



On spectra of Hankel operators on the polydisc

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Abstract. We give sufficient conditions for the essential spectrum of the Hermitian square of a class of Hankel operators on the Bergman space of the polydisc to contain intervals. We also compute the spectrum in case the symbol is a monomial.

The study of spectral properties of Toeplitz and Hankel operators acting on the Bergman space is a difficult topic. In recent years, some progress has been made in understanding spectral properties of Toeplitz operators. We highlight the result by Sundberg and Zheng [SZ10] who have proved that the spectra and essential spectra of Toeplitz operators on the unit disc \mathbb{D} need not be connected. Their result shows a sharp difference with the spectra of Toeplitz operators on the Hardy space. As is well-known, Widom [Wid64] showed that for any L^∞ symbol ψ on the unit circle, the spectrum of the Toeplitz operator T_ψ is a connected subset of \mathbb{C} . Similarly Douglas [Dou98, Theorem 7.45] proved that the essential spectrum of T_ψ is also connected. In this context, we also mention papers [ZZ16, GZZ23] which study spectra of certain classes of the Bergman Toeplitz operators on the unit disk. On the other hand, we are not aware of much work done about the spectral properties of Hankel operators.

Since the Hankel operator H_ψ does not map the Bergman space into itself, we will consider the Hermitian square $H_\psi^* H_\psi$ and we will obtain some initial results about the spectrum. In this paper, we will only be concerned with symbols that are continuous up to the boundary of the domain. In the case of the unit disc, H_ψ is compact for $\psi \in C(\overline{\mathbb{D}})$ (see, for instance, [ACM82, Proposition 8]); hence the spectrum of $H_\psi^* H_\psi$ is discrete. Like the situation on the unit disc, on bounded strongly pseudoconvex domains in \mathbb{C}^n , H_ψ is compact [ČŠ09]. Then on such domains the spectrum of $H_\psi^* H_\psi$ is discrete as well. In this paper, we focus on the polydisc and find some sufficient conditions in terms of behavior ψ on the boundary so that the spectrum of $H_\psi^* H_\psi$ contains intervals. One of the reasons for this departure from the one-dimensional and strongly pseudoconvex case is the fact that Hankel operators with symbols continuous on \mathbb{D}^2 may not be compact (see [CŞ18a, Le10]). We note that, in case the symbol is smooth on the closure the same result was proven in [ČŠ09]. With regard to Sundberg–Zheng result, from the same papers mentioned in this paragraph,

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we know that if the symbol ψ is holomorphic along any disc in the boundary of a convex domain, then H_ψ is compact [ČŠ09, ÇŞ23, Zim23]; hence the spectrum of $H_\psi^*H_\psi$ is disconnected.

At this point we would like to mention the well known formula connecting Hankel and Toeplitz operators

$$H_\psi^*H_\psi = T_{|\psi|^2} - T_\psi^*T_\psi.$$

Thus our results could shed a new light on the spectra of semicommutators of Toeplitz operators. For more information about Hankel and Toeplitz operators on the unit disc, we refer the reader to a standard reference [Zhu07].

In the rest of the paper, we prove two sufficient conditions for the spectrum of $H_\psi^*H_\psi$ to contain intervals. We also compute the spectrum of $H_\psi^*H_\psi$ in the case ψ is monomial.

1 Main results

Let Ω be a bounded domain in \mathbb{C}^n and $A^2(\Omega)$ denote the Bergman space, the set of square integrable holomorphic functions on Ω . We denote the Bergman projection by P^Ω . Then for a bounded measurable function ψ on Ω , Hankel operator H_ψ^Ω on $A^2(\Omega)$ with symbol ψ is defined as

$$H_\psi^\Omega f = (I - P^\Omega)(\psi f),$$

for $f \in A^2(\Omega)$. Here I denotes the identity operator. For simplicity, we will simply write H_ψ when there is no confusion about the domain. We note that the Toeplitz operator T_ψ is defined as

$$T_\psi f = P(\psi f),$$

for $f \in A^2(\Omega)$.

