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The period map for cubic threefolds

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To Jozef Steenbrink, for his 60th birthday

Abstract

Allcock, Carlson and Toledo defined a period map for cubic threefolds which takes values in a ball quotient of dimension 10. A theorem of Voisin implies that this is an open embedding. We determine its image and show that on the algebraic level this amounts to identification of the algebra of $\mathrm{SL}(5, \mathbb{C})$-invariant polynomials on the representation space $\mathrm{Sym}^3(\mathbb{C}^5)^*$ with an explicitly described algebra of meromorphic automorphic forms on the complex 10-ball.

Introduction

The polarized Hodge structure of nonsingular cubic threefold $X \subset \mathbb{P}^4$ is (according to Clemens and Griffiths [CG72]) encoded by its intermediate Jacobian, which is a principally polarized abelian variety of dimension five. It seems hard to characterize the abelian varieties that so appear and this is perhaps the reason that Allcock–Carlson–Toledo instead proposed to consider the cyclic order three cover of $\mathbb{P}^4$ that is totally ramified over X and take the polarized Hodge structure (with μ_3-action) of that cover. This cover is a smooth cubic fourfold and such a variety has an interesting and well-understood Hodge structure: its primitive cohomology sits in the middle dimension and has as its nonzero Hodge numbers $h^{3,1} = h^{1,3} = 1$ and $h^{2,2} = 20$. What makes this so tractable is that the period map for nonsingular cubic fourfolds is very much like that for polarized K3 surfaces: such polarized Hodge structures are parameterized by a bounded symmetric domain of type IV in the Cartan classification and of dimension 20 (so one more than for polarized K3 surfaces). Moreover, the period map is a local isomorphism and even more is true: according to a theorem of Voisin, it is an open embedding. Here, however, we are dealing with rather special cubic fourfolds, namely those that come with a μ_3-action whose fixed point set is a cubic threefold. Their primitive cohomology inherits that action and one finds that such data are parameterized by a symmetric subdomain of dimension 10 that is isomorphic to a complex ball. Thus, the period map of Allcock–Carlson–Toledo is a map from the moduli space of nonsingular cubic threefolds to a complex 10-ball modulo an arithmetic group. Voisin's theorem implies that this is an open embedding and so an issue that remains is the determination of its image. This is the subject of the present paper.

Our main result, Theorem 3.1, states among other things that this image is the complement of a locally symmetric divisor. It also identifies the algebra of $\mathrm{SL}(5, \mathbb{C})$-invariant polynomials on the representation $\mathrm{Sym}^3(\mathbb{C}^5)^*$ with an algebra of meromorphic automorphic forms on the complex 10-ball. Thus, the situation is very much like that of the period map for quartic plane curves (which assigns to such a curve the Hodge structure of the μ_4-cover of $\mathbb{P}^2$ totally ramified along that curve). Indeed, this fits in the general framework that we developed for that purpose in [LS05]. The same

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construction applied to a cubic surface yields a cubic threefold with μ_3-action (and, hence, gives rise to a cubic fourfold with $\mu_3 \times \mu_3$-action). In this manner from our theorem one can deduce Allcock–Carlson–Toledo's identification [ACT02] of the moduli space of cubic surfaces as a four-dimensional ball quotient.

After we finished this paper, we received from Allcock–Carlson–Toledo a manuscript [ACT06] in which they also determine the image of the period map. Their proof is clearly different from ours.

1. Cubic hypersurfaces of dimension three and four

1.1 Cubic fourfolds and their residual Hodge lines

The middle dimensional cohomology group of a nonsingular cubic fourfold $Y \subset \mathbb{P}^5$ is free of rank 23 and comes with an intersection form that is unimodular (because of Poincaré duality) and of signature $(21, 2)$. If $\eta \in H^2(Y, \mathbb{Z})$ is the hyperplane class, then $\langle \eta^4, [Y]\rangle = 3$ and so 3 is also the self-intersection of $\eta^2 \in H^4(Y, \mathbb{Z})$. In particular, the intersection form is odd. The primitive part $H^4_o(Y, \mathbb{Z}) \subset H^4(Y, \mathbb{Z})$, which is by definition the orthogonal complement of η^2, is generated by vanishing cycles. As a vanishing cycle has self-intersection $+2$, this is an even lattice. According to the theory of quadratic forms (see, for instance, Nikulin [Nik80]) this characterizes the vector η^2 up to an orthogonal transformation of $H^4(Y, \mathbb{Z})$ and we may conclude that we the have a lattice isomorphism

$$H^4_o(Y, \mathbb{Z}) \cong 2E_8 \perp 2U \perp A_2,$$

where, as usual, U denotes the hyperbolic plane (a lattice spanned by two isotropic vectors which have inner product 1).

The nonzero Hodge numbers in dimension four are $h^{3,1}(Y) = h^{1,3}(Y) = 1$, $h^{2,2}(Y) = 21$ and hence $H^4_o(Y)$ has a Hodge structure of type IV. We can represent $H^4_o(Y)$ by means of regular 5-forms on $\mathbb{P}^5 - Y$. This follows from the fact that $\mathbb{P}^5 - Y$ is affine and the proposition below.

Proposition 1.1. *If Y is a cubic fourfold whose singular set is nonsingular of dimension at most one, then we have an exact sequence*

$$0 \to H^5(\mathbb{P}^5 - Y) \to H^4(Y_{\mathrm{reg}})(-1) \to \mathbb{Z} \to 0,$$

where $H^5(\mathbb{P}^5 - Y) \to H^4(Y_{\mathrm{reg}})(-1)$ is the residue map and $H^4(Y_{\mathrm{reg}}) \to \mathbb{Z}$ is integration over a general linear section of dimension two.

Proof. First consider the exact sequence

$$H^5(\mathbb{P}^5) \to H^5(\mathbb{P}^5 - Y) \to H^6_Y(\mathbb{P}^5) \to H^6(\mathbb{P}^5) \to H^6(\mathbb{P}^5 - Y).$$

We have $H^5(\mathbb{P}^5) = 0$ and $H^6(\mathbb{P}^5 - Y) = 0$ because $\mathbb{P}^5 - Y$ is affine. Hence, $H^5(\mathbb{P}^5 - Y)$ maps isomorphically to the primitive part of $H^6_Y(\mathbb{P}^5)$. In the exact sequence

$$H^6_{Y_{\mathrm{sg}}}(\mathbb{P}^5) \to H^6_Y(\mathbb{P}^5) \to H^4(Y_{\mathrm{reg}})(-1) \to H^7_{Y_{\mathrm{sg}}}(\mathbb{P}^5)$$

the extremal terms are zero because Y_{sg} is smooth of codimension at least four and hence the middle map is an isomorphism. The proposition follows. $\square$

Thus, an equation $G \in \mathbb{C}[Z_0, \ldots, Z_5]$ (a homogeneneous form of degree three) defines an element $[\alpha(G)] \in H^4(Y_{\mathrm{reg}}, \mathbb{C})$ by taking the image of the class

$$\left[\mathrm{Res}_{\mathbb{P}^5} \frac{dZ_0 \wedge \cdots \wedge dZ_5}{G^2}\right] \in H^5(\mathbb{P}^5 - Y; \mathbb{C})$$

under the map of Proposition 1.1. It will be important for us to verify that in certain cases $[\alpha(G)]$ is nonzero. In cases where Y is smooth that is certainly so, as according to Griffiths [Gri69a, Gri69b],

$[\alpha(G)]$ is then a generator of $H^{3,1}(Y)$. It is clear that the span of $[\alpha(G)]$ in $H^4(Y_{\mathrm{reg}};\mathbb{C})$ only depends on Y. We write $\mathcal{F}(Y)$ for the one-dimensional vector space spanned by $G^{-2}dZ_0 \wedge \cdots \wedge dZ_5$. We often identify $\mathcal{F}(Y)$ with its image of under the Griffiths residue map in $H^4(Y_{\mathrm{reg}};\mathbb{C})$ if that image is nonzero.

