\delta\nu_2+\nu_1\delta\mu_2)+\ldots+\ldots\\=K_2(\lambda_1\delta\lambda_2+\mu_1\delta\mu_2+\nu_1\delta\nu_2)+\chi_2(\lambda_1\delta L+\mu_1\delta M+\nu_1\delta N).\end{aligned}$$

On adding, and remembering that by virtue of the conjugate relation

$$\begin{aligned}\delta a\lambda_1\lambda_2+\ldots+\ldots+\delta f(\mu_1\nu_2+\mu_2\nu_1)+\ldots+\ldots+a\delta(\lambda_1\lambda_2)+\ldots+\ldots\\+f\delta(\mu_1\nu_2+\mu_2\nu_1)+\ldots+\ldots=0,\end{aligned}$$

we get

$$\begin{aligned}\delta a\lambda_1\lambda_2+\ldots+\ldots+\delta f(\mu_1\nu_2+\mu_2\nu_1)+\ldots+\ldots=(K_1-K_2)R\frac{\partial\theta}{\partial\nu}\\+\frac{2}{R^2}(\lambda_1\delta L+\mu_1\delta M+\nu_1\delta N)(\lambda_2\delta L+\mu_2\delta M+\nu_2\delta N).\end{aligned}$$

The values of $\lambda_1\lambda_2$, etc., can be obtained without difficulty from the K, χ equations: we have

$$\begin{aligned}K_1\lambda_1&=a\lambda_1+h\mu_1+g\nu_1-\chi_1L,\\K_2\lambda_2&=a\lambda_2+h\mu_2+g\nu_2-\chi_2L.\end{aligned}$$

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$$\therefore \quad (K_1 - K_2)\lambda_1\lambda_2 = h(\lambda_2\mu_1 - \lambda_1\mu_2) + g(\lambda_2\nu_1 - \lambda_1\nu_2) - (\lambda_2\chi_1 - \lambda_1\chi_2)L$$
$$= \frac{Mg - Nh}{R} - (\lambda_2\chi_1 - \lambda_1\chi_2)L.$$

Now $$R^2\chi_1 = \lambda_1\delta L + \mu_1\delta M + \nu_1\delta N,$$
$$R^2\chi_2 = \lambda_2\delta L + \mu_2\delta M + \nu_2\delta N.$$

$$\therefore R^2(\lambda_2\chi_1 - \lambda_1\chi_2) = (\lambda_2\mu_1 - \lambda_1\mu_2)\delta M + (\lambda_2\nu_1 - \lambda_1\nu_2)\delta N = \frac{1}{R}(M\delta N - N\delta M).$$

$$\therefore \quad R^3(K_1 - K_2)\lambda_1\lambda_2 = (Mg - Nh)R^2 + L(N\delta M - M\delta N)$$
$$= \tfrac{1}{2}\mathfrak{A}.$$

Again $$K_1\mu_1 = h\lambda_1 + b\mu_1 + f\nu_1 - \chi_1 M,$$
$$K_1\nu_1 = g\lambda_1 + f\mu_1 + c\nu_1 - \chi_1 N.$$

$$\therefore \quad K_1(\mu_1\nu_2 + \mu_2\nu_1) = h\lambda_1\nu_2 + b\mu_1\nu_2 + f\nu_1\nu_2 + g\lambda_1\mu_2 + f\mu_1\mu_2 + c\nu_1\mu_2 - \chi_1\nu_2 M - \chi_1\mu_2 N,$$

and similarly

$$K_2(\mu_1\nu_2 + \mu_2\nu_1) = h\lambda_2\nu_1 + b\mu_2\nu_1 + f\nu_2\nu_1 + g\lambda_2\mu_1 + f\mu_2\mu_1 + c\nu_2\mu_1 - \chi_2\nu_1 M - \chi_2\mu_1 N.$$

$$\therefore \quad R^3(K_1 - K_2)(\mu_1\nu_2 + \mu_2\nu_1) = R^2(-hM + bL + gN - cL) + M(M\delta L - L\delta M) + N(L\delta N - N\delta L)$$
$$= \mathfrak{F}.$$

