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The limits of time stepping when used to solve the long wave equations numerically

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The two approaches (the hydrodynamical-numerical and the harmonic methods) to the solution of the long wave equations are compared, enabling an analysis of the limits imposed by time stepping. The minimum time scale of motions, which may be modelled by the time stepping scheme, is determined from the stability condition derived for the harmonic method.

1. Introduction

The problems of tidal dynamics may be solved numerically by at least two methods. These use finite-difference methods to describe the dependence of the variables on position. The first method, proposed by Hansen [2], uses a suitably stable finite-difference approximation for the time derivative and the dependent fields are stepped forward in time. This method is often described as the *HN*-method. The transformation of the dependent variables from a function of time to a function of frequency provides the basis of an alternative method usually referred to as the harmonic method.

Since the spatial dependence may be treated in the same way for these two methods, a detailed comparison provides an insight into the effect of the discretization of the time dependence. Transformation of the stability condition from the time to the frequency domain and vice versa provides "sensible" lower and upper limits on the time step and frequency respectively.

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The results, expressed in matrix notation, are illustrated by a numerical example.

2. Comparison of *HN* and harmonic methods

The linearized long wave equations integrated over depth may be written as

$$\frac{\partial U}{\partial t} - fV = -gh\,\frac{\partial \zeta}{\partial x} + \frac{1}{\rho}\left(F_s - F_B\right) \,, \tag{2.1}$$

$$\frac{\partial V}{\partial t} + fU = -gh\,\frac{\partial \zeta}{\partial y} + \frac{1}{\rho}\left(G_s - G_B\right) \,, \tag{2.2}$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial \zeta}{\partial t} = 0 \,, \tag{2.3}$$

using the following notation:

t	the time;
ζ	the elevation of the sea surface;
U, V	the components of the total stream;
F_s, G_s	the components of surface stress;
F_B, G_B	the components of bottom stress;
h	the undisturbed depth of the water;
f	the coriolis parameter;
g	the acceleration of the earth's gravity.

For this analysis, the bottom stress is assumed to take the form

$$\left(F_B,\, G_B\right) = \frac{\rho r}{h}\,(U,\, V)$$

where r is the linear coefficient of friction.

The boundary conditions are that the normal velocity on a closed boundary (the shoreline) is zero, and the elevation is given at an open boundary (an artificial boundary dividing the region of interest from the open sea).

By assuming values for U, V and ζ at $t = 0$, the problem becomes an initial value problem. It has been shown that the solution after a time, T , is independent of the initial conditions if frictional dissipation is included. The value of T depends on the rate of dissipation.

The equations may be expressed in a discrete form, so that the U, V and ζ fields may be stepped forward in time. The system is either explicit or implicit, depending on the form of finite-difference used. An example of an explicit scheme as given by Heaps [3] is

$$U_i(t+\tau) = U_i(t)\left(1 - \frac{r\tau}{h_i}\right) + \tau f V_i(t) - \tau g h_i D_i(t) + \frac{\tau}{\rho} F_{si}(t) ,$$

$$V_i(t+\tau) = -\tau f U_i(t) + \left(1 - \frac{r\tau}{h_i}\right) V_i(t) - \tau g h_i E_i(t) + \frac{\tau}{\rho} G_{si}(t) ,$$

$$\zeta_i(t+\tau) = \zeta(t) - \tau A_i(t+\tau) - \tau B_i(t+\tau) ,$$

for $i = 1, \ldots, n$, where τ is the time step, and A, B, C and D are the finite-difference forms of the derivatives $\frac{\partial U}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial \zeta}{\partial x}$ and $\frac{\partial \zeta}{\partial y}$ respectively. These equations written in matrix notation are

$$\mathsf{U}(t+\tau) = (I-\tau K)\mathsf{U}(t) + \tau F\mathsf{V}(t) - \tau g H D \boldsymbol{\zeta}(t) + \frac{\tau}{\rho} \mathsf{F}_s(t) ,$$

$$\mathsf{V}(t+\tau) = -\tau F\mathsf{U}(t) + (I-\tau K)\mathsf{V}(t) - \tau g H E \boldsymbol{\zeta}(t) + \frac{\tau}{\rho} \mathsf{G}_s(t) ,$$

$$\boldsymbol{\zeta}(t+\tau) = \boldsymbol{\zeta}(t) - \tau A\mathsf{U}(t+\tau) - \tau B\mathsf{V}(t+\tau) ,$$