The spectrum for $H_\psi^*H_\psi$, for symbols continuous up to the boundary, is a discrete set for a large class of domains on which the operator is compact [ČŠ09]. So it would be interesting to know the sufficient conditions for the spectrum to contain intervals.

Let $\sigma(T)$ denote the spectrum of a linear map T . The set of eigenvalues is called the point spectrum $\sigma_p(T)$ [Con90, VII Definition 6.2]. The discrete spectrum, $\sigma_d(T)$ is composed of eigenvalues with finite (algebraic) multiplicity that are isolated points of $\sigma(T)$. Finally, the essential spectrum $\sigma_e(T)$ is defined as $\sigma_e(T) = \sigma(T) \setminus \sigma_d(T)$. A characterization of the essential spectrum in our set-up, called Weyl's Criterion, is presented in Theorem A below. We note that for self-adjoint operators T_1 and T_2 we have $\sigma_e(T_1) = \sigma_e(T_2)$ if $T_1 - T_2$ is compact (see, for instance, [HS96, Theorems 5.10 and 7.2] or [Con90, XI Proposition 4.2]).

Our first result is about the essential spectrum of $H_\psi^*H_\psi$ on $A^2(\mathbb{D}^n)$ when ψ is a product of two functions that depend on different variables. We note that, $H_\varphi^{\mathbb{D}^{n-1}*}$ in the theorem below denotes the adjoint of the operator $H_\varphi^{\mathbb{D}^{n-1}}$.

Theorem 1.1 Let $\varphi \in C(\overline{\mathbb{D}^{n-1}})$, $\chi \in C(\overline{\mathbb{D}})$ and $\psi(z', z_n) = \varphi(z')\chi(z_n)$ for $z' \in \overline{\mathbb{D}^{n-1}}$, and $z_n \in \overline{\mathbb{D}}$. Then the essential spectrum of $H_\psi^*H_\psi$ contains the set

$$\left\{ |\chi(q)|^2 \mu : q \in b\mathbb{D}, \mu \in \sigma(H_\varphi^{\mathbb{D}^{n-1}*} H_\varphi^{\mathbb{D}^{n-1}}) \right\}.$$

In case the symbol is separable we have the following corollary, proof of which will be presented in Section 2.

Corollary 1.2 Let $\chi_j \in C(\overline{\mathbb{D}})$ for $j = 1, \dots, n$ and $\psi(z_1, \dots, z_n) = \chi_1(z_1)\chi_2(z_2)\dots\chi_n(z_n)$. Then the essential spectrum of $H_\psi^*H_\psi$ contains the set

$$\bigcup_{j=1}^n \left\{ \mu \prod_{k \neq j} |\chi_k(q_k)|^2 : q_k \in b\mathbb{D}, \mu \in \sigma(H_{\chi_j}^{\mathbb{D}^*} H_{\chi_j}^{\mathbb{D}}) \right\}.$$

In the next theorem, we give a sufficient condition for the spectrum to contain an interval for more general symbols than the ones in Theorem 1.1.

Theorem 1.3 Let $\psi \in C(\overline{\mathbb{D}^n})$, $1 \leq k \leq n - 1$, and $\psi_q(z') = \psi(z', q)$ for $z' \in \mathbb{D}^{n-1}$ and $q \in b\mathbb{D}$. Assume that $q \rightarrow \|H_{\psi_q}^{\mathbb{D}^{n-1}}\|$ is non-constant. Then the essential spectrum of $H_\psi^*H_\psi$ contains an open interval.

The following corollary can be proved using Theorems 1.1 or 1.3. We will mention both in Section 2.

Corollary 1.4 Let $\varphi \in C(\overline{\mathbb{D}^{n-1}})$ and $\chi \in C(\overline{\mathbb{D}})$ such that $|\chi|$ is not constant on the unit circle and φ is not holomorphic. Then the essential spectrum of $H_\psi^*H_\psi$ contains an open interval where $\psi(z', z_n) = \varphi(z')\chi(z_n)$ for $z' \in \overline{\mathbb{D}^{n-1}}$ and $z_n \in \overline{\mathbb{D}}$.