Hence the value of $\frac{\partial\theta}{\partial\nu}$ is given by

$$\mathfrak{A}\left\{\delta a - \frac{2(\delta L)^2}{R^2}\right\} + .. + .. + 2\mathfrak{F}\left\{\delta f - \frac{2\delta M\delta N}{R^2}\right\} + .. + .. = 2R^4(K_1 - K_2)^2\frac{\delta\theta}{\delta\nu},*$$

or

$$\mathfrak{A}\{\delta a - 2A\} + \dots + \dots + 2\mathfrak{F}\{\delta f - 2F\} + \dots + \dots = 2R^4(K_1 - K_2)^2\frac{\partial\theta}{\delta\nu},$$

and Cayley's equation may be deduced by putting $\frac{\partial\theta}{\partial\nu}$ equal to zero.

---

* Mr Johnson's original form is

$$2\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^2\frac{\partial\theta}{\partial\nu} = 2\left(m\frac{dl}{dz} - n\frac{dl}{dy}\right)\left\{\frac{d}{d\nu}\frac{dl}{dx} - \left(\frac{dl}{d\nu}\right)^2\right\} + \dots + \dots$$
$$+ \left\{l\left(\frac{dm}{dy} - \frac{dn}{dz}\right) - m\frac{dm}{dx} + n\frac{dn}{dx}\right\}\left\{\frac{d}{d\nu}\left(\frac{dm}{dz} + \frac{dn}{dy}\right) - 2\frac{dm}{d\nu}\frac{dn}{d\nu}\right\} + \dots + \dots$$

where l, m, n are the actual direction cosines of the normal. His reduction to Cayley's form I do not understand, but it is easy by ordinary means.

3. *An extension of Dupin's theorem to the case in which a family of surfaces is cut orthogonally by two other families which intersect at a constant angle, with the condition that a family may be capable of being cut in this manner.*

Cayley has shown that if OA, OB (Fig. 5) be elements of lines of curvature on a surface, and if the normals at O, A, B meet a consecutive surface at C, E, D, then the elements CE and CD are at right angles: and conversely if OA and OB are at right angles, and CE and CD are at right angles, then OA, OB must be elements of lines of curvature.

I propose here to enquire into the relation which OA and OB bear to the lines of curvature when they are inclined to one another at an angle γ, and CE and CD are inclined to one another at the same angle.

We have as before $d_1x' = d_1x + \rho d_1\mathrm{L} + \mathrm{L}d_1\rho$, etc.,
but the orthogonal condition is replaced by

$$d_1xd_2x + d_1yd_2y + d_1zd_2z = d_1sd_2s\cos\gamma.$$

Now $$(d_1s')^2 = (d_1s)^2 + 2\rho(d_1xd_1\mathrm{L} + d_1yd_1\mathrm{M} + d_1zd_1\mathrm{N}),$$

so that $$d_1s' = d_1s + \frac{\rho}{d_1s}\mathrm{I}_1,$$

where $$\mathrm{I}_1 \equiv d_1xd_1\mathrm{L} + d_1yd_1\mathrm{M} + d_1zd_1\mathrm{N}$$
$$\equiv d_1x(ad_1x + hd_1y + gd_1z) + \dots + \dots$$
$$\equiv (a, b, c, f, g, h)(d_1x, d_1y, d_1z)^2.$$

Hence the condition that CE and CD may contain an angle γ is

$$(d_1x + \rho d_1\mathrm{L} + \mathrm{L}d_1\rho)(d_2x + \rho d_2\mathrm{L} + \mathrm{L}d_2\rho) + \dots + \dots = \left(d_1s + \frac{\rho\mathrm{I}_1}{d_1s}\right)\left(d_2s + \frac{\rho\mathrm{I}_2}{d_2s}\right)\cos$$

or $$2\mathrm{I}_{12} = \left(\frac{d_2s}{d_1s}\mathrm{I}_1 + \frac{d_1s}{d_2s}\mathrm{I}_2\right)\cos\gamma,$$

where $$\mathrm{I}_{12} \equiv d_1x(ad_2x + hd_2y + gd_2z) + d_1y(hd_2x + bd_2y + fd_2z) + d_1z(gd_2x + fd_2y + cd_2z).$$

This condition may be written

$$2\mathrm{J}_{12} = (\mathrm{J}_1 + \mathrm{J}_2)\cos\gamma,$$

where $\mathrm{J}_1 \equiv (a, b, c, f, g, h)(l_1, m_1, n_1)^2$, etc., l_1, m_1, n_1 being the direction cosines of OA, and l_2, m_2, n_2 those of OB.

