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DIAGONALS OF DOUBLY STOCHASTIC MATRICES

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Let A be an additive abelian group and $D_n(A)$ the set of those $n \times n$ matrices over A all of whose row and column sums are equal. Such matrices can be regarded as a possible generalization of doubly stochastic real matrices; alternatively, if A is a commutative ring, it turns out that $D_n(A)$ is exactly the image of the permutation representation of S_n, the symmetric group of degree n, over A.

We recall that each $\sigma \in S_n$ determines a "σth diagonal sum" of an $n \times n$ matrix X, namely $\sum_{j=1}^n x_{\sigma(j),j}$; there are thus $n!$ diagonal sums of X. Our aim is to prove

THEOREM. *Suppose A has no n-torsion. If X and Y are elements of $D_n(A)$ such that more than $n!-(n-1)!$ corresponding diagonal sums of X and Y are equal, then $X = Y$. This bound is the best possible if $A \neq 0$.*

In the doubly stochastic case, this was proved by Wang [1], using a result of Marcus and Minc [2]. However, his method involves the taking of logarithms and does not seem to generalize. In our attempt to find a purely algebraic proof of this fact, we were led to the above theorem and other interesting algebraic properties of $D_n(A)$.

It is easy to check that $D_n(A)$ is an abelian group and that if A is a ring, so is $D_n(A)$. We shall find it convenient to identify a permutation $\sigma \in S_n$ with the matrix in $D_n(Z)$, where Z denotes the ring of integers, whose (i, j)th coefficient is 1 if $\sigma(j) = i$ and 0 otherwise.

PROPOSITION 1. *Suppose P is a set of permutations in S_n with more than $n!-(n-1)!$ elements. Then P generates $D_n(Z)$ as an abelian group.*

Proof. We use induction on n; the assertion clearly holds if n is 1 or 2.

Let $P_k = \{\sigma \in P \mid \sigma(1) = k\}$; each P_k has at most $(n-1)!$ elements and $P = P_1 \cup \ldots \cup P_n$. If some P_k were empty, P would have at most $(n-1)(n-1)! = n!-(n-1)!$ elements, contrary to assumption; we can therefore choose a permutation σ_k from each P_k. On the other hand, there exists an index i such that P_i has more than

$$(n!-(n-1)!)/n > (n-1)!-(n-2)!$$

elements; subtracting P_i from P takes away at most $(n-1)!$ elements, so that there must exist another index j such that P_j has more than

$$(n!-2(n-1)!)/(n-1) = (n-1)!-(n-2)!$$

elements.

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Let $D^k_{n-1}(Z)$ be the image in $D_n(Z)$ of the imbedding $D_{n-1}(Z) \to D_n(Z)$ given by

$$X \mapsto (1k)\begin{pmatrix} s & 0 \\ 0 & X \end{pmatrix}$$

where s is the common row and column sum of X. The induction assumption implies that P certainly generates $D^i_{n-1}(Z)$ and $D^j_{n-1}(Z)$.

Now suppose $M \in D_n(Z)$; subtracting $\sum_{k=1}^n m_{k1}\sigma_k$ from M allows us to assume that the first column of M is zero. Since $n \geq 3$, one can choose an index $r \neq i, j$ and write $M = M_1 + M_2$, where $M_2 = \sum_{q=2}^n m_{iq}(E_{iq} - E_{rq})$ and the E_{pq} are the usual matrix units. Since $M_1 \in D^i_{n-1}(Z)$ and $M_2 \in D^j_{n-1}(Z)$, we conclude that P generates all of $D_n(Z)$.

In particular, it follows that elements of $D_n(Z)$ can be written as sums of permutation matrices. More explicitly, if $X \in D_n(A)$, we have the formula

$$X = x_{11}I + \sum_{i=2}^{n} (x_{i1} + x_{1i} + x_{ii} - s)(1i) + \sum_{\substack{i,j=2 \\ i \neq j}}^{n} x_{ij}(1i)(1j),$$

where s is the common row and column sum of X. Furthermore, this representation is unique; therefore the $(n-1)^2+1$ permutations I, $(1i)$, $(1i)(1j)$ form a basis of $D_n(Z)$ and in general

$$D_n(A) \cong A \otimes_Z D_n(Z) \cong A^{(n-1)^2+1}.$$

It is interesting to note the formula for a permutation $\sigma \in S_n$,

$$\sigma = I + \sum_{j \notin F} ((1\sigma(j))(1j) - (1j)),$$

where F is the set of fixed points of σ.

Consider the bilinear map

$$B: D_n(Z) \times D_n(A) \to A$$

defined by $B(M, X) = \operatorname{tr}(MX)$. Since $B(\sigma, X) = \sigma$th diagonal sum of X, saying that more than $n! - (n-1)!$ corresponding diagonal sums of X and Y in $D_n(A)$ are equal amounts to saying that $B(\sigma, X - Y) = 0$ for more than $n! - (n-1)!$ permutations σ. By Proposition 1, this implies that $B(M, X-Y) = 0$ for all $M \in D_n(Z)$, i.e. $X - Y \in D_n(Z)^\perp$.

PROPOSITION 2. *$D_n(Z)^\perp$ consists of matrices of the form*

$$(i)\begin{pmatrix} a & c_2+a & \cdots & c_n+a \\ b_2+a & & \vdots & \\ \vdots & \cdots & b_i+c_j+a & \\ b_n+a & & & \end{pmatrix} \quad \text{(column } (j)\text{)}$$

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where (i) $nb_i = nc_i = 0$

(ii) $b_2 + \cdots + b_n = c_2 + \cdots + c_n$

(iii) $na + b_2 + \cdots + b_n + c_2 + \cdots + c_n = 0.$

Proof. Suppose $X \in D_n(Z)^\perp$; then if $i \neq 1, j \neq 1, i \neq j$,

$$B((1j)(1i)-(1j), X) = x_{ij} + x_{1i} - x_{1j} - x_{ii} = 0$$

$$B((1i)-I, X) = x_{1i} + x_{i1} - x_{11} - x_{ii} = 0.$$

Therefore $x_{ij} = x_{1j} + x_{i1} - x_{11}$ and $x_{ii} = x_{1i} + x_{i1} - x_{11}$. Letting $a = x_{11}$, $b_i = x_{i1} - x_{11}$ and $c_j = x_{1j} - x_{11}$, we conclude that X has the required form. Conditions (i) and (ii) result from the fact that $X \in D_n(A)$, while (iii) is just the fact that $B(I, X) = 0$. The converse is clear.

COROLLARY.

(i) *$D_n(Z)^\perp$ is annihilated by $n^2 I$.*

(ii) *If A has no n-torsion, $D_n(Z)^\perp = 0$.*

(iii) *Suppose A is a field of characteristic $p \mid n$. Then*

$$\dim_A D_n(Z)^\perp = \begin{cases} 2n-3 & \text{if } p \text{ is odd} \\ 2n-2 & \text{if } p = 2. \end{cases}$$

In view of the argument preceding Proposition 2, the theorem now follows from part (ii) of the Corollary. To see that the bound is the best possible, choose a non-zero $x \in A$ and consider the matrix (due to Wang)

$$x \cdot \begin{pmatrix} n^2-2n+2 & 2-n & \ldots & 2-n \\ 2-n & 2 & \ldots & 2 \\ \vdots & & & \\ 2-n & 2 & \ldots & 2 \end{pmatrix}.$$

The $n! - (n-1)!$ diagonal sums corresponding to permutations σ for which $\sigma(1) \neq 1$ are all zero, while the matrix is not zero since A has no n-torsion.

REFERENCES

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