ON COMPLEX HOMOGENEOUS MANIFOLDS

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Compact complex homogeneous manifolds have been studied in great detail by Borel, Goto, Remmert and Wang (cf., (13)): it was shown that every compact, connected complex homogeneous manifold M is a holomorphic fiber bundle over a projective algebraic homogeneous manifold B with a connected, complex parallelizable fiber F. Goto (4) has shown that if M has a compact transformation group, then M is homogeneous projective rational. Aeppli [1] has studied these manifolds using a differential geometric method and has obtained some interesting results. In § 1, we supplement his remarks using some rather elementary and well-known results. In § 2, we prove that there is only one homogeneous complex structure on $S_2 \times S_2$.

1. Let M be a compact complex homogeneous manifold which is simply connected; we may assume that $M = G/H$ where G is a connected Lie group acting effectively on M and H is a closed subgroup.

It is well-known that the Euler-Poincaré characteristic E(M) of M is non-negative; it is strictly positive if and only if H is of maximal rank. We prove the following result which generalizes a result of Wang:

THEOREM 1. Let M be a compact complex homogeneous manifold which is simply connected. If the Euler-Poincaré characteristic E(M) of M is different from zero, then M is a Kähler-Einstein manifold of positive Ricci curvature; moreover, M is projective algebraic.

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Proof. Since M is simply connected and E(M) is different from zero, we may assume that a maximal compact subgroup K of G acts transitively on M by a well-known theorem of Montgomery [10]; since K acts effectively, the center of K reduces to identity and hence K is semi-simple. We may assume that M = K/L where L is a closed subgroup of K ; it is known [8] that L is the centralizer of a torus and M is a Kähler-Einstein manifold[1] [8]. Hence the Ricci curvature of M is either positive, zero or negative; since M = K/L is simply connected and K is compact semi-simple, the Ricci curvature of M is different from zero [9]. M being compact complex homogeneous, it follows that the Ricci curvature of M is necessarily positive (cf., Theorem 6.11.2, p.225, [3]). Consequently M is projective algebraic by a well-known theorem of Kadaira [6a].

We have the following remarks:

COROLLARY 1. Let M be a simply connected, compact complex homogeneous manifold and let M be non-Kählerian; then the Euler-Poincaré characteristic of M vanishes.

Let M be as in Theorem 1; since the Ricci curvature of M is strictly positive, M has no holomorphic p-forms, $0 < p \leq n$. In particular, $h^{2,0}(M) = 0$ and hence every Kähler metric on M is a Hodge[2] metric by a theorem of Kodaira[3] . Consequently this remark and a result of Aeppli (Theorem 3 [1]) imply:

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[1] This can also be proved as follows: M = K/L has an invariant Kähler metric (cf., Theorem 3 [1]); since M = K/L is homogeneous, it is complete and has constant scalar curvature [3]. Consequently, the Ricci 2-form of M is coclosed; but the Ricci 2-form of M is always closed. Thus it is harmonic and hence M is a Kähler-Einstein space.

[2] Recall that a Kähler metric is a Hodge metric if its exterior 2-form represents an integer cohomology class.

[3] Kodaira proved this theorem from any compact Kähler surface but his proof works for any compact Kähler manifold (cf., Theorem [6b]).

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COROLLARY 2. If M is a simply connected compact, complex homogeneous manifold whose Euler-Poincaré characteristic is different from zero, then every invariant hermitian metric on M is a Hodge metric.

Let M be a complex homogeneous manifold with a compact transformation group K and let K act effectively. Suppose that $E(M) \neq 0$; then the center of K is trivial and K is semi-simple; the argument in the proof of Theorem 1 above shows that M is homogeneous Kähler-Einstein; consequently, the Ricci curvature of M is strictly positive and hence M is simply connected [6]. Thus we have (cf., [2]).

THEOREM 2. Let M be a complex homogeneous manifold with a compact effective transformation group; if $E(M) \neq 0$ then M is simply-connected.

REMARK. In fact, it is enough to assume in Theorem 1 (and its corollaries) that the fundamental group $\pi_1(M)$ of M is finite; since M is Kähler-Einstein and has a strictly positive Ricci curvature, it follows that M is simply connected [7].

2. Hirzebruch [5] has shown the existence of an infinity of inequivalent complex structures Σ_n on $V = S_2 \times S_2$ which are algebraic; in fact, these structures are all rational by a classical theorem of Castelnuova-Enriques since these surfaces are all simply-connected and the arithmetic genus p_a and (pluri) 2-genus P_2 vanish. By a result of Kodaira [6b], any complex structure on V is algebraic since $c_1^2 > 0$, where c_1 is the first chern class of V ; it is not known whether it is rational. We prove the following:*

THEOREM 1. Any homogeneous complex structure on $V = S_2 \times S_2$ is isomorphic to the usual complex structure on the complex quadric.

Proof. Let $V = G/H$, where G is a complex Lie group and assume that G acts effectively on V ; since the Euler-Poincaré characteristic of V is different from zero and it is

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simply-connected, there exists a maximal compact subgroup K of G which acts transitively by a well-known theorem of Montgomery [10]. K is semi-simple and $V = K/L$ admits an invariant Kähler-Einstein metric; moreover, L is the connected component of the centralizer B of a 1-parameter subgroup of K. Let $K = K_1 \times \ldots \times K_m$ where each K_i is compact, connected and simple; since L is of maximal rank, we have $L = L_1 \times \ldots \times L_m$ where $L_i \subset K_i$ and $V = \prod_i (K_i/L_i)$.

Since $V = K/L$ is compact homogeneous Kählerian, each factor $V_i = K_i/L_i$ is also compact homogeneous Kählerian [9]. Since V is of complex dimension 2, we have necessarily either $V = K/L$ with K compact simple or $V = V_1 \times V_2$ where each V_i is a complex manifold of complex dimension 1. If $V = K/L$ with K compact simple, then V is hermitian symmetric irreducible [9] and hence its second Betti number will be 1, a contradiction. Thus we have $V = V_1 \times V_2$ and each V_i, being a compact and simply connected complex manifold of dimension 1, is isomorphic to the Riemann sphere S_2. Consequently $V = K/L$ is isomorphic to the complex quadric.

REMARK. In fact, since $V = K/L$ is simply connected with K compact and semi-simple, its Ricci curvature is different from zero [9]; moreover, since V is Kähler-Einstein, and V is compact homogeneous complex, it follows that the Ricci curvature of V is necessarily positive (cf. § 1). Thus $p_g = 0$ and $P_2 = \dim H^2(V, \Omega^0(2K)) = 0$ and hence V is rational by a classical theorem of Castelnuova-Enriques (cf. [4]). Thus it is birationnally equivalent without exceptions to one of the models Σ_n of Hirzebruch by a theorem of Andreotti-Nagata [11].

3. Let M be a complex manifold (not necessarily simply connected) with vanishing second Betti number b_2; if $\dim_c M = 2$, that is M is a compact complex surface with $b_2 = 0$, [1] then

[1] Aeppli has given (cf. p.67 [1]) examples of non-Kähler complex manifolds with $b_2 \neq 0$ and having a non-zero Euler-Poincaré characteristic; these manifolds are not simply connected (cf. Theorem 1).

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the Euler-Poincaré characteristic of M vanishes (cf. Lemma [10]). If M is a compact complex homogeneous, simply connected (cf., Theorem 1), manifold with $b_2 = 0$, then $E(M) = 0$. It will be very interesting to know if this is true for an arbitrary compact complex manifold with $b_2 = 0$.

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