On Deducing the Properties of the Trigonometrical Functions from their Addition Equations.

By R. F. Muirhead, M.A., B.Sc.

1. Take first the Addition-Formula of the Tangent:

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y} \quad - \quad - \quad - \quad \text{I.}$$

Take $y=0$, $\quad \therefore \tan x=\dfrac{\tan x+\tan 0}{1-\tan x\tan 0}$ - - - (1)

Assume that $1-\tan x\tan 0$ is not zero for all value of x - - II.

and we have $\quad \tan x(1-\tan x\tan 0)=\tan x+\tan 0$

$$\therefore \tan 0(1+\tan^2 x)=0 \quad - \quad - \quad - \quad (2)$$

Assume that there is *one* value at least for x for which

$$\tan^2 x \neq -1 \quad \text{and} \quad 1-\tan x\tan 0 \neq 0 \quad - \quad - \quad \text{III.}$$

and we have $\quad \tan 0=0$ - - - - - - (3)

Hence $\quad 0=\tan(x-x)=\dfrac{\tan x+\tan(-x)}{1+\tan^2 x}$

$$\therefore \tan x=\tan(-x) \quad - \quad - \quad - \quad (4)$$

hence, writing $-y$ for y in I.

$$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y} \quad - \quad - \quad - \quad (5)$$

Again $\quad \tan(x+h)-\tan x=\tan h(1+\tan^2 x)\div(1-\tan x\tan h)$ - (6)

By taking h small enough, the denominator may be made positive, so long as $\tan x$ is finite. Hence $\tan(x+h)>\tan x$ and as x increases from 0, $\tan x$ increases at an increasing rate, and must eventually become $=1$

Let $\dfrac{R}{2}$ be the value of x which makes $\tan\dfrac{R}{2}=1$ - - - IV.

Then $\quad \tan R=\tan\left(\dfrac{R}{2}+\dfrac{R}{2}\right)=\dfrac{1+1}{1-1}=\infty$

$$\therefore \tan 2R=\frac{2\tan R}{1-\tan^2 R}=2\div\left(\frac{1}{\tan R}-\tan R\right)=0$$

Hence $\quad \tan(2R+x)=\dfrac{0+\tan x}{1+0}=\tan x$ - - - (7)

Thus $\tan x$ is periodic, and the period is $2R$.

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Again $$\tan(2R - x) = \tan(-x) = -\tan x.$$

And $$\tan(R - x) = \frac{\tan R - \tan x}{1 + \tan R \tan x} = \frac{1}{\tan x}$$

Hence the graph of $\tan x$ might be roughly sketched.

Again $$1 = \tan\frac{R}{2} = \tan\left(\frac{R}{4} + \frac{R}{4}\right) = \frac{2\tan\frac{R}{4}}{1 - \tan^2\frac{R}{4}}$$

Hence $$\tan\frac{R}{4} = \pm\sqrt{2} - 1 \quad \therefore \tan\frac{R}{4} = \sqrt{2} - 1$$

In this manner we can calculate $\tan\frac{R}{2^n}$ where n is any integer.

Hence by the Addition Formula we can find $\tan\frac{mR}{2^n}$ where m is any integer, and in this way we can approximate to the value of the tangent of any angle between 0 and R; and then, using (4) and (7) to that of any angle whatever.

R depends on the unit of angular measurement; or *vice versa*, if R be arbitrarily chosen, the unit of measurement depends on it.

If we take the limit when $x = 0$ of $\tan x/x$ to be $= 1$ as in radian measure, we get from (6)

$$\frac{d\tan x}{dx} = 1 + \tan^2 x, \text{ or } \frac{d\tan^{-1}x}{dx} = \frac{1}{1+x^2}$$

whence $$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5}$$

$$\therefore \frac{R}{2} = \tan^{-1}1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{ etc.}$$

2. It will appear that the Addition Formulae of the sine and cosine are not sufficient to determine all their properties, or in other words, that the trigonometrical functions are particular cases of more general functions having the same Addition Equations. I have therefore, in what follows, extended the scope of the paper so far as to determine what these more general functions are.

Let $\phi(x)$ and $\psi(x)$ be two functions of x such that

$$\phi(x+y)=\phi(x)\psi(y)+\psi(x)\phi(y) \quad - \quad - \quad - \quad \text{I.}$$

$$\psi(x+y)=\psi(x)\psi(y)-\phi(x)\phi(y) \quad - \quad - \quad - \quad \text{II.}$$

so that ϕ and ψ are functions having the same Addition-Formulae as the sine and the cosine respectively; we shall seek to deduce from I. and II. the nature of ϕ and ψ, pointing out as we proceed, any further assumptions that we make as to the nature of ϕ and ψ.