Let OC_1, OC_2 be the projections of OA on the lines of curvature at O, and let their direction cosines be λ_1, μ_1, ν_1 and λ_2, μ_2, ν_2: also let the angle AOC_1 be θ, and put $p_1 = \cos\theta$, $q_1 = \sin\theta$.

Then by projecting OA on the axis of x, we obtain

$$l_1 = p_1\lambda_1 + q_1\lambda_2,$$

and similarly
$$m_1 = p_1\mu_1 + q_1\mu_2,$$
$$n_1 = p_1\nu_1 + q_1\nu_2,$$

with, of course, like equations for l_2, m_2, n_2 in terms of p_2, q_2.

Hence if $\quad K_1 \equiv (a, b, c, f, g, h)(\lambda_1, \mu_1, \nu_1)^2$, etc.,

we have
$$J_1 = p_1^2K_1 + 2p_1q_1K_{12} + q_1^2K_2$$
$$= p_1^2K_1 + q_1^2K_2,$$

since $K_{12} = 0$ by the conjugate relation for lines of curvature:

and
$$J_{12} = p_1p_2K_1 + (p_1q_2 + p_2q_1)K_{12} + q_1q_2K_2$$
$$= p_1p_2K_1 + q_1q_2K_2.$$

Hence the condition that CE and CD may contain an angle γ is

$$2(p_1p_2K_1 + q_1q_2K_2) = \{(p_1^2 + p_2^2)K_1 + (q_1^2 + q_2^2)K_2\}\cos\gamma.$$

This equation, in conjunction with $p_1p_2 + q_1q_2 = \cos\gamma$, is sufficient to determine p_1, p_2, q_1, q_2.

On multiplying the latter by $2K_2$ and subtracting from the former, we get

$$2p_1p_2(K_1 - K_2) = \{(p_1^2 + p_2^2)K_1 - (2 - q_1^2 - q_2^2)K_2\}\cos\gamma$$
$$= (p_1^2 + p_2^2)(K_1 - K_2)\cos\gamma,$$

so that, unless $K_1 = K_2$,

$$2p_1p_2 = (p_1^2 + p_2^2)\cos\gamma.$$

Thus the inclinations of OA, OB to the lines of curvature are independent of the position of O on the surface: further, on putting $p_1 = \cos\theta$, $p_2 = \cos(\theta + \gamma)$, we get

$$2\cos\theta\cos(\theta + \gamma) = \cos\gamma\{\cos^2\theta + \cos^2(\theta + \gamma)\}$$
$$\cos(\theta + \gamma)\{\cos\theta - \cos\gamma\cos(\theta + \gamma)\} = \cos\theta\{\cos\gamma\cos\theta - \cos(\theta + \gamma)\}$$
$$\cos(\theta + \gamma)\sin\gamma\sin(\theta + \gamma) = \cos\theta\sin\gamma\sin\theta$$
$$\sin2(\theta + \gamma) = \sin2\theta\,;$$

$\therefore$
$$2\theta + 2\gamma = \pi - 2\theta \text{ or } 3\pi - 2\theta$$
$$\theta = \frac{\pi}{4} - \frac{\gamma}{2} \text{ or } \frac{3\pi}{4} - \frac{\gamma}{2}$$

and
$$\theta + \gamma = \frac{\pi}{4} + \frac{\gamma}{2} \text{ or } \frac{3\pi}{4} + \frac{\gamma}{2}.$$

Hence OA and OB make complementary angles with the lines of curvature at O, or, what is the same thing, equal angles $\frac{\gamma}{2}$ with one or other of the bisectors of the angles between the principal tangents.