Remark 1.2. On the form level the residue map can be defined in terms of the inner product on $\mathbb{C}^6$ (or, equivalently, in terms of the Fubini–Study metric). Concretely, consider the normalized gradient vector field of G in $\mathbb{C}^6$

$$N(G) := \frac{\|Z\|^2}{\|dG\|} \sum_{i=0}^{5} \left(\overline{\frac{\partial G}{\partial Z_i}}\right) \frac{\partial}{\partial Z_i}.$$

Letting ι stand for the inner product (the contraction of a vector with a form), then

$$\iota_{N(G)} d\iota_{N(G)} \left(\frac{dZ_0 \wedge \cdots \wedge dZ_5}{G^2}\right)$$

is a 5-form on $\mathbb{C}^6 - \{0\}$ of Hodge level at least four on whose restriction to the zero set of G is closed. It is also invariant under scalar multiplication and so it has a residue at infinity: this is a closed 4-form on $\mathbb{P}^5$ whose restriction $\alpha(G)$ to Y_{reg} is closed. It is a sum of a form of type $(3,1)$ and one of type $(4,0)$ that represents the above residue up to a universal nonzero scalar. We might therefore also think of $\mathcal{F}(Y)$ as a line of forms on Y_{reg} (which, however, depends on the Fubini–Study metric).

The simple hypersurface singularities in dimension four, A_k, $D_{k\geqslant 4}$, E_6, E_7, E_8, are the 'double suspensions' of the Kleinian singularities that bear the same name. We recall that a suspension of a hypersurface singularity adds to its equation a square in a new variable. Doing this twice does not affect the monodromy group of its miniversal deformations and so in the present case we have finite monodromy groups: such singularities go largely unnoticed by the period map.

We wish to establish that for some reduced cubic fourfolds Y, $(Y, \mathcal{F}(Y))$ is a *boundary pair* (in the sense of [LS05]) and we will also want to know its type. Here we use that notion in a slightly more general sense than [LS05, Definition (2.3)] in that we allow (at least in principle) the possibility that $\mathcal{F}(Y)$ maps to zero in $H^4(Y_{\mathrm{reg}};\mathbb{C})$, but then require that the map from $H_4^o(Y_{\mathrm{reg}};\mathbb{C})$ (or the part that matters to us, in the present case an eigenspace for an action of the cyclic group of order three) to the primitive homology of a smoothing of Y be nontrivial. In that case Y still imposes (via [LS05, Lemma 1.2]) a nontrivial linear constraint on the limiting behavior of the period map.

The first class of examples is furnished by the isolated hypersurface singularities in dimension four that come after the simple ones: the double suspensions of the simple elliptic singularities $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$.

Proposition 1.3. *Let Y be a cubic fourfold with a singular point of type $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$. Then $(Y, \mathcal{F}(Y))$ defines a boundary pair of type II.*

Proof. It is enough to verify that if G is an equation for Y, then $\alpha(Y) \wedge \bar{\alpha}(Y)$ is not integrable and that $H_4(Y_{\mathrm{reg}})$ contains an isotropic lattice of rank two that is mapped by $\alpha(Y)$ to a lattice in $\mathbb{C}$. We will see that this is essentially a local issue that has been dealt with in singularity theory. Let us for concreteness assume that Y has a singularity o of type $\tilde{E}_8$. We may choose local complex-analytic coordinates $(z_1, \ldots, z_5)$ at o such that Y is given there as $f(z) = z_1^6 + z_2^3 + \lambda z_1 z_2 z_3 + z_3^2 + z_4 z_5$. Note that f is weighted homogenenous of degree 6 with weights $(1,2,3,3,3)$. The residue α of $f^{-2} dz_1 \wedge \cdots \wedge dz_5$ on the smooth part of the zero set of $f = 0$ is homogeneous of degree zero (equivalently, $\mathbb{C}^\times$-invariant). The form $\alpha \wedge \bar{\alpha}$ is positive everywhere and $\mathbb{C}^\times$-invariant also, and hence will not be integrable near o. It is well known (and implied by the work of Steenbrink [Ste77]) that the link L of this singularity has the property that $H_4(L)$ is free of rank two and that the periods of α on it are the periods of the elliptic curve defined by $z_1^6 + z_2^3 + \lambda z_1 z_2 z_3 + z_3^2$ in a weighted projective space. It is also known (see [Ste77]) that if we multiply α by an element of the maximal ideal of $\mathbb{C}\{z_1, \dots, z_5\}$, then it becomes exact on the germ of Y_{reg} at o. Since $\alpha(Y)$ equals α up to a unit in $\mathbb{C}\{z_1, \dots, z_5\}$, the image of $H_4(L) \to H_4(Y_{\mathrm{reg}})$ is as required and the proposition follows. $\square$

In the following lemma we use the notion of boundary pair in the above more general sense.

Lemma 1.4. *Let Y be a cubic fourfold whose singular locus has as an irreducible component a curve such that Y has a transversal singularity of type A_2 along the generic point of that curve. If the primitive homology $H_4^o(Y_{\mathrm{reg}})$ has nontrivial intersection pairing, then $(Y, \mathcal{F}(Y))$ defines a boundary pair of type I and $H_4^o(Y_{\mathrm{reg}})$ is positive semi-definite.*

Proof. Choose complex-analytic coordinates $(z_1, \dots, z_5)$ at a generic point of the curve in question such that Y is there given by $f(z) = z_1^3 + z_2^2 + z_3^2 + z_4^2$. So f is weighted homogenenous of degree six with weights $(2, 3, 3, 3)$. The residue α of $f^{-2} dz_1 \wedge \cdots \wedge dz_5$ on the smooth part of the zero set of $f = 0$ is homogeneous of degree -1 and hence the form $\alpha \wedge \bar{\alpha}$ will not be integrable near the origin. Arguing as in the proof of Proposition 1.3 we find that $\alpha(G)$ is not integrable. Now let Z be a 4-cycle on Y_{reg} that is perpendicular to the hyperplane class and has nonzero self-intersection number $Z \cdot Z$. If $\mathcal{Y}/\Delta \subset \mathbb{P}^5_\Delta$ is any smoothing of Y, then Z extends as a relative cycle $\mathcal{Z}/\Delta$. Clearly, $\int_{Z_t} \alpha(Y_t)$ is bounded. On the other hand, $\int_{Y_t} \alpha(Y_t) \wedge \overline{\alpha(Y_t)}$ tends to infinity as $t \to 0$. This implies that any limiting point of the line in $H^4(Y_t; \mathbb{C})$ spanned by $\alpha(Y_t)$ is in the hyperplane defined by $[Z_t]$. Since $Z_t \cdot Z_t \neq 0$, this hyperplane must be of type I: $Z_t \cdot Z_t > 0$. $\square$

Remark 1.5. In the preceding lemma, the assumption that Y_{sg} contains a curve along which we have a transversal A_2-singularity can be weakened to the following: Y_{sg} contains an irreducible component of dimension one (the above argument also works for transversal singularity type A_1 and hence for any singularity type that is worse).