where it is implied that

$$\left(A\mathsf{U}(t+\tau)\right)_i = A_i(t+\tau) .$$

The matrices H, F and K are diagonal matrices such that $H_{ii} = h_i$, $F_{ii} = f$ and $K_{ii} = r/h_i$. All non-active elements should be removed from the vectors U, V and $\boldsymbol{\zeta}$, so that the matrices are irreducible. An active grid point is defined as a point on which the dependent variable is not defined by a boundary condition and is within the region of interest. The terms due to outer boundary forcing can be absorbed into the surface forcing terms. The matrices H and K remain diagonal although there are now two forms of each. Nothing can be said about the structure of the two forms of F , except that they are not likely to be diagonal. The final form of the equation is

$$\mathsf{z}(t+\tau) = G\mathsf{z}(t) + \mathsf{w}(t) , \qquad (2.4)$$

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where

$$\mathsf{z}(t)^T = \left[\mathsf{U}(t)^T,\ \mathsf{V}(t)^T,\ \zeta(t)^T\right] ,$$

$$\mathsf{w}(t)^T = \frac{\tau}{\rho}\left[\mathsf{F}_s(t)^T,\ \mathsf{G}_s(t)^T,\ -\tau\left(A\mathsf{F}_s(t)+B\mathsf{G}_s(t)\right)^T\right] ,$$

and

$$(2.5)\qquad G = \begin{bmatrix} I-\tau K_1 & \tau F_1 & -\tau g H_1 D \\ -\tau F_2 & I-\tau K_2 & -\tau g H_2 E \\ -\tau\left(A-\tau AK_1-\tau BF_2\right) & -\tau\left(\tau AF_1+B-\tau BK_2\right) & I+\tau^2 g\left(AH_1D+BH_2E\right) \end{bmatrix}$$

The equations which form the basis of the harmonic method are the Fourier transforms of (2.1), (2.2) and (2.3). The resulting equations are

$$\hat{U}\left(i\omega + \frac{r}{h}\right) - U(x,\ 0) - f\hat{V} = -gh\,\frac{\partial\hat{\zeta}}{\partial x} + \frac{1}{\rho}\,\hat{F}_s ,$$

$$\hat{V}\left(i\omega + \frac{r}{h}\right) - V(x,\ 0) + f\hat{U} = -gh\,\frac{\partial\hat{\zeta}}{\partial y} + \frac{1}{\rho}\,\hat{G}_s ,$$

$$i\omega\hat{\zeta} - \zeta(x,\ 0) + \frac{\partial\hat{U}}{\partial x} + \frac{\partial\hat{V}}{\partial y} = 0 ,$$

where $\hat{}$ represents the Fourier transform. The discrete form of these equations may be written as

$$(2.6)\qquad Q\hat{\mathsf{z}} = \hat{\mathsf{s}} ,$$

where

$$\hat{\mathsf{s}}^T = \left[\frac{1}{\rho}\,\hat{\mathsf{F}}_s^T+\mathsf{U}(0)^T,\ \frac{1}{\rho}\,\hat{\mathsf{G}}_s^T+\mathsf{V}(0)^T,\ \zeta(0)^T\right]$$

and

$$(2.7)\qquad Q = \begin{bmatrix} i\omega I+K_1 & -F_1 & gH_1D \\ F_2 & i\omega I+K_2 & gH_2E \\ A & B & i\omega I \end{bmatrix}$$

Setting $F_s(t)$ and $G_s(t)$ to be periodic, with period $N\tau$, causes the field $\mathsf{z}(t)$ to oscillate with the same frequency after a time, T, enabling a comparison between the methods, since

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$$\sum_{j=1}^{N} e^{-i2\pi j/N}(I-\kappa G)\mathrm{z}(t+j\tau) = \kappa \sum_{j=1}^{N} e^{-i2\pi j/N}\mathrm{w}(t+j\tau) ,$$

where $\kappa = e^{-i2\pi/N}$. For $\omega = 2\pi/N\tau$,

$$\frac{(I-\kappa G)}{\tau\kappa} = \begin{bmatrix} I\kappa^{-1}\left(i\omega + \frac{\omega^2\tau}{2}\right) + K_1 & -F_1 & gH_1D \\ F_2 & I\kappa^{-1}\left(i\omega + \frac{\omega^2\tau}{2}\right) + K_2 & gH_2E \\ A-\tau(AK_1+BF_2) & B-\tau(BK_2-AF_1) & I\kappa^{-1}\left(i\omega + \frac{\omega^2\tau}{2}\right) \\ & & -\tau g(AH_1D+BH_2E) \end{bmatrix}$$