Example 1.5 Let $\psi(z_1, z_2) = \bar{z}_1(\bar{z}_2 + 1)$. Since \bar{z}_1 is not holomorphic and $|\bar{z}_2 + 1|$ is not constant on the unit circle, Corollary 1.4 implies that $\sigma_e(H_\psi^*H_\psi)$ contains an interval. It is worth noting that (see Theorem 1.6 below) the spectrum of the Hermitian square of a Hankel operator with a monomial symbol is a discrete set; yet for the a quadratic polynomial $\bar{z}_1(\bar{z}_2 + 1)$ the spectrum contains an interval.

In Theorem 1.6 below, we give a complete characterization of the spectrum on the polydisc for monomial symbols.

We note that \mathbb{N}_0 denotes the set of non-negative integers, $\{0, 1, 2, \dots\}$. For $\mathbf{m}, \mathbf{n}, \alpha \in \mathbb{N}_0^n$ and $\emptyset \neq B \subset B_n = \{1, 2, 3, \dots, n\}$ we define

$$\lambda_{\mathbf{n}, \mathbf{m}, \alpha, B} = \prod_{k \in B} \frac{\alpha_k + 1}{\alpha_k + n_k + m_k + 1},$$

whenever $\alpha_k < m_k - n_k$ for some $k \in B$ and

$$\lambda_{\mathbf{n}, \mathbf{m}, \alpha, B} = \prod_{k \in B} \frac{\alpha_k + 1}{\alpha_k + n_k + m_k + 1} - \prod_{k \in B} \frac{(\alpha_k + 1)(\alpha_k + n_k - m_k + 1)}{(\alpha_k + n_k + 1)^2},$$

whenever $\alpha_k \geq m_k - n_k$ for all $k \in B$.

Theorem 1.6 Let $\psi(z) = z^n \bar{z}^m$ for some $\mathbf{m}, \mathbf{n} \in \mathbb{N}_0^n$. Then $H_\psi^* H_\psi$ on $A^2(\mathbb{D}^n)$ has the following spectrum

$$\sigma(H_\psi^* H_\psi) = \{0\} \cup \{\lambda_{\mathbf{n}, \mathbf{m}, \alpha, B} : \alpha \in \mathbb{N}_0^n, \emptyset \neq B \subset B_n\}.$$

Furthermore, if $n_k + m_k \geq 1$ for all $k \in B_n$ and $m_k \geq 1$ for some $k \in B_n$ then all of the eigenvalues have finite multiplicities. On the other hand, if $n_k + m_k = 0$ for some $k \in B_n$ then all of the eigenvalues have infinite multiplicities.

Since the spectrum of $H_\psi^* H_\psi$ on $A^2(\mathbb{D}^n)$ for $\psi(z) = z^n \bar{z}^m$ contains countably many points it has empty interior.

Corollary 1.7 Let $\psi(z) = z^n \bar{z}^m$ for some $\mathbf{m}, \mathbf{n} \in \mathbb{N}_0^n$. Then the spectrum of $H_\psi^* H_\psi$ on $A^2(\mathbb{D}^n)$ has empty interior.

In case the symbol is a pure conjugate holomorphic monomial in \mathbb{C}^2 , we have the following corollary.

Corollary 1.8 Let $\psi(z_1, z_2) = \bar{z}_1^n \bar{z}_2^m$ on \mathbb{D}^2 for some positive integers n, m . Then