The geometric invariant theory (GIT) of cubic fourfolds is probably not sufficiently worked out yet to make it feasible at present to verify whether every semistable cubic fourfold yields a boundary pair, but we shall see that we can do this for cubic fourfolds attached to semistable cubic threefolds.

Let $X \subset \mathbb{P}^4$ be a nonsingular cubic threefold. Following Allcock–Carlson–Toledo, its period map is best studied by passing to the μ_3-cover $Y \to \mathbb{P}^4$ which ramifies over X. This cover is a cubic fourfold. To be more precise, let an equation for X be $F \in \mathbb{C}[Z_0, Z_1, Z_2, Z_3, Z_4]$. Then $G := F - Z_5^3$ is an equation for Y. Moreover, $H^{3,1}(Y)$ comes with the generator $[\alpha(G)]$.

The GIT for cubic hypersurfaces $X \subset \mathbb{P}^4$ has been carried out independently by Allcock [All03] and Yokoyama [Yok02]. They found that such an $X \subset \mathbb{P}^4$ is stable if and only its singularities are of type A_1, A_2, A_3 or A_4. This means that Y has singularities of type A_2, D_4, E_6 or E_8, respectively (add a cube in a new variable). The minimal strictly semistable cubic threefolds X are the following.

(D_4^3) In this case, X_{sg} consists of three D_4-singularities. Such an X is unique up a linear transformation. The associated fourfold Y has three $\tilde{E}_6$ singularities (double suspensions of degree three simply elliptic singularities of zero j-invariant). Hence, Y yields a boundary pair of type II.

(A_5^2) In this case, X_{sg} consists of two A_5-singularities, perhaps also with a singularity of type A_1. This makes up a one parameter family of PGL(5)-orbits. The associated fourfold Y has two $\tilde{E}_8$ singularities (in fact, double suspensions of degree one simply elliptic singularities of zero j-invariant) and, possibly, an A_2-singularity. Hence, Y yields a boundary pair of type II.

(A_1^∞) In this case, X is a chordal cubic: it is the secant variety of a rational normal curve in $\mathbb{P}^4$ of degree four; this curve equals X_{sg} and the transversal singularity is of type A_1. It lies in the closure of the curve that parameterizes the (A_5^2)-case.

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So the GIT boundary consists of an isolated point (D_4^3) and an irreducible curve that is the union of (A_5^2) and (A_1^∞).

2. Eisenstein lattices

2.1 Generalities

We fix a generator T of μ_3 so that the group ring $\mathbb{Z}\mu_3$ is identified with $\mathbb{Z}[T]/(T^3-1)$. This identifies the number field $\mathbb{Q}(\mu_3)$ with $\mathbb{Q}[T]/(T^2+T+1)$ and its ring of integers with $\mathbb{Z}[T]/(T^2+T+1)$ (which is therefore a quotient of $\mathbb{Z}\mu_3$). The latter is called the *Eisenstein ring* and we shall denote it by $\mathcal{E}$. If we substitute for T the standard choice (relative to a choice of $\sqrt{-1}$) of a primitive third root of unity, $\zeta = -\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, then $\mathcal{E}$ gets identified with the set of $\frac{1}{2}(a+b\sqrt{-3})$ with $a, b \in \mathbb{Z}$ of the same parity.

If μ_3 operates on a finitely generated free abelian group A, then $\mathcal{E} \otimes_{\mathbb{Z}\mu_3} A$ can be identified with the quotient of A by the fixed point subgroup A^{μ_3}. If the latter happens to be trivial (so that A is an $\mathcal{E}$-module), then $\mathbb{C} \otimes A$ splits according to the characters of μ_3 as

$$\mathbb{C} \otimes A = (\mathbb{C} \otimes A)_\chi \oplus (\mathbb{C} \otimes A)_{\bar{\chi}},$$

where $\chi : \mu_3 \subset \mathbb{C}^\times$ is the tautological character. The first summand may be identified with $\mathbb{C} \otimes_{\mathcal{E}} A$ and the second summand is the complex conjugate of the first. If A also comes with an integral μ_3-invariant symmetric bilinear form $(\,\cdot\,) : A \times A \to \mathbb{Z}$, then

$$\phi : A \times A \to \mathcal{E}, \quad \phi(a, a') = -(a \cdot a')\zeta + (a \cdot Ta')$$

is skew-Hermitian over $\mathcal{E}$. It is such that $\phi(a, a) = -\frac{1}{2}\sqrt{-3}(a \cdot a)$ (so $(\,\cdot\,)$ had to be even). Multiplication by $\sqrt{-3}$ turns ϕ into a Hermitian form

$$h(a, a') := \sqrt{-3}\phi(a, a') = \tfrac{3}{2}(a \cdot a') + \sqrt{-3}(a \cdot \tfrac{1}{2}a' + Ta')$$

with $h(a, a) = \frac{3}{2}(a \cdot a)$. Conversely, every finitely generated torsion free $\mathcal{E}$-module equipped with an $\mathcal{E}\sqrt{-3}$-valued Hermitian form (or, equivalently, an $\mathcal{E}$-valued skew-Hermitian form ϕ) so arises and that is why we call these data an *Eisenstein lattice*.

We are concerned with certain Eisenstein lattices denoted Λ_k and so we recall their definition: Λ_k is a free $\mathcal{E}$-module with generators $r_1, \dots, r_k$, whose Hermitian form is characterized by

$$h(r_i, r_j) = \begin{cases} 3 & \text{if } j = i, \\ \sqrt{-3} & \text{if } j = i+1, \\ 0 & \text{if } j > i+1. \end{cases} \tag{2.1}$$

This is equivalent to $r_i \cdot r_i = 2$, $r_i \cdot r_{i+1} = 0$, $r_i \cdot Tr_{i+1} = 1$ and $r_i \cdot T^k r_j = 0$ for $j > i+1$ and all k. This lattice is isomorphic to its conjugate, for the matrix of h on the basis $((-1)^i r_i)_i$ is conjugate to the matrix of h on $(r_i)_i$.

We now give a few cases of special interest to us. The A_2-lattice has just two rotations of order three which are each others inverse (these are also its Coxeter transformations) and the resulting μ_3-action makes it an Eisenstein lattice isomorphic to Λ_1. If we do something similar to an E_8-lattice by letting $T \in \mu_3$ act as the tenth power of a Coxeter transformation (which has order 30), then the resulting Eisenstein lattice is isomorphic to Λ_4. The hyperbolic Eisenstein lattice $U_{\mathcal{E}}$ is, by definition, spanned by two isotropic vectors with inner product $\sqrt{-3}$; its underlying integral lattice is U^2. It is known that $\Lambda_{10} \cong \Lambda_4 \perp U_{\mathcal{E}} \perp \Lambda_4$.