to the order ω^2 . The HN analogues of the fields $\hat{\mathrm{z}}$ and $\hat{\mathrm{s}}$ are, respectively,

$$\overline{\mathrm{z}} = \sum_{j=1}^{N} \frac{e^{-i2\pi j/N}}{N} \mathrm{z}(t+j\tau) , \tag{2.8}$$

and

$$\mathrm{s} + \tau\mathrm{b} = \frac{1}{\tau} \sum_{j=1}^{N} e^{-i2\pi j/N}\mathrm{w}(t+j\tau) , \tag{2.9}$$

where

$$\mathrm{b}^T = \left[0,\ 0,\ -(A\overline{\mathrm{F}}_s+B\overline{\mathrm{G}}_s)^T\right] .$$

Equation (2.4) may now be written as

$$(Q+\tau R)\overline{z} = s + \tau b , \tag{2.10}$$

where

$$R = \begin{bmatrix} -\omega^2 I & 0 & 0 \\ 0 & -\omega^2 I & 0 \\ -(AK_1+BF_2) & -(BK_2-AF_1) & -\omega^2 I-g(AH_1D+BH_2E) \end{bmatrix} . \tag{2.11}$$

Writing $\overline{\mathrm{z}} = \hat{\mathrm{z}} + \mathrm{h}_1 + \mathrm{h}_2$, equation (2.10) may be divided into

$$(Q+\tau R)(\hat{\mathrm{z}}+\mathrm{h}_1) = \mathrm{s} , \tag{2.12}$$

and

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$$(Q+\tau R)\mathbf{h}_2 = \tau \mathbf{b} \ . \tag{2.13}$$

The HN and harmonic formulations are equivalent for $\tau \to 0$, as

$$\frac{\|\mathbf{h}_1\|}{\|\mathbf{2}\|} \le \frac{\tau\|Q\|\|Q^{-1}\|\|R\|/\|Q\|}{1-\tau\|Q\|\|Q^{-1}\|\|R\|/\|Q\|} ,$$

and

$$\|\mathbf{h}_2\| \le \frac{\tau\|Q\|\|Q^{-1}\|\|b\|/\|Q\|}{1-\tau\|Q\|\|Q^{-1}\|\|R\|/\|Q\|} ,$$

which tend to zero as $\tau \to 0$, for Q not singular.

3. Stability

The stability condition for the time stepping scheme may be written as

$$|\lambda_k| \le 1 \tag{3.1}$$

where the λ_k's are the eigenvalues of G . Consider the matrices G_1 and G_2 , where

$$G = G_1 + O(\tau^2) , \tag{3.2}$$

and

$$G_2 = (G_1 - I)/\tau \ .$$

Platzman [4] has shown that, for a Richardson grid, a particular structure of F_1 and F_2 , and zero friction, the matrix G_2 has pure imaginary eigenvalues $i\beta_k$. The inclusion of friction would produce eigenvalues of the form

$$-\alpha_k + i\gamma_k ,$$

with

$$\alpha_k \ge 0 \quad \text{and} \quad |\gamma_k| \le |\beta_k| \ .$$

Hence the eigenvalues of G_1, $\lambda_k^{G_1}$, could be written as

$$\lambda_k^{G_1} = \left(1-\tau\alpha_k\right) + i\tau\gamma_k .$$

The approximation for λ_k gives

$$\left|\lambda_k^{G_1}\right|^2 = \left|1-2\tau\alpha_k+\tau^2\alpha_k^2+\tau^2\gamma_k^2\right| , \tag{3.3}$$

and for zero friction $\left(\alpha_k = 0\right)$ (3.3) becomes

$$\left|\lambda_k^{G_1}\right| \geq 1 . \tag{3.4}$$

Since the time stepping scheme is stable, the elements in G of the order of τ^2 must be sufficient to reduce $|\lambda_k|$ below 1 .

The more usual stability condition, as stated by Fischer [1] is

$$\tau \leq \frac{c_1\delta}{\max(gh)^{\frac{1}{2}}} , \tag{3.5}$$

and

$$\tau \leq \frac{r}{f^2\max(h)} , \quad f \neq 0 , \tag{3.6}$$

where δ is the minimum distance between adjacent grid points and c_1 is a constant; usually $c_1 = 1, 2$, or $\sqrt{2}$, depending on the grid scheme. Unlike condition (3.1), no account has been taken of the boundary.