$$\begin{aligned} \sigma(H_\psi^* H_\psi) = \{0\} \cup & \bigcup_{\alpha_1=0}^{n-1} \bigcup_{\alpha_2=0}^{m-1} \left\{ \frac{\alpha_1 + 1}{\alpha_1 + n + 1}, \frac{\alpha_2 + 1}{\alpha_2 + m + 1} \right\} \\ & \bigcup_{\alpha_1=n}^{\infty} \bigcup_{\alpha_2=m}^{\infty} \left\{ \frac{n^2}{(\alpha_1 + n + 1)(\alpha_1 + 1)}, \frac{m^2}{(\alpha_2 + m + 1)(\alpha_2 + 1)} \right\} \\ & \bigcup_{\alpha_1=n}^{\infty} \bigcup_{\alpha_2=m}^{\infty} \left\{ \frac{n^2(\alpha_2 + 1)^2 + m^2(\alpha_1 + 1)^2 - n^2 m^2}{(\alpha_1 + n + 1)(\alpha_2 + m + 1)(\alpha_1 + 1)(\alpha_2 + 1)} \right\} \\ & \bigcup_{\alpha_1=0}^{n-1} \bigcup_{\alpha_2=0}^{\infty} \left\{ \frac{(\alpha_1 + 1)(\alpha_2 + 1)}{(\alpha_1 + n + 1)(\alpha_2 + m + 1)} \right\} \\ & \bigcup_{\alpha_1=0}^{\infty} \bigcup_{\alpha_2=0}^{m-1} \left\{ \frac{(\alpha_1 + 1)(\alpha_2 + 1)}{(\alpha_1 + n + 1)(\alpha_2 + m + 1)} \right\}. \end{aligned}$$

Furthermore, all of the eigenvalues are of finite multiplicity.

A precursor to the next corollary has appeared in [Č18b, Remark 4.2].

Corollary 1.9 Let $\psi(z_1, z_2) = \bar{z}_1^n$ on \mathbb{D}^2 for some positive integer n . Then

$$\sigma(H_\psi^* H_\psi) = \{0\} \cup \bigcup_{\alpha_1=0}^{n-1} \left\{ \frac{\alpha_1 + 1}{\alpha_1 + n + 1} \right\} \bigcup_{\alpha_1=n}^{\infty} \left\{ \frac{n^2}{(\alpha_1 + n + 1)(\alpha_1 + 1)} \right\}.$$

Furthermore, all of the eigenvalues are of infinite multiplicity.

2 Proof of Theorems 1.1 and 1.3

In the proofs, below K^Ω and $k_z^\Omega = K_z^\Omega / \|K_z^\Omega\|$ denote the Bergman kernel and the normalized Bergman kernel of Ω , respectively. We will drop the superscript Ω when the domain is clear.

To prove Theorems 1.1 and 1.3, we will need the following result. We refer the reader to [HS96] for a proof.

Theorem A (Weyl’s Criterion) *Let A be a bounded self-adjoint linear operator on a Hilbert space H . Then*

- i. λ is in the spectrum of A if and only if there exists a sequence $\{u_n\} \subset H$ such that $\|u_n\| = 1$ for all n and $\|(A - \lambda)u_n\| \rightarrow 0$ as $n \rightarrow \infty$.
- ii. λ is in the essential spectrum of A if and only if there exists a sequence $\{u_n\} \subset H$ such that $\|u_n\| = 1$ for all n , $u_n \rightarrow 0$ weakly, and $\|(A - \lambda)u_n\| \rightarrow 0$ as $n \rightarrow \infty$.

The following lemma is probably well known. We include it here for completeness. We would like to thank Tomas Miguel P. Rodriguez for the proof.

Lemma 2.1 *Let Ω be a domain in \mathbb{C}^n and $\{f_j\}$ be a sequence in $A^2(\Omega)$. Then $f_j \rightarrow 0$ weakly as $j \rightarrow \infty$ if and only if $\{f_j\}$ is bounded in $A^2(\Omega)$ and $f_j \rightarrow 0$ as $j \rightarrow \infty$ uniformly on compact subsets in Ω .*

Proof. Let us assume that $\{f_j\}$ is bounded in $A^2(\Omega)$ and $f_j \rightarrow 0$ as $j \rightarrow \infty$ uniformly on compact subsets in Ω . Let $f \in A^2(\Omega)$ and $\varepsilon > 0$. Then there exists a compact set $K \subset \Omega$ such that $\|f\|_{L^2(\Omega \setminus K)} < \varepsilon$. Then for large j , we have $\sup\{|f_j(z)| : z \in K\} < \varepsilon$ and

$$\begin{aligned} |\langle f, f_j \rangle| &\leq \|f\|_{L^2(K)} \|f_j\|_{L^2(K)} + \|f\|_{L^2(\Omega \setminus K)} \|f_j\|_{L^2(\Omega)} \\ &\leq \varepsilon \left(\|f\|_{L^2(K)} \sqrt{V(K)} + \sup\{\|f_j\|_{L^2(\Omega)} : j \in \mathbb{N}\} \right), \end{aligned}$$

where $V(K)$ denotes the Lebesgue volume of K . Hence $\langle f, f_j \rangle \rightarrow 0$. That is, $f_j \rightarrow 0$ weakly as $j \rightarrow \infty$.