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2.2 The vanishing Eisenstein lattice attached to a chordal cubic

If $X \subset \mathbb{P}^4$ is a nonsingular cubic threefold and $\mathbb{P}^5 \supset Y \to \mathbb{P}^4$ the associated cubic fourfold with μ_3-action, then the μ_3-invariant part of $H^4(Y;\mathbb{Q})$ can be identified with $H^4(\mathbb{P}^4;\mathbb{Q})$. This is also the image of $H^4(\mathbb{P}^5;\mathbb{Q}) \to H^4(Y;\mathbb{Q})$ and, hence, is spanned by η^2. It follows that $H^4_o(Y,\mathbb{Z})$ is in a natural manner an $\mathcal{E}$-module. The intersection pairing turns it into an Eisenstein lattice. We use the degeneration of X into a chordal cubic to determine its isomorphism type.

Jim Carlson has determined the limiting Hodge structure for a linear smoothing of the chordal cubic as well as for the associated smoothing of cubic fourfolds. The following lemma can also be derived from his computations.

Lemma 2.1. *Let $X \subset \mathbb{P}^4$ be the chordal cubic, $\mathcal{X}/\Delta \subset \mathbb{P}^4_\Delta$ a general linear smoothing of X over the unit disk Δ and X' a general fiber of this smoothing. Denote by $Y \subset \mathbb{P}^5$, $\mathcal{Y}/\Delta \subset \mathbb{P}^5_\Delta$ and Y' the associated (relative) cubic fourfolds. Then the kernel of a natural map $H_4(Y';\mathcal{E}) \to H_4(Y;\mathcal{E})$ equipped with the intersection pairing and the μ_3-action contains an Eisenstein lattice isomorphic to Λ_{10}.*

Proof. By assumption the smoothing of X has the form $F_t = F + tF'$, where F is an equation for X. We suppose that $(F' = 0)$ meets the singular set C of X transversally and that for $0 < |t| < 1$, $X_t = (F_t = 0)$ is smooth. For such t, $C \cap X_t$ consists of $4 \times 3 = 12$ distinct points. Near a point of $C - C \cap X_t$ (respectively, $C \cap X_t$) we can find local analytic coordinates $(z_1, \dots, z_5)$ on $\mathbb{P}^5$ such that the smoothing $\mathcal{Y}$ is given by $z_1^3 + z_2^2 + z_3^2 + z_4^2 = t$ (respectively, $z_1^3 + z_2^2 + z_3^2 + z_4^2 = tz_5$) with μ_3 affecting only the first coordinate.

Choose an oriented embedded circle γ in C that contains $C \cap X_t$, label the points of $C \cap X_t$ in a corresponding cyclic manner $\{p_i\}_{i \in \mathbb{Z}/12}$, and denote by γ_i the part of γ that goes from p_{i-1} to p_i. In what follows, only the isotopy class of γ_i matters.

For $\epsilon > 0$ small and $i \in \mathbb{Z}/12$ given, we construct over γ_i a cycle Γ_i in $Y' := Y_\epsilon$ as follows. Since this essentially only involves the topology, we may suppose that γ_i is very small so that p_{i-1} and p_i are about to coalesce. This allows us to assume that γ_i is contained in a complex-analytic coordinate patch $(U; z_1, \dots, z_5)$ such that $C \cap U$ is open in the z_5-axis, $z_5(p_{i-1}) = -1, z_5(p_i) = 1$ and the smoothing is of the form

$$z_1^3 + z_2^2 + z_3^2 + z_4^2 = t(1 - z_5^2),$$

with, as before, μ_3 affecting the first coordinate only, and that $\gamma|U$ is simply the intersection of U with the real part of the z_5-axis (so that γ_i is the interval $[-1, 1]$ in that axis).

Let D_i be the chain on Y' defined by

$$x_1^3 + x_2^2 + x_3^2 + x_4^2 = \epsilon(1 - x_5^2), \quad x_i \in \mathbb{R},\ x_1 \geqslant 0,\ -1 \leqslant x_5 \leqslant 1.$$

We note that D_i is a topological 4-disk, for its projection on the x_5-axis has as the fiber over $x_5 \in [-1, 1]$ the 3-disk in the case $|x_5| \neq 1$ (with $\epsilon(1 - x_5)^2 - x_1^3$ as the radial parameter and $(x_2, x_3, x_4)/|(x_2, x_3, x_4)|$ as the angular parameter), and as the fiber over ± 1 the singleton $\{(0, 0, 0, 0, \pm 1)\}$. We orient D_i by taking the orientation defined by the coordinates $(x_1, \dots, x_4)$ at the point $(\sqrt[3]{\epsilon}, 0, 0, 0, 0)$ so that we may regard it as a chain. It has the same boundary as TD_i and hence $\Gamma_i := (1 - T)D_i$ is the cycle defined by an oriented 4-sphere.

Claim 1. The intersection numbers $(\Gamma_i \cdot T^k \Gamma_{i+1})_{k \in \mathbb{Z}/3}$ are up to a cyclic permutation equal to $(1, 0, -1)$.

Near $(0, 0, 0, 0, 1)$, $(z_1, \dots, z_4)$ is a coordinate system for Y'. In terms of that system, D_i is simply the set for which all coordinates are real and $x_1 \geqslant 0$. Hence, Γ_i is there defined by z_2, z_3, z_4 real and z_1 or $\zeta^{-1} z_1$ real and at least zero. After a possible isotopy, the sphere D_{i+1} meets U in the set

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where z_1 is a primitive sixth root of unity and z_2, z_3, z_4 are purely imaginary (so that $1 - z_5^2$ is real and at most zero). These are oriented topological submanifolds meeting in $(0,0,0,0,1)$ only. Their intersection number is the same as that of an arc a in $\mathbb{C}$ composed of two half rays making an angle $2\pi/3$ and the transform of $-a$ under a primitive sixth root of unity. It is clear that if we replace Γ_{i+1} by $T^k\Gamma_{i+1}$, then we must multiply this primitive sixth root of unity by ζ_3^k. Now the claim follows from the easily seen fact that if $\zeta_6 = \frac{1}{2} + \frac{1}{2}\sqrt{-3}$ and a' is $\zeta_6 a$ with its opposite orientation, then $a \cdot a' = 1$, $a \cdot \zeta_3 a' = 0$ and $a \cdot \zeta_3^2 a' = -1$.

CLAIM 2. The class of Γ_i generates in $H_4(Y')$ a copy of Λ_1: $(1 + T + T^2)\Gamma_i = 0$ and $\Gamma_i \cdot \Gamma_i = 2$.

It is clear that $(1 + T + T^2)\Gamma_i = 0$. On $U \cap Y'$ we have a flow of the form

$$\Phi_\alpha(z_1, \dots, z_5) = (e^{\sqrt{-1}\alpha/3} z_1, e^{\sqrt{-1}\alpha/2} z_2, e^{\sqrt{-1}\alpha/2} z_3, e^{\sqrt{-1}\alpha/2} z_4, \phi_\alpha(z_5)),$$

where α is small and $1 - \phi_\alpha(z_5)^2 = e^{\sqrt{-1}\alpha}(1 - z_5^2)$ (since $1 - z_5^2$ has only simple zeroes, this is well defined). It has $(0,0,0,0,\pm 1)$ as fixed points. We see that for nonzero α, $\Phi_\alpha(\Gamma_i)$ meets Γ_i in these fixed points only. The argument above shows that at each of these points the intersection number is 1: it is the intersection number of the real part of $\mathbb{C}^4$ with its transform under Φ_α (as acting on the first four coordinates).