In I. put $y=0$ $\therefore$ $\phi(x)=\phi(x)\psi(0)+\psi(x)\phi(0)$

$$\therefore \phi x(1-\psi(0))=\psi(x)\phi(0) \quad - \quad - \quad (1)$$

Similarly from II. $\psi(x)(1-\psi(0))=-\phi(x)\phi(0)$ - - - (2)

Multiplying across and transposing,

$$[\{\psi(x)\}^2+\{\phi(x)\}^2\{1-\psi(0)\}\phi(0)=0 \quad - \quad - \quad (3)$$

Assume that $(\phi x)^2+(\psi x)^2$ is not $=0$ for all values of x - - III.

This is certainly true if $\phi(x)$ and $\psi(x)$ are real and not both always $=0$

Hence either $\phi(0)=0$ or $1-\psi(0)=0$

Now if $\phi(0)=0$ then by (1) $\psi(0)=1$ unless $\phi(x)=0$ for all values of x or $\psi(x)=\infty$ for all values of x.

And by (2) $\psi(0)=1$ unless $\psi(x)=0$ for all values of x or $\phi(x)=\infty$ for all values of x.

On the whole then, the alternative $\phi(0)=0$ involves also that $\psi(0)=1$ if we assume:

There is some value of x for which ϕ or ψ is neither ∞ nor 0. - - - - - - - - - - IV.

And it is clear that, on this assumption, the equation $\phi(0)=0$ can be deduced from the other alternative $\psi(0)=1$. Hence finally

$$\left.\begin{array}{l}\phi(0)=0\\ \psi(0)=1\end{array}\right\} \quad - \quad - \quad - \quad - \quad (4)$$

Put $y=-x$ in I. and II. and we have

$$0=\phi(0)=\phi(x-x)=\phi(x)\psi(-x)+\psi(x)\phi(-x) \quad - \quad (5)$$

$$1=\psi(0)=\psi(x-x)=\psi(x)\psi(-x)-\phi(x)\phi(-x) \quad - \quad (6)$$

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Eliminating successively $\psi(-x)$ and $\phi(-x)$ we have

$$\phi(x)=\{(\phi x)^2+(\psi x)^2\}\phi(-x) \quad - \quad - \quad - \quad (7)$$

$$\psi(x)=\{(\phi x)^2+(\psi x)^2\}\psi(-x) \quad - \quad - \quad - \quad (8)$$

Assume $$(\psi x)^2+(\phi x)^2\equiv 1 \quad - \quad - \quad - \quad - \quad \text{V.}$$

(Note that this includes Assumption III.)

Hence by (7) and (8), $$\phi(x)=-\phi(-x) \quad - \quad - \quad - \quad - \quad (9)$$

$$\psi(x)=\quad\psi(-x) \quad - \quad - \quad - \quad - \quad (10)$$

Hence from I. and II., by putting $-y$ for y, we have the Subtraction Formulae:

$$\left.\begin{aligned}\phi(x-y)&=\phi x\psi y-\psi x\phi y\\ \psi(x-y)&=\psi x\psi y+\phi x\phi y\end{aligned}\right\} \quad - \quad - \quad - \quad (11)$$

It is clear that at this stage V., (9), (10), and (11) are all equivalent, so that any one of them being assumed, the others would follow.

As in books on Elementary Trigonometry we can now deduce from I., II., and (11) all the formulae for functions of multiples or submultiples of x, and such formulae as

$$\phi(x)+\phi(y)=2\phi\left(\frac{x+y}{2}\right)\psi\left(\frac{x-y}{2}\right).$$

Assume now that $\phi(x)$ is not always $=0$ and is real and continuous, and that it becomes positive* at first as x increases from 0 - - - - - - - - - - - VI.

and consider the identities

$$\left.\begin{aligned}\phi(x+h)-\phi(x)&=\quad 2\phi\left(\frac{h}{2}\right)\psi\left(x+\frac{h}{2}\right)\\ \psi(x+h)-\psi(x)&=-2\phi\left(\frac{h}{2}\right)\phi\left(x+\frac{h}{2}\right)\end{aligned}\right\} \quad (12)$$

We see that as x increases from 0, so long as ψ is positive, ϕ is increasing, and ψ is decreasing at an accelerating rate.
Hence there must be a value of x for which ψ first becomes $=0$.
Let that value be R.