For the converse, we assume that $f_j \rightarrow 0$ weakly as $j \rightarrow \infty$. We define $S_{f_j}(f) = \langle f, f_j \rangle$ for $f \in A^2(\Omega)$. Then $S_{f_j}(f) \rightarrow 0$ as $j \rightarrow \infty$. That is $\sup\{|S_{f_j}(f)| : j \in \mathbb{N}\} < \infty$ for all $f \in A^2(\Omega)$. Then by the uniform boundedness principle $\sup\{\|S_{f_j}\| : j \in \mathbb{N}\} = \sup\{\|f_j\| : j \in \mathbb{N}\} < \infty$. That is $\{f_j\}$ is bounded in $A^2(\Omega)$. Furthermore, $f_j(z) = \langle f_j, K_z \rangle \rightarrow 0$ as $j \rightarrow \infty$ for all $z \in \Omega$. We will use this fact to conclude that $f_j \rightarrow 0$ uniformly on compact subsets as follows. Let K be a compact subset of Ω and $\{f_{j_k}\}$ be a subsequence of $\{f_j\}$. Then Montel’s theorem implies that there is a further subsequence $\{f_{j_{k_l}}\}$ that is convergent to a holomorphic function f uniformly on K . However, $f_j(z) = \langle f_j, K_z \rangle \rightarrow 0$ as $j \rightarrow \infty$. Then $f = 0$. Therefore, every subsequence $\{f_{j_k}\}$ has a further subsequence $\{f_{j_{k_l}}\}$ converging to zero uniformly on K . Then $f_j \rightarrow 0$ uniformly on K as $j \rightarrow \infty$. Since K is arbitrary we conclude that $f_j \rightarrow 0$ uniformly on compact subsets. ■

Proof of Theorem 1.1 Let $q \in b\mathbb{D}$. For $p \in \mathbb{D}$ and $j \in \mathbb{N}$ we define

$$f_{j,p}(z', z_n) = g_j(z') k_p^{\mathbb{D}}(z_n),$$

where $z' = (z_1, \dots, z_{n-1})$, $k_p^{\mathbb{D}}$ is the normalized kernel for $A^2(\mathbb{D})$ centered at p and $\{g_j\} \subset A^2(\mathbb{D}^{n-1})$, to be determined later, such that $\|g_j\|_{L^2(\mathbb{D}^{n-1})} = 1$. We note that $\{g_j\}$ is uniformly bounded on compact subsets of \mathbb{D}^{n-1} . Then $\sup\{|f_{j,p}(z)| : j \in \mathbb{N}, z \in$

$K\} \rightarrow 0$ as $p \rightarrow q$ for any compact set $K \subset \mathbb{D}^n$. Hence, by Lemma 2.1, for any $p_j \rightarrow q$ we conclude that $f_{j,p_j} \rightarrow 0$ weakly $j \rightarrow \infty$.

Let us fix j and denote $\psi_q(z', z_n) = \psi(z', q) = \chi(q)\varphi(z')$. Since ψ_q is independent of z_n and

$$P(fg)(z) = (P^{\mathbb{D}^{n-1}} f)(z') \cdot (P^{\mathbb{D}} g)(z_n),$$

whenever f is a function of z' and g is a function of z_n , we have

$$\begin{aligned} H_{\psi_q} f_{j,p} &= \psi_q g_j k_p^{\mathbb{D}} - P(\psi_q g_j k_p^{\mathbb{D}}) \\ (2.1) \qquad &= \chi(q)\varphi g_j k_p^{\mathbb{D}} - \chi(q) \left(P^{\mathbb{D}^{n-1}}(\varphi g_j) \right) k_p^{\mathbb{D}} \\ &= \chi(q) (H_{\varphi}^{\mathbb{D}^{n-1}} g_j) k_p^{\mathbb{D}}. \end{aligned}$$