In the transform method, condition (3.5) becomes

$$\omega \geq \frac{2\pi\max(gh)^{\frac{1}{2}}}{c_1N\delta} > 0 , \tag{3.7}$$

as there is no restriction on the maximum value of N .

Providing Q is not singular, equation (2.6) may be solved for all values of ω . For large values of ω the solution has no physical significance and an upper limit needs to be determined. Equation (2.6) may be written as

$$\left(i\omega I-G_2\right)\hat{z} = \hat{s} .$$

If H is formed by the eigenvectors of G_2 , then

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$$\left(i\omega I - H^{-1}G_2 H\right)H^{-1}\hat{z} = H^{-1}\hat{s} \ ,$$

giving

$$\hat{z} = -iH\left[\operatorname{diag}\left(\frac{1}{\omega-\beta_k}\right)\right]H^{-1}\hat{s} \ .$$

To avoid aliasing, the condition

$$\omega \le \max\left(|\beta_k|\right) \tag{3.8}$$

must be satisfied. This condition may be developed intuitively. At least two grid points are needed to define a wave with a maximum frequency of

$$\omega \le \frac{\pi \min(gh)^{\frac{1}{2}}}{c_1\delta} \simeq \max\left(|\beta_k|\right) \ . \tag{3.9}$$

In the time stepping regime, this becomes

$$\tau_{\mathrm{MTS}} \ge \frac{2\pi}{N\max\left(|\beta_k|\right)} \ , \tag{3.12}$$

which is not a stability condition in the sense of equation (3.5), but a measure of the minimum time scale (MTS) which is consistent with the spatial discretization.

Let p be the fraction of the total number of time steps which are sufficient to completely describe the HN solution. The ratio of (3.5) to (3.10) with $N = 2$ provides an approximate estimate of p ,

$$p = \frac{\tau_{\mathrm{STAB}}}{\tau_{\mathrm{MTS}}} \simeq \left(\frac{\min h}{\max h}\right)^{\frac{1}{2}} \ .$$

It is assumed that the time step used would be τ_{STAB} . A smaller value of τ would reduce p but provide no additional information.

4. Numerical example

To measure the errors arising from the time discretization, it is necessary to disentangle the effects of space discretization. To provide a basis for comparison an analytic solution is required. Accordingly, a simple example has been chosen and the analytic, HN , and harmonic solutions found.

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The model used was a rectangular basin (206 Km long) of constant depth (40 m), open at one end. The open boundary was assumed to oscillate with an amplitude of 0.1 m and a period of 12 hours. Figure 1 provides a comparison of the three solutions for the amplitude of the surface elevation oscillations as a function of position.