Then $\psi(\text{R})=0$. Hence by V., $\phi(\text{R})=1$ - - - - (13)

* The contrary supposition, that it is negative at first, would lead to similar results.

$$
\left.\begin{aligned}
\text{Then } \psi(R-x) &= \psi R\psi x + \phi R\phi x\\
&= \phi x\\
\text{Similarly } \phi(R-x) &= \psi x\\
\text{,, } \phi(2R-x) &= \phi(x)\\
\text{,, } \psi(2R-x) &= -\psi(x)\\
\text{,, } 0 &= \phi(2R) = \phi(4R) = \phi(6R) =\\
\text{,, } 0 &= \psi(R) = \psi(3R) = \psi(5R) =\\
\text{,, } 1 &= -\phi(3R) = \phi(5R) = -\phi(7R) =\\
\text{,, } 1 &= -\psi(2R) = \psi(4R) = -\psi(6R) =
\end{aligned}\right\} \quad (14)
$$

Hence $\phi(4R+x) = \phi x$ and $\psi(4R+x) = \psi x$ - - - - (15)

Thus ϕ and ψ are both periodic functions, the period being $=4R$, and it is clear that if the values of ϕ from $x=0$ to $x=R$ were tabulated, we could at once find the value of ϕ or ψ for any other x by means of (14) and (15).

We are in a position to trace the ϕ and ψ curves roughly, and they are obviously similar to $\sin x$ and $\cos x$, the unit angle being $\frac{1}{R}$ of a right angle.

The actual values of ϕ and ψ for as many values of x as we please can easily be calculated, *e.g.*,

$$\frac{R}{2} = R - \frac{R}{2} \quad \therefore \text{ by (14) } \phi\left(\frac{R}{2}\right) = \psi\left(\frac{R}{2}\right)$$

Hence by V. $\psi\left(\frac{R}{2}\right) = \pm\frac{1}{\sqrt{2}}$, and we must take the upper sign since ψ is positive till $x = R$.

Hence also $\phi\left(\frac{R}{2}\right) = \frac{1}{\sqrt{2}}$ - - - - - - - - (16)

By repeated application of the formula

$$\psi\left(\frac{x}{2}\right) = \sqrt{\left(\frac{1+\psi x}{2}\right)}$$

we can get the value of $\psi(x)$ for as small a value of x as we please: thence finding $\phi(x)$ for the same value of x by means of V., we could, by the Addition Formulae, interpolate as many values as we pleased of ϕ and ψ between the values found, of which (16) is a specimen.

Thus we have shown that the two functions defined by I., II., III., IV., V., VI. are one-valued.

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Since the sine and cosine satisfy these assumptions, ϕ and ψ must be identical with them.

If we suppose the sine and cosine defined geometrically, then to prove their identity with the functions ϕ and ψ as above conditioned, it would only be necessary to prove geometrically the Addition Equations and the Subtraction Equations and verify the assumptions III., IV., VI.

R is arbitrary, and its choice determines the unit of angular measurement. To connect R with the limiting value of $\phi(x)/x$ we might use the formula

$$1=\phi(\mathrm{R})=2^n\psi\left(\frac{\mathrm{R}}{2}\right)\psi\left(\frac{\mathrm{R}}{2^2}\right)\ \ldots\ \psi\left(\frac{\mathrm{R}}{2^n}\right)\phi\left(\frac{\mathrm{R}}{2^n}\right)$$

$$\therefore\ 1=\sqrt{\frac{1+0}{2}}\cdot\sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}}\cdot\sqrt{\frac{1+\sqrt{\left(\frac{1}{2}+\frac{1}{2\sqrt{2}}\right)}}{2}}\ \ldots\ \times\mathrm{R}\frac{\phi\left(\frac{\mathrm{R}}{2^n}\right)}{\frac{\mathrm{R}}{2^n}}$$

$\therefore\ \frac{1}{\mathrm{R}}=$ a certain number $\times k$, where k is the limiting value when $x=0$ of $\phi(x)/x$. The number referred to is of course $\frac{2}{\pi}$. Instead of choosing R arbitrarily we might choose k, and so fix R.

3. In order to prove that the assumptions III., IV., V., VI. are all *necessary* we should have to show that if any one is omitted, the functions ϕ and ψ may be different from the sine and cosine.