It is clear that if $j - i \neq \pm 1$, then $\Gamma_i \cdot T^k\Gamma_j = 0$ for every k. Upon replacing Γ_{i+1} for $i = 1, \dots, 9$ successively by an element of $\pm\mu_3\Gamma_{i+1}$, we can arrange that $\Gamma_i \cdot \Gamma_{i+1} = 0$ and $\Gamma_i \cdot T\Gamma_{i+1} = 1$ for $i = 1, \dots, 9$ so that $\Gamma_1, \dots, \Gamma_{10}$ span in $H_4(Y'; \mathcal{E})$ a sublattice of type Λ_{10}. □

The following proposition plays a central role in this paper and will be proved in § 4.

PROPOSITION 2.2. *Let $X \subset \mathbb{P}^4$ be a chordal cubic and Y the cubic fourfold that is a μ_3-cover ramified over X. Then $H_\bullet(Y_{\mathrm{reg}}; \mathbb{C})_\chi = H_4(Y_{\mathrm{reg}}; \mathbb{C})_\chi$ and the latter is of dimension one. Moreover, the intersection pairing defines a positive Hermitian form on this space so that $(Y, \mathcal{F}(Y))$ is a boundary pair of type I.*

Remark 2.3. Let $F \in \mathbb{C}[Z_0, \dots, Z_4]$ define the chordal cubic and put $G := F - Z_5^3$. If there is a simple way to prove that the 6-form $G^{-2} dZ_0 \wedge \cdots \wedge dZ_5$ is not exact, then Proposition 1.1 implies that the Griffiths residue $\alpha(G)$ defines a nonzero class in $H^4(Y_{\mathrm{reg}}; \mathbb{C})$. We then could use that fact to bypass the preceding proposition, resulting in the elimination of § 4 and, hence, in a substantial shortening of the proof of our main Theorem 3.1.

We now give an interesting corollary to Lemma 2.1. It seems to suggest that $H_4(Y_{\mathrm{reg}}; \mathcal{E})$ is isomorphic to Λ_1.

COROLLARY 2.4. *For Y' as in Lemma 2.1, $H_o^4(Y')$ is an Eisenstein lattice and is as such isomorphic to $\Lambda_{10} \perp \Lambda_1$.*

Proof. Since $H^4(Y'; \mathbb{C})^{\mu_3} \cong H^4(Y'^{\mu_3}; \mathbb{C}) = H^4(\mathbb{P}^4; \mathbb{C})$ is spanned by η^2, it follows that $H_o^4(Y')$ is an Eisenstein lattice. The integral lattice underlying Λ_{10} is $E_8^2 \perp U^2$, hence is unimodular. If we identify $H^4(Y')$ with $H_4(Y')$ via Poincaré duality, and use the fact that Λ_{10} is isomorphic to its conjugate, then the lemma in question proves that $H^4(Y')$ contains a copy of Λ_{10}.

The orthogonal complement of this copy of Λ_{10} in $H^4(Y')$ is unimodular, odd, positive definite, of rank three and, hence, isomorphic to $\mathbb{Z}^3$ equipped with its standard form $x_1^2 + x_2^2 + x_3^2$. The vectors with self-product three are those that have each coordinate ± 1 and, hence, all lie in the same orbit of the integral orthogonal group. We may therefore arrange that the isomorphism takes $(1,1,1)$ to η^2, so that the orthogonal complement of the Λ_{10}-copy in $H_o^4(Y')$ is identified with the orthogonal complement of $(1,1,1)$ in $\mathbb{Z}^3$. The latter is an A_2-lattice and any $\mathcal{E}$-structure on that lattice makes it isomorphic to Λ_1. The corollary follows. □

3. The main result

In this section we fix a complex vector space U of dimension five and abbreviate the $\mathrm{GL}(U)$-representation $\mathrm{Sym}^3 U^*$ by S. Let $\mathcal{X} \subset \mathbb{P}(U)_S$ be the universal cubic. The latter is given by a single equation $F \in \mathbb{C}[S \times U]$. Denote by $f : \mathcal{Y} \subset \mathbb{P}_S(U \oplus \mathbb{C}) \to S$ the μ_3-cover defined by $G := F + w^3$. It is invariant under the obvious $\mathrm{SL}(U)$-action. If we fix a generator $\mu \in \det(U^*)$, then the Griffiths residue construction applied to $G^{-2}\mu$ yields a relative $(3,1)$-form on the part $\mathcal{Y}^\circ$ of $\mathcal{Y}$ where $\mathcal{Y}$ is smooth over S. We denote by S° the locus where $\mathcal{Y}$ is smooth over S so that $\mathcal{Y}_{S^\circ} \subset \mathcal{Y}^\circ$.

3.1 The refined period map

We fix an odd unimodular lattice L of signature $(21,2)$, a vector $v_o \in L$ with $v_o \cdot v_o = 3$ whose orthogonal complement L_o is even, and a μ_3-action on L whose fixed point set is $\mathbb{Z}v_o$ and for which L_o is isomorphic to $\Lambda_{10} \perp \Lambda_1$ as an $\mathcal{E}$-lattice. We shall write Λ for L_o as an $\mathcal{E}$-lattice. The quadratic form on L_o becomes a Hermitian form on Λ that takes values in $\sqrt{-3}\mathcal{E}$; we denote that form by h. We denote the underlying $\mathbb{C}$-vector space of Λ by H (so $H := \mathbb{C} \otimes_{\mathcal{E}} \Lambda$ and $\mathbb{C} \otimes L_o = H \perp \bar{H}$) and we let $H_+ \subset H$ be the set of $v \in H$ with $h(v,v) < 0$. Then $\mathbb{P}(H_+)$ is the symmetric domain of the unitary group of H and is isomorphic to a complex 10-ball. The restriction of $\mathcal{O}_{\mathbb{P}(H)}(-1)$ to $\mathbb{P}(H_+)$ is our basic automorphic line bundle (its 11th tensor power is the equivariant canonical bundle of $\mathbb{P}(H_+)$); we will denote that line bundle by $\mathcal{A}(1)$. The subgroup $\Gamma \subset O(L)$ of μ_3-automorphisms of L that fix v_o is arithmetic and acts properly on H_+ and $\mathbb{P}(H_+)$. The Baily–Borel theory asserts that

$$\bigoplus_{k \geqslant 0} H^0(\mathbb{P}(H_+), \mathcal{A}(k))^\Gamma$$

is a finitely generated graded algebra (of automorphic forms) whose Proj defines a normal projective completion $\Gamma\backslash\mathbb{P}(H_+) \subset \Gamma\backslash\mathbb{P}(H_+)^{\mathrm{bb}}$ of the orbit space by adding one point (a *cusp*) for every Γ-orbit in $\partial\mathbb{P}(H_+)$ whose points are defined over $\mathbb{Q}(\zeta)$.

We shall interpret $\mathbb{P}(H_+)$ as a classifying space for certain Hodge structures on L_o of weight four polarized by $(\cdot)$ and invariant under μ_3: giving a point of $\mathbb{P}(H_+)$ amounts to giving a line $F^3 \subset H$ on which h is negative and such a line determines a weight four polarized Hodge structure on L_o:

$$\mathbb{C} \otimes_{\mathbb{Z}} L_o = H^{3,1} \perp H^{2,2} \perp H^{1,3},$$

with $H^{3,1} := F^3$, $H^{1,3} := \overline{F^3}$ and $H^{2,2}$ the orthogonal complement of $F^3 \perp \overline{F^3}$. Conversely, any Hodge structure on L_o with $(h^{3,1}, h^{2,2}, h^{1,3}) = (1, 20, 1)$, polarized by the quadratic form and with $H^{3,1}$ in the eigenspace of the tautological character $\chi : \mu_3 \subset \mathbb{C}^\times$ is thus obtained.