We have thus the theorem :—*If a family of surfaces A can be cut orthogonally by two other families B and C which intersect at a constant angle, the families B and C determine on any member of A two systems of curves which intersect the lines of curvature at constant angles and are equally inclined at any point to one or other of the bisectors of the angles between the principal tangents.*

The system for which $\theta = \frac{\pi}{4} - \frac{\gamma}{2}$ and $2pq = \cos\gamma$ (the suffixes may be dropped since $p_1 = q_2$ and $p_2 = q_1$) I shall refer to as the η lines. The other system, for which $\theta = \frac{3\pi}{4} - \frac{\gamma}{2}$ and $2pq = -\cos\gamma$, I will call the ϵ lines. It will be seen later that if the family A can be cut orthogonally by two families following the η lines, a certain differential equation of the third order must be satisfied, and that in this case the family A cannot be cut orthogonally by two families following the ϵ lines. In fact, the differential equations are different in the two cases, though, of course, each reduces to Cayley's equation when $\gamma = \frac{\pi}{2}$.

I now proceed to find these equations.

We have as before

$$d_1x' = d_1x + \rho d_1\mathrm{L} + \mathrm{L}d_1\rho,$$
etc.,

and
$$d_1s' = d_1s(1 + \rho\mathrm{J}_1).$$

$$\therefore \qquad \frac{d_1x'}{d_1s'} = \frac{d_1x}{d_1s} + \rho\frac{d_1\mathrm{L}}{d_1s} + \mathrm{L}\frac{d_1\rho}{d_1s} - \rho\frac{d_1x}{d_1s}\mathrm{J}_1,$$

or $$l_1' = l_1 + \rho(al_1 + hm_1 + gn_1) + \mathrm{L}\left(l_1\frac{\partial\rho}{\partial x} + m_1\frac{\partial\rho}{\partial y} + n_1\frac{\partial\rho}{\partial z}\right) - \rho l_1\mathrm{J}_1,$$

and similarly

$$l_2' = l_2 + \rho(al_2 + hm_2 + gn_2) + \mathrm{L}\left(l_2\frac{\partial\rho}{\partial x} + m_2\frac{\partial\rho}{\partial y} + n_2\frac{\partial\rho}{\partial z}\right) - \rho l_2\mathrm{J}_2.$$

Now $$l_1 = p\lambda_1 + q\lambda_2$$
$$l_2 = q\lambda_1 + p\lambda_2,$$

and we will put
$$l_1' = p\lambda_1' + q\lambda_2'$$
$$l_2' = q\lambda_1' + p\lambda_2',$$

so that λ_1', μ_1', ν_1' and λ_2', μ_2', ν_2' are the direction cosines of lines at C related to CE, CD in the same way as the principal tangents at O are related to OA, OB. But these lines at C are not necessarily principal tangents, and in fact the equation we are seeking is obtained by expressing the condition that λ_1', μ_1', ν_1' and λ_2', μ_2', ν_2' may satisfy the conjugate relation on the surface DCE. It is already sufficiently obvious that they are at right angles.

On multiplying the equations for l_1' and l_2' by p and q respectively, subtracting and dividing by $p^2 - q^2$, and remembering that
$$(p^2 - q^2)\lambda_1 = pl_1 - ql_2$$
$$(p^2 - q^2)\lambda_2 = -(ql_1 - pl_2),$$

we get
$$\lambda_1' = \lambda_1 + \rho(a\lambda_1 + h\mu_1 + g\nu_1) + \mathrm{L}\left(\lambda_1\frac{\partial p}{\partial x} + \mu_1\frac{\partial \rho}{\partial y} + \nu_1\frac{\partial \rho}{\partial z}\right) - \rho\frac{pl_1\mathrm{J}_1 - ql_2\mathrm{J}_2}{p^2 - q^2},$$

or $$\lambda_1' = \lambda_1 + \rho\frac{d\mathrm{L}}{d\sigma_1} + \mathrm{L}\frac{d\rho}{d\sigma_1} - \rho\frac{pl_1\mathrm{J}_1 - ql_2\mathrm{J}_2}{p^2 - q^2},$$

where $$\frac{d}{d\sigma_1} \equiv \lambda_1\frac{\partial}{\partial x} + \mu_1\frac{\partial}{\partial y} + \nu_1\frac{\partial}{\partial z}.$$

Similarly $$\lambda_2' = \lambda_2 + \rho\frac{d\mathrm{L}}{d\sigma_2} + \mathrm{L}\frac{d\rho}{d\sigma_2} + \rho\frac{ql_1\mathrm{J}_1 - pl_2\mathrm{J}_2}{p_2 - q_2}.$$