The assumptions III., IV. are obviously necessary, and the generality gained by dropping them is not of interest. It is otherwise with assumption V., viz.: $(\phi x)^2+(\psi x)^2\equiv 1$. Suppose this assumption dropped, so that ϕ and ψ are now conditioned only by I., II., III., IV., VI., and let ϕ_1 and ψ_1 be thus defined:

$$\left.\begin{aligned}\phi_1(x)&\equiv\phi(x)\div\left[\left\{\phi\left(\frac{x}{2}\right)\right\}^2+\left\{\psi\left(\frac{x}{2}\right)\right\}^2\right]\\ \psi_1(x)&\equiv\psi(x)\div\left[\left\{\phi\left(\frac{x}{2}\right)\right\}^2+\left\{\psi\left(\frac{x}{2}\right)\right\}^2\right]\end{aligned}\right\}\quad - \quad - \quad (17)$$

We can easily verify that ϕ_1 and ψ_1 satisfy I., II., III., IV., VI.

Again $\{\phi_1(2x)\}^2+\{\psi_1(2x)\}^2=\{\phi(2x)^2+\psi(2x)^2\}\div\{\phi(x)^2+\psi(x)^2\}^2$

$$=[\{2\phi x.\psi x\}^2+\{(\phi x)^2-(\psi x)^2\}^2]\div\{\phi(x)^2+\psi(x)^2\}^2$$

$$=\ 1$$

Thus the functions ϕ_1 and ψ_1 satisfy the condition $\phi_1^2+\psi_1^2=1$, and are therefore, as we saw, identical with the sine and cosine of x.

To find the nature of the more general functions ϕ and ψ let us put $\qquad \phi(x)\equiv f(x)\phi_1(x)$

then by (17) we get $\qquad \psi(x)=f(x)\psi_1(x)$

Substituting in the Addition Formulae we get

$$f(x+y)=f(x)f(y)$$

Hence, as in the proof of the Exponential Theorem,

$$f(x)\equiv a^x, \text{ where } a \text{ is an arbitrary constant.}$$

Thus the most general functions conditioned by I., II., III., IV., VI. are $a^x\sin x$ and $a^x\cos x$. The special case $a=0$ giving $\sin x$ and $\cos x$ is got by introducing either V. or any of its equivalents.

If we drop Condition VI. as well as V., *i.e.*, if we admit imaginaries, the present mode of treatment becomes inconvenient.

In VI. the assumption of continuity is required by the occurrence of $\phi\left(x+\frac{h}{2}\right)$ and $\psi\left(x+\frac{h}{2}\right)$ on the right hand sides of equations (12). By writing (12) in the form

$$\phi(x+h)-\phi(x)=2\phi\left(\frac{h}{2}\right)\left\{\psi(x)\psi\left(\frac{h}{2}\right)-\phi(x)\phi\left(\frac{h}{2}\right)\right\} \text{ etc.,}$$

we see that if $\phi\left(\frac{h}{2}\right)\div\frac{h}{2}$ has a finite limit, the functions are continuous, *i.e.*, if $\phi(x)$ *begins* by being continuous, it must remain so, and ψ also.

4. The Addition Equations I. and II. can also be discussed as follows:

Take
$$\left.\begin{aligned} f(x)&\equiv i\phi(x)+\psi(x)\\ g(x)&\equiv -i\phi(x)+\psi(x)\end{aligned}\right\} \qquad (18)$$

$$\begin{aligned}\therefore f(x+y)&= i(\phi x\psi y+\psi x\phi y)+\psi x\psi y-\phi x\phi y\\ &=(\psi x+i\phi x)\,(\psi y+i\phi y)\\ &=f(x)f(y) \qquad (19)\end{aligned}$$

Similarly $g(x+y)=g(x)\,.\,g(y) \qquad (20)$

Hence $f(x)\equiv a^x$ and $g(x)\equiv b^x$, where a and b are arbitary constants.

Hence

$$\left.\begin{aligned}\phi(x) &\equiv \frac{1}{2i}(a^x - b^x)\\ \psi(x) &\equiv \frac{1}{2}(a^x + b^x)\end{aligned}\right\} \qquad (21)$$

There is no restriction as to the values of a and b, as we can verify by substituting in I. and II.

But if ϕ and ψ are to be *real* functions of x, since by (21) $a^x = \psi + \phi i$, $b^x = \psi - \phi i$, it follows that a and b must be complex quantities having the same modulus, say A, and equal and opposite amplitudes, $\pm$B.

Hence $\qquad \psi(x) = A^x \cos Bx, \;\; \phi(x) = A^x \sin Bx \qquad (22)$

Although this result agrees with the general result obtained by the previous method, it is to be noted that we have not proved exactly the same thing. Here we have identified $A^x \cos Bx$ and $A^x \sin Bx$ *as defined analytically* with the functions ϕ and ψ. In the previous method the analytical expressions for sine and cosine were not assumed.

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