We have of course arranged that if $Y \subset \mathbb{P}(U \oplus \mathbb{C})$ is a smooth fiber of f, then there is a μ_3-isomorphism $H^4(Y) \cong L$ that takes η^2 to v_o (this follows from Corollary 2.4). Such isomorphisms (also called *markings* of Y) are permuted simply transitively by $\Gamma \subset O(L)$. A marking carries $\alpha(Y) \in H^{3,1}(Y)$ to a point of H_+. This defines a refined period map $S^\circ \to \Gamma\backslash H_+$. It is evidently constant on the $\mathrm{SL}(U)$-orbits. However, there is, in fact, $\mathrm{GL}(U)$-equivariance: if $F \in S^\circ$, then let F' be its transform under the scalar $t \in \mathbb{C}^\times \subset \mathrm{GL}(U)$: $F' = t^{-3}F$. If $(z', w') = (tz, w)$, then clearly, $F'(z') - (w')^3 = F(z) - w^3$ and, hence, we get an isomorphism $Y_F \cong Y_{F'}$. This isomorphism pulls back $(F' - (w')^3)^{-2}\mu$ to $t^5(F - w^3)^{-2}\mu$ and, hence, sends $\alpha(F')$ to $t^5\alpha(F)$. Thus, the refined period map defines a morphism

$$P : \mathrm{GL}(U)\backslash S^\circ \to \Gamma\backslash\mathbb{P}(H_+)$$

that is covered by a morphism of line bundles that sends $p^*\mathcal{A}$ to the fifth power of the line bundle over the left-hand side defined by the determinant character $\det : \mathrm{GL}(U) \to \mathbb{C}^\times$ (which is the geometric quotient of $\mathcal{O}_{S^\circ}(3)$ by $\mathrm{SL}(U)$). In this way we get a $\mathbb{C}$-algebra homomorphism from $\bigoplus_{k\geqslant 0} H^0(\mathbb{P}(H_+), \mathcal{A}(k))^\Gamma$ to the part of $\mathbb{C}[S]^{\mathrm{SL}(U)}$ spanned by the summands of degree a multiple of five. Since the center of $\mathrm{SL}(U)$ is μ_5 and acts on $\mathrm{Sym}^3 U^*$ faithfully by scalar multiplication, $\mathbb{C}[S]^{\mathrm{SL}(U)}$ only lives in degrees that are multiples of five. For a similar reason, $\bigoplus_{k\geqslant 0} H^0(\mathbb{P}(H_+), \mathcal{A}(k))^\Gamma$ only lives in degrees that are multiples of three: the center of Γ contains μ_3, which acts in the obvious manner on $\mathcal{A}$ as scalar multiplication. Thus, we find a $\mathbb{C}$-algebra homomorphism

$$p : \bigoplus_{k\geqslant 0} H^0(\mathbb{P}(H_+), \mathcal{A}(k))^\Gamma \to \mathbb{C}[\mathrm{Sym}^3 U^*]^{\mathrm{SL}(U)}$$

of which the source (respectively, range) is graded by multiples of three (respectively, five) and which is homogeneneous of degree $\frac{5}{3}$. The Proj of the right-hand side is the GIT compactification of $\mathrm{GL}(U)\backslash S^\circ$, namely $\mathrm{GL}(U)\backslash\backslash S^{\mathrm{ss}}$, where S^{ss} denotes the semistable locus and $\backslash\backslash$ indicates that we take the geometric quotient. It contains $\mathrm{GL}(U)\backslash S^{\mathrm{st}}$ as a dense-open subset, where S^{st} denotes the $\mathrm{SL}(U)$-stable locus in S. Following Allcock [All03], S^{st} is precisely the set of $s \in S$ for which the fiber Y_s has only simple singularities in the sense of Arnol'd (double suspensions of Kleinian singularities). These have the property that the monodromy of the fibration $\mathcal{Y}_{S^\circ}/S^\circ$ near such a fiber is finite. It is well known [GT84] that this implies that the period mapping extends across such singularities as a map

$$P : \mathrm{GL}(U)\backslash S^{\mathrm{st}} \to \Gamma\backslash\mathbb{P}(H_+).$$

A local Torelli theorem tells us that P is a local isomorphism. The GIT boundary of $\mathrm{GL}(U)\backslash S^{\mathrm{st}}$ is of dimension one and has as a distinguished point the isomorphism type of the chordal cubic. Away from this point the variation of polarized Hodge structure defined by our family has degenerations of type II only.

3.2 The image of the period map

We recall from [LS05] that a boundary pair defines a Γ-orbit $\mathcal{K}$ of hyperplanes of H. If it is of type I, then each $K \in \mathcal{K}$ meets H_+. Otherwise, it is of type II and $\mathbb{P}(K)$ intersects the closure of $\mathbb{P}(H^+)$ in a single point only; this point lies on the boundary and is defined over $\mathbb{Q}(\zeta)$ (hence, defines a cusp).

THEOREM 3.1. *The period map defines an isomorphism from the moduli space of stable cubic threefolds onto a* 10*-dimensional ball quotient minus an irreducible locally symmetric hypersurface. To be precise, let $\mathcal{H}$ be the collection of hyperplanes in H that are the complex span of an Eisenstein sublattice of Λ isomorphic to Λ_{10}. Then $\mathcal{H}$ is a Γ-orbit and if we write H_+° for $H_+ - \bigcup_{K\in\mathcal{H}} K_+$, then P maps* $\mathrm{GL}(U)\backslash S^{\mathrm{st}}$ *isomorphically onto $\Gamma\backslash\mathbb{P}(H_+^\circ)$. Moreover, P induces an isomorphism (of degree $\frac{5}{3}$) from the $\mathbb{C}$-algebra of meromorphic Γ-automorphic forms $\bigoplus_{k\in\mathbb{Z}} H^0(H_+^\circ, \mathcal{A}(k))^\Gamma$ (we allow arbitrary poles along the hyperplane sections indexed by $\mathcal{H}$) onto the $\mathbb{C}$-algebra of invariants $\mathbb{C}[\mathrm{Sym}^3 U^*]^{\mathrm{SL}(U)}$. In particular, $H^0(H_+^\circ, \mathcal{A}(k))^\Gamma = 0$ for $k < 0$ and the GIT compactification* $\mathrm{GL}(U)\backslash\backslash S^{\mathrm{ss}}$ *gets identified with* $\mathrm{Proj} \bigoplus_{k\geqslant 0} H^0(H_+^\circ, \mathcal{A}(k))^\Gamma$.