The terms in J_1, J_2 can be expressed in terms of K_1, K_2, for it has been proved that
$$\mathrm{J}_1 = p^2\mathrm{K}_1 + q^2\mathrm{K}_2$$
$$\mathrm{J}_2 = q^2\mathrm{K}_1 + p^2\mathrm{K}_2:$$

in short we obtain
$$\frac{pl_1\mathrm{J}_1 - ql_2\mathrm{J}_2}{p^2 - q^2} = \lambda_1\mathrm{K}_1 + pq(\mathrm{K}_1 - \mathrm{K}_2)\lambda_2,$$
$$\frac{ql_1\mathrm{J}_1 - pl_2\mathrm{J}_2}{p^2 - q^2} = pq(\mathrm{K}_1 - \mathrm{K}_2)\lambda_1 - \lambda_2\mathrm{K}_2.$$

Thus $$\lambda_1' = \lambda_1 + \rho\frac{dL}{d\sigma_1} + L\frac{d\rho}{d\sigma_1} - \rho\{\lambda_1 K_1 + pq(K_1 - K_2)\lambda_2\}$$

$$\lambda_2' = \lambda_2 + \rho\frac{dL}{d\sigma_2} + L\frac{d\rho}{d\sigma_2} - \rho\{-pq(K_1 - K_2)\lambda_1 + \lambda_2 K_2\}.$$

It remains to express the condition that the conjugate relation

$$\lambda_1(a\lambda_2 + h\mu_2 + g\nu_2) + \mu_1(h\lambda_2 + b\mu_2 + f\nu_2) + \nu_1(g\lambda_2 + f\mu_2 + c\nu_2) = 0$$

may be satisfied when λ_1' is put for λ_1, and $a + \rho\delta a$ for a, with similar substitutions for the other letters involved. The work is identical with that already carried out in the case of Cayley's equation, except for the part played by the terms involving K_1 and K_2. These terms give

$$\rho pq(K_1 - K_2)K_1 - \rho pq(K_1 - K_2)K_2$$

or $$\rho pq(K_1 - K_2)^2.$$

The condition required is therefore

$$\{\delta a - 2A\}\lambda_1\lambda_2 + .. + .. + \{\delta f - 2F\}(\mu_1\nu_2 + \mu_2\nu_1) + .. + .. + pq(K_1 - K_2)^2 = 0,$$

and since

$$\frac{\lambda_1\lambda_2}{\mathfrak{A}} = \ldots = \ldots = \frac{\mu_1\nu_2 + \mu_2\nu_1}{2\mathfrak{F}} = \ldots = \ldots = \frac{1}{2R^3(K_1 - K_2)},$$

we obtain finally for the condition that the family A may be cut orthogonally by two families B and C which intersect at a constant angle γ

$$\mathfrak{A}\{\delta a - 2A\} + \mathfrak{B}\{\delta b - 2B\} + \mathfrak{C}\{\delta c - 2C\} + 2\mathfrak{F}\{\delta f - 2F\} + 2\mathfrak{G}\{\delta g - 2G\}$$
$$+ 2\mathfrak{H}\{\delta h - 2H\} + 2R^3(K_1 - K_2)^3 pq = 0,$$

where $2pq$ must be put equal to $\cos\gamma$ if B and C follow the η lines on A, and equal to $-\cos\gamma$ if they follow the ϵ lines.

These results can be obtained geometrically.

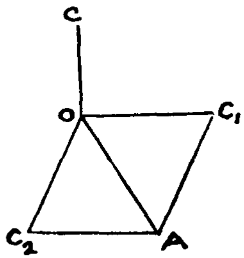

Fig. 7.

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Let OC_1AC_2 be a mesh bounded by lines of curvature on a member of the family A. Then the rotation which will bring the axes $O(C_1C_2C)$ into a position parallel to the corresponding axes at A has components

$$\frac{\theta_{C_21}}{OC_2}\,.\,OC_2 \text{ about } OC_1$$

and $$\frac{\theta_{C_12}}{OC_1}\,.\,OC_1 \text{ about } OC_2,$$

i.e. $$-\frac{1}{R}K_2OAq_1 \text{ about } OC_1{}^*$$

and $$\frac{1}{R}K_1OAp_1 \text{ about } OC_2,$$

where $\cos AOC_1 = p_1$, $\sin AOC_1 = q_1$.