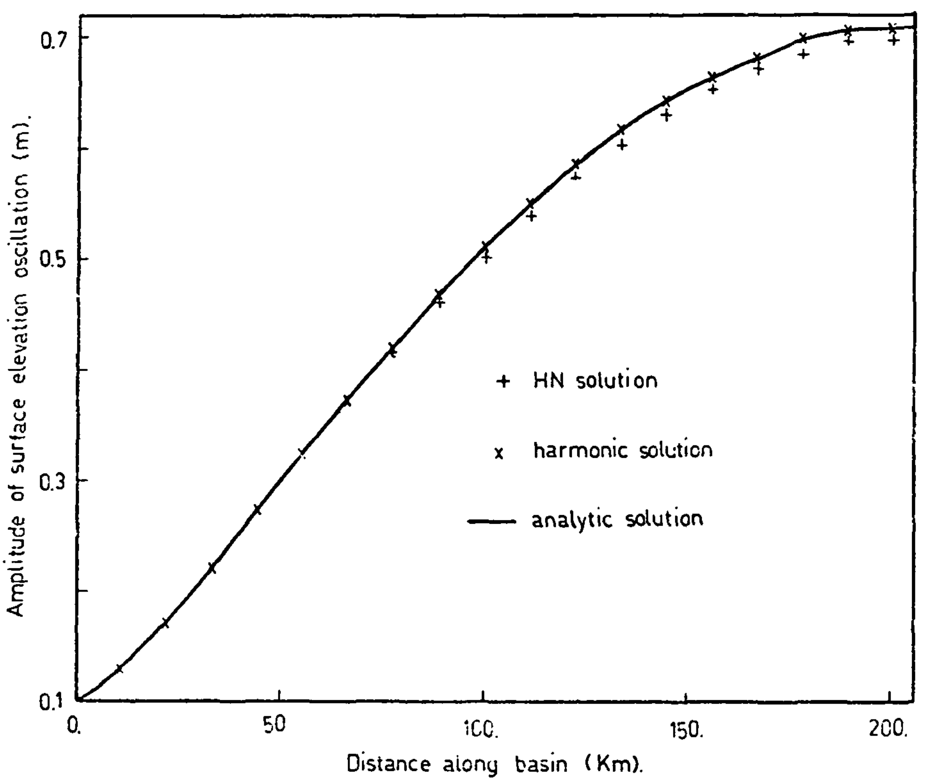

Figure 1. The amplitude of the surface elevation oscillations given by the *HN*, harmonic, and analytic methods for a simple example.

A similar comparison for the phases of the transport and elevation are shown in Figure 2 (page 330). The maximum value of τ given by equation (3.5) is 9.3 minutes.

No significant error is apparent due to spatial discretization, as the harmonic and analytic solutions are equivalent. The error in the *HN* solution for surface elevation increases away from the open end, while the error in the phase of the transport is approximately constant along the

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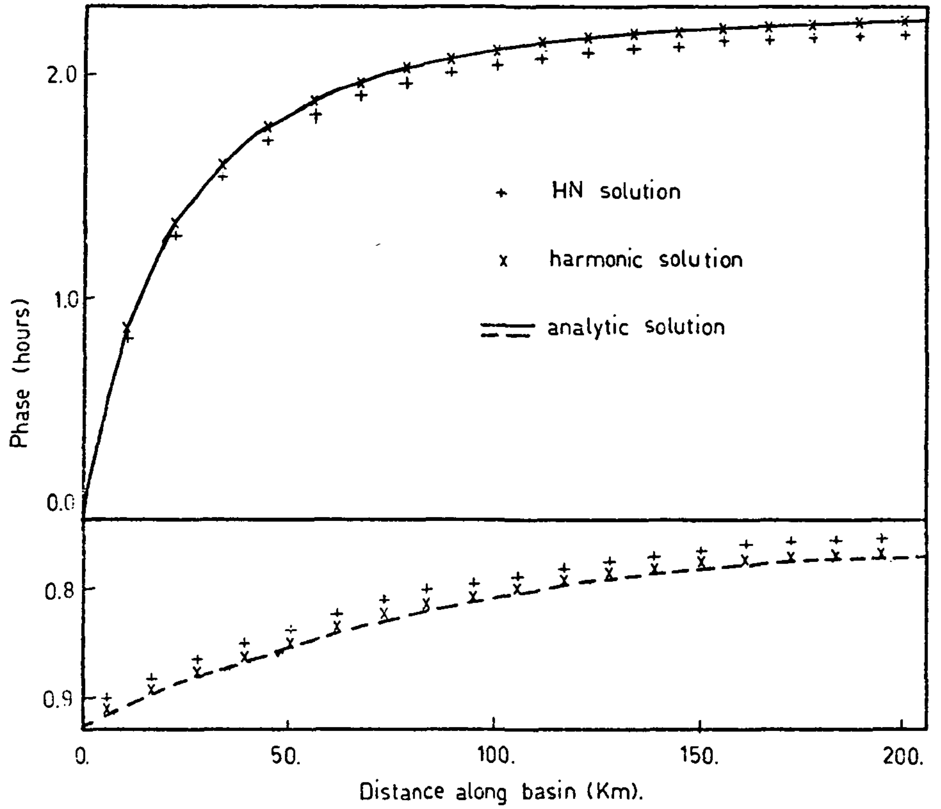

Figure 2. The analytic solution for the phase of the surface elevation oscillations (——) and the phase of the transport (---) compared with the respective solutions given by the *HN* and harmonic methods.

basin. The phases of the surface elevation and transport, found by time stepping, lag the analytic solution.

From numerical experiments it seems that the effect of violating condition (3.9) is to give a solution which decays very rapidly away from the open boundary.

5. Conclusions

The analysis shows that the error in the numerical solution using the *HN* method arises from the need to ensure stability, as well as from the

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discretization of the time. For the example considered, the error arising from the use of finite-difference methods used to describe the spatial variations is negligible compared to the errors induced by time stepping.

The analysis of the stability conditions provides a measure of the number of time steps which are unnecessary except to ensure stability. To keep these to a minimum, it is necessary to minimize the range of depths within the model.

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