Proof. Proposition 2.2 shows that the chordal cubic defines a boundary pair $(Y, \mathcal{F}(Y))$ of type I with $H_4(Y, \mathbb{C})_\chi$ of dimension one. According to [LS05], such a pair determines a Γ-orbit $\mathcal{K}_1$ of hyperplane sections of $\mathbb{P}(H_+)$. Elsewhere on the GIT boundary, the degeneration of the variation of polarized Hodge structure is of type II and so the period map is proper over the complement of $D(\mathcal{K}_1)$ in $\Gamma\backslash\mathbb{P}(H_+)$. The local Torelli theorem says that P is a local isomorphism. Voisin's Torelli theorem [Voi86] says that if X_1 and X_2 are smooth cubic threefolds with the same image under this period map, then their associated cubic fourfolds Y_1, Y_2 are projectively equivalent. We claim that for generic X_1, this isomorphism identifies the μ_3-actions. The two actions certainly coincide on $H^{3,1}$ (it is there scalar multiplication), and hence they coincide on the transcendental lattices of the two fourfolds (the transcendental lattice of Y_i is the smallest primitive sublattice in $H^4(Y_i)$ whose complex span contains $H^{3,1}(Y_i)$). However, for generic X_1, the transcendental lattice is all of $H^4_\circ(Y_1)$. We conclude that P is of degree one and hence is an open embedding.

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According to Lemma 2.1, some $K \in \mathcal{K}_1$ contains a sublattice of Λ isomorphic to Λ_{10}, in other words, $K \in \mathcal{H}$. Since $\mathcal{K}_1$ is a single Γ-orbit, it follows that $\mathcal{H} = \mathcal{K}_1$.

The image of P is as asserted if we prove that its image is disjoint with $D(\mathcal{H})$. If P meets $D(\mathcal{H})$, then the complement of the image of P is a closed subset of $\Gamma\backslash\mathbb{P}(H_+)$ of codimension at least two everywhere. Then p will be an isomorphism and, hence, identifies the GIT compactification with the Baily–Borel compactification. This is a contradiction since the former has a one-dimensional boundary, whereas the latter's boundary is finite. The isomorphism P is covered by an isomorphism of line bundles: the Γ-quotient of $\mathcal{A}(3)|H_+^\circ$ gets identified with the orbifold line bundle on $\mathrm{GL}(U)\backslash S^{\mathrm{st}}$ defined by $\mathcal{O}_{\mathbb{P}(S)}(1)$ and this implies the last statement. □

Remarks 3.2. The theory developed in [Loo03, LS05] tells us a bit more. For example, it interprets $\mathrm{Proj} \bigoplus_{k\geqslant 0} H^0(H_+^\circ, \mathcal{A}(k))^\Gamma$ in terms of arithmetic data: the boundary it adds to $\Gamma\backslash\mathbb{P}(H_+^\circ)$ has a stratification whose members are described in § 1. Since this is also the GIT boundary, which, according to Allcock, consists of an isolated point and a curve, we are able to recover these arithmetic data without any extra effort:

(i) Γ has two orbits of cusps in $\partial\mathbb{P}(H_+)$, corresponding to the cases (D_4^3) and (A_5^2), respectively;

(ii) a (D_4^3)-cusp does not lie on any $\mathbb{P}(K)$ with $K \in \mathcal{H}$;

(iii) the common intersection of the $K \in \mathcal{H}$ for which $\mathbb{P}(K)$ contains a given (A_5^2)-cusp is a codimension two linear subspace of H;

(iv) the members of $\mathcal{H}$ become disjoint when intersected with H_+.

It also tells us what the graph of the period map is like when regarded as a rational map from the GIT compactification to the Baily–Borel compactification: on the GIT side there is a hypersurface lying over the chordal cubic and on the Baily–Borel side there is a curve lying over the (A_5^2)-cusp. The hypersurface and the curve meet in a single point.

If we are able to verify the first three of these four properties in the context of lattice theory (which we have not done, although standard methods make this feasible), then the use of Voisin's Torelli theorem may be eliminated as follows: if we view the period map as a rational map from the GIT compactification to the Baily–Borel compactification, then (by the Zariski connectedness theorem) the preimages of the two cusps will be disjoint. They are evidently contained in the GIT-boundary. One component of the GIT-boundary is the singleton represented by a cubic threefold X with three D_4-singularities. It gives rise to a cubic fourfold Y with three $\tilde{E}_6$-singularities and the period map sends this point to the (D_4^3)-cusp. Hence, this singleton appears as a fiber of the rational period map. It therefore suffices to verify that the latter is of degree one near X. Now if Y_t is a smooth cubic fourfold close to Y, then the orthogonal complement of the vanishing homology in $H_4^o(Y_t)$ is an isotropic plane and for this reason the behavior of the period map near Y is essentially that of its restriction to the vanishing homology, that is, the period map used in singularity theory in the sense of [Loo78]. So we only need to know that the latter is of degree one and this is indeed the case (see [Loo78]).

4. The boundary pair defined by the chordal cubic

In this section $X \subset \mathbb{P}^4$ denotes a chordal cubic. That is, X is the secant variety of a normal rational curve $C \subset \mathbb{P}^4$ (since C is unique up to projective equivalence, so is X). This is indeed a cubic hypersurface. A geometric argument might run as follows: if $\ell \subset \mathbb{P}^4$ is a general line, then any point of $\ell \cap X$ lies on a secant of C by definition. If we project away from ℓ, we map to a projective plane and the image of C is an irreducible rational quartic curve in that plane. It will have three ordinary double points and these double points define the secants of C which meet ℓ. So $\ell \cdot X = 3$.

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Rather than making this argument rigorous, we derive an explicit equation for X which is visibly of degree three: suppose that C is parameterized by $[1:t] \mapsto [1:t:t^2:t^3:t^4]$. Then for a general point $[x_0:\cdots:x_4] \in X$ there exist s, t, λ, μ such that $x_i = \lambda t^i + \mu s^i$. Elimination of s, t, λ, μ is straightforward and we find that

$$x_0(x_3^2 - x_2x_4) + (x_2^3 + x_1^2x_4 - 2x_1x_2x_3) = 0 \tag{4.1}$$

is the cubic equation that defines X. Note that X contains the union T_C of tangent lines of C (as the secants of the infinitesimally near points).

4.1 Orbit decomposition of the chordal cubic

It is convenient here to take a more abstract approach and construct everything in terms of a complex vector space W of dimension two. Let us write P_k for $\mathbb{P}(\mathrm{Sym}^k W)$, allowing ourselves to suppress the subscript when $k=1$. We identify P_k with the linear system of effective degree k divisors on P and identify P with its image in P_k (for $k>0$) by means of $p \in P \to kp \in P_k$.

Our curve C is now identified with $P \subset P_4$ and $X \subset P_4$ parameterizes the quartics in P that can be written as the sum of two fourth powers, or at least infinitesimally so (these are quartic forms that are divisible by a third power). In the last case this means that the point in question lies on a tangent line of C, i.e. in T_C. If we interpret P_4 as the linear system of effective degree four divisors x on P, then $x \in X$ if and only if it is invariant under an involution which fixes at least one point of $\mathrm{supp}(x)$. We then see that $\mathrm{PGL}(W)$ has three orbits in X: C (divisors of the form $4p$), $T_C - C$ (divisors of the form $3p+q$ with $q \neq p$) and $X - T_C$ (reduced divisors). The singular locus of X is C.