Now if OA is an element of a curve of continuous curvature on the surface and we choose as axes OA, the line at right angles to OA in the tangent plane at O, and the normal OC, the components in the tangent plane of the rotation which will bring the axes at O into a position parallel to those at A are the same as the components that we have just obtained. The components about OC are, however, different in the two cases, unless the curve intersects the lines of curvature at a constant angle.

The component of the rotation about OA is

$$\frac{1}{R}(K_1 - K_2)p_1q_1OA.\dagger$$

Let OA, OB be elements of the lines of intersection of a member of the family A by members of the families B and C, and CE, CD the corresponding elements on the consecutive surface, so

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* For θ_{C_12} = cosine of the angle between OC_1 and the normal at C_1

$$= \Sigma\lambda_1\left\{\frac{L}{R} + d_1\left(\frac{L}{R}\right)\right\} = \Sigma\frac{\lambda_1 d_1 L}{R} = \frac{K_1}{R}d_1s\,;$$

and similarly $\theta_{C_21} = -\frac{K_2}{R}d_2s$.

† M. Fouché's expression for the geodesic torsion is $\frac{1}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\sin 2\phi$.

that each of the angles AOB and ECD is γ; and let θ_C be the component about OC of the rotation which will bring the axes O(A, B, C) into a position parallel to those at C.

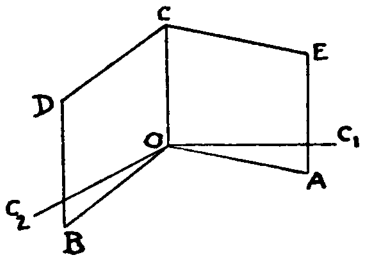

Fig. 8.

Then the condition that CE may meet the normal at A is

$$\theta_C \cdot OA + \frac{1}{R}(K_1 - K_2)p_1q_1 OA \cdot OC = 0,$$

and the condition that CD may meet the normal at B is

$$\theta_C \cdot OB + \frac{1}{R}(K_1 - K_2)p_2q_2 OB \cdot OC = 0.$$

Hence $$p_1q_1 = p_2q_2,$$

or $$\sin 2\theta = \sin 2(\theta + \gamma).$$

$\therefore$ $$\theta = \frac{\pi}{4} - \frac{\gamma}{2} \text{ or } \frac{3\pi}{4} - \frac{\gamma}{2}.$$

Thus θ is independent of the position of the point O on the surface, so that the lines of intersection of the families B and C with any member of the family A cut the lines of curvature at constant angles, and further make equal angles at any point with one or other of the bisectors of the angles between the principal tangents. There are two possible sets of lines of intersection, one of which we have called the η lines, and the other the ϵ lines. For the former $\theta = \frac{\pi}{4} - \frac{\gamma}{2}$ and $2pq = \cos\gamma$, while for the latter $\theta = \frac{3\pi}{4} - \frac{\gamma}{2}$ and $2pq = -\cos\gamma$.

Also since the angle AOC_1 is constant, $\frac{\theta_C}{OC}$ is the rate at which the principal tangents rotate about the normal as O describes an

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orthogonal trajectory of the family A, and is therefore the same as $\frac{\partial\theta}{\partial\nu}$, which Mr Johnson has evaluated, and for which we have found an equivalent expression. On substituting its value, we have as the condition that a family of surfaces A may be cut orthogonally by two families B and C which intersect at a constant angle γ

$$\mathfrak{A}\{\delta a - 2\mathrm{A}\} + \mathfrak{B}\{\delta b - 2\mathrm{B}\} + \mathfrak{C}\{\delta c - 2\mathrm{C}\} + 2\mathfrak{F}\{\delta f - 2\mathrm{F}\} + 2\mathfrak{G}\{\delta g - 2\mathrm{G}\} + 2\mathfrak{H}\{\delta h - 2\mathrm{H}\} + 2\mathrm{R}^3(\mathrm{K}_1 - \mathrm{K}_2)^3 pq = 0,$$

where $2pq = \cos\gamma$ when the families B and C follow the η lines, and $2pq = -\cos\gamma$ when they follow the ϵ lines.