4.2 Cohomology of the smooth part

We use the classical theory of Lefschetz pencils to determine the χ-Betti numbers of Y_{reg}. Both X and Y have a smooth singular locus with a transversal singularity whose link is a rational homology sphere. So either is a rational homology manifold and therefore its cohomology satisfies Poincaré duality over $\mathbb{Q}$. The Gysin sequence for the pair (Y, C),

$$\cdots \to H^{k-4}(C;\mathbb{Q}) \to H^k(Y;\mathbb{Q}) \to H^k(Y_{\mathrm{reg}};\mathbb{Q}) \to H^{k-3}(C;\mathbb{Q}) \to \cdots,$$

shows that $H^k(Y;\mathbb{C})_\chi \to H^k(Y_{\mathrm{reg}};\mathbb{C})_\chi$ is an isomorphism (and likewise for $\bar\chi$, of course), so that either comes with a nondegenerate $\mathbb{Q}(\zeta)$-valued Hermitian form. Let us now perform an Euler characteristic computation. The projectively completed tangent bundle T_C of C has Euler characteristic equal to $e(C).e(\mathbb{P}^1) = 4$. Since $X - T_C$ is fibered (over $P_2 - P$) with fiber $\mathbb{C}^\times$, its Euler characteristic is zero. Hence, $e(X) = 4$ and $e(P_4 - X) = 5 - 4 = 1$. Since $Y \to P_4$ is a μ_3-cover totally ramified over X, we have $e(Y) = 3e(P_4 - X) + e(X) = 3 + 4 = 7$. The μ_3-orbit space of Y can be identified with P_4, which has Euler characteristic five. Hence, $e_\chi(Y) = e_{\bar\chi}(Y) = \frac{1}{2}(7-5) = 1$.

In order to prove the more precise Proposition 2.2 we choose a Lefschetz pencil for X. Such a pencil is defined as a generic 2-plane $A \subset P_4$ (its axis). The genericity assumption entails that it avoids C and meets X_{reg} transversally. A generic hyperplane H through A meets X in a cubic surface X_H whose singular points are $C \cap H$. The latter intersection is transversal and so each of the four points of $C \cap H$ is a singularity of type A_1. The cubic surfaces with four A_1-singularities form a single projective equivalence class.

Lemma 4.1. *Let $Z \subset \mathbb{P}^3$ be a cubic surface with four A_1-singularities and let $K \to \mathbb{P}^3$ be the normal μ_3-cover that totally ramifies over Z. Then $H^\bullet(K;\mathbb{C})_\chi = H^3(K;\mathbb{C})_\chi$ and the latter has dimension one.*

Proof. We first observe that K is a cubic threefold with four A_2-singularities. Since the vanishing homology of an A_2-singularity is a symplectic unimodular lattice of rank two, we find that the primitive cohomology of K is concentrated in degree three and is of rank $4 \times 2 = 8$ less than the rank of the primitive cohomology of a smooth cubic in that dimension (which is 10). It follows that $H_\bullet(K;\mathbb{C})_\chi = H_3(K;\mathbb{C})_\chi$ has dimension one. $\square$

Lemma 4.2. *If H is a tangent hyperplane of X at some $p \in X - T_C$, then $H^\bullet(Y_H;\mathbb{C})_\chi$ is trivial.*

Proof. Since $\mathrm{SL}(W)$ is transitive on $X - T_C$, any $p \in X - T_C$ will do. We use (4.1) of X, $x_0x_3^2 - x_0x_2x_4 + x_2^3 + x_1^2x_4 - 2x_1x_2x_3$, and we take $p = [1:0:0:0:1]$. Then the tangent hyperplane at p is given by $x_2 = 0$ and so X_H is given by $x_0x_3^2 + x_1^2x_4$. Hence, Y_H is given by $y^3 = x_0x_3^2 + x_1^2x_4$. The singular set of this threefold is the line ℓ defined by $y = x_1 = x_3 = 0$. Consider the $\mathbb{C}^\times$-action on Y_H defined by

$$\lambda[x_0 : x_1 : x_3 : x_4 : y] := [x_0 : \lambda^3 x_1 : \lambda^3 x_3 : x_4 : \lambda^2 y].$$

We note that the fixed point set is the union of ℓ and the line ℓ' defined by $y = x_0 = x_4 = 0$. This action provides a contraction of $Y_H - \ell'$ onto ℓ so that $\ell \subset Y_H - \ell'$ is a homotopy equivalence. This shows that $H^\bullet(Y_H - \ell';\mathbb{C})_\chi = 0$. We also have $H^\bullet(\ell';\mathbb{C})_\chi = 0$, of course. Since ℓ' lies in the smooth part of Y_H, the Gysin sequence can be applied to the pair (Y_H, ℓ'). We thus find that $H^\bullet(Y_H;\mathbb{C})_\chi = 0$. $\square$

Proof of Proposition 2.2. Since Y is a rational homology manifold, $H^\bullet(Y;\mathbb{C})_\chi$ satisfies Poincaré duality in the sense that $H^\bullet(Y;\mathbb{C})_\chi$ pairs nondegenerately with $H^{8-\bullet}(Y;\mathbb{C})_{\bar\chi}$. In particular, $H^4(Y;\mathbb{C})_\chi$ comes with a nondegenerate Hermitian form. As to the assertions concerning $H^\bullet(Y:\mathbb{C})_\chi$, since we already verified that $e_\chi(Y) = 1$, it suffices to prove that $H^k(Y;\mathbb{C})_\chi = 0$ for $k \leqslant 3$.

Let L_A be the pencil of hyperplanes in $\mathbb{P}^4$ passing through A and denote by $\tilde X \subset \mathbb{P}^4 \times L_A$ (respectively, $\tilde Y \subset \mathbb{P}^5 \times L_A$) the corresponding Lefschetz pencils. The projection $\tilde Y \to Y$ contracts $Y_A \times L_A$ along its projection onto L_A and so $H^\bullet(Y;\mathbb{C}) \to H^\bullet(\tilde Y;\mathbb{C})$ is injective. We prove that $H^k(\tilde Y;\mathbb{C})_\chi = 0$ for $k \leqslant 3$. To this end, consider the χ-Leray spectral sequence of the projection $\pi : \tilde Y \to L_A$. Lemma 4.1 shows that $(R^q\pi_*\mathbb{C}_{\tilde Y})_\chi$ is zero unless $q = 3$ so that the Leray spectral sequence for χ-cohomology degenerates and

$$H^k(\tilde Y;\mathbb{C})_\chi = H^{k-3}(L_A;(R^3\pi_*\mathbb{C}_{\tilde Y})_\chi).$$

In particular, $H^k(\tilde Y;\mathbb{C})_\chi = 0$ for $k \leqslant 2$. Lemma 4.2 implies that a stalk of $(R^q\pi_*\mathbb{C}_{\tilde Y})_\chi$ is zero if the associated hyperplane is tangent to X_{reg}. Since there are such hyperplanes, $H^3(\tilde Y;\mathbb{C})_\chi = H^0(L_A;(R^3\pi_*\mathbb{C}_{\tilde Y})_\chi) = 0$ also. The rest of the proposition follows from an application of Lemma 1.4. $\square$

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We happily dedicate this paper to Jozef Steenbrink on the occasion of his 60th birthday. One of us (Looijenga) has known Jozef since he was a graduate student. Many mathematical exchanges between us have taken place since (a great deal more than the page count of our joint work (3 pp) might suggest) and of course to mutual benefit. For me, one of these benefits has been to acquire a

good understanding of a subject that also plays a role in the present paper, namely mixed Hodge theory.

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