Then $\|H_{\psi_q} f_{j,p}\| = |\chi(q)| \|H_{\varphi}^{\mathbb{D}^{n-1}} g_j\|_{L^2(\mathbb{D}^{n-1})}$. Now we write $\psi = \psi - \psi_q + \psi_q$ and

$$H_{\psi}^* H_{\psi} f_{j,p} = H_{\psi_q}^* H_{\psi_q} f_{j,p} + H_{\psi}^* H_{\psi - \psi_q} f_{j,p} + H_{\psi - \psi_q}^* H_{\psi_q} f_{j,p}.$$

Hence for $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} (2.2) \qquad &\left\| H_{\psi_q}^* H_{\psi_q} f_{j,p} - \lambda f_{j,p} \right\| - \left\| H_{\psi}^* H_{\psi - \psi_q} f_{j,p} + H_{\psi - \psi_q}^* H_{\psi_q} f_{j,p} \right\| \\ &\leq \left\| H_{\psi}^* H_{\psi} f_{j,p} - \lambda f_{j,p} \right\| \\ &\leq \left\| H_{\psi_q}^* H_{\psi_q} f_{j,p} - \lambda f_{j,p} \right\| \\ &\quad + \left\| H_{\psi}^* H_{\psi - \psi_q} f_{j,p} + H_{\psi - \psi_q}^* H_{\psi_q} f_{j,p} \right\|. \end{aligned}$$

Let $\{h_j\}$ be a bounded sequence in $L^2(\mathbb{D}^{n-1})$. Then $\|(\psi - \psi_q)h_j k_p^{\mathbb{D}}\| \rightarrow 0$ as $p \rightarrow q$ for any fixed j . This can be seen as follows. By continuity of ψ , for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\psi(z', z_n) - \psi_q(z', z_n)| < \varepsilon \text{ for } |z_n - q| < \delta.$$

Then we have

$$\begin{aligned} &\|(\psi - \psi_q)h_j k_p^{\mathbb{D}}\|^2 \\ &= \|(\psi - \psi_q)h_j k_p^{\mathbb{D}}\|_{L^2(\{(z', z_n) \in \mathbb{D}^n : |z_n - q| < \delta\})}^2 + \|(\psi - \psi_q)h_j k_p^{\mathbb{D}}\|_{L^2(\{(z', z_n) \in \mathbb{D}^n : |z_n - q| \geq \delta\})}^2 \\ &\leq \varepsilon^2 \|h_j\|_{L^2(\mathbb{D}^{n-1})}^2 \|k_p^{\mathbb{D}}\|_{L^2(\mathbb{D})}^2 \\ &\quad + \pi \|h_j\|_{L^2(\mathbb{D}^{n-1})}^2 \sup \{ |(\psi(z', z_n) - \psi_q(z', z_n))k_p^{\mathbb{D}}(z_n)|^2 : z' \in \mathbb{D}^{n-1}, |z_n - q| \geq \delta \}. \end{aligned}$$

However,

$$\sup \{ |k_p^{\mathbb{D}}(z_n)| : |z_n - q| \geq \delta \} \rightarrow 0 \text{ as } p \rightarrow q.$$

Then, for any j we have $\limsup_{p \rightarrow q} \|(\psi - \psi_q)h_j k_p^{\mathbb{D}}\| \leq \varepsilon \|h_j\|$ for all $\varepsilon > 0$. Therefore,

$$(2.3) \qquad \lim_{p \rightarrow q} \|(\psi - \psi_q)h_j k_p^{\mathbb{D}}\| = 0 \text{ for any } j.$$

This fact together with $h_j = 1$ imply that for any j we have

$$\|H_{\psi}^* H_{\psi - \psi_q} f_{j,p}\| = \|H_{\psi}^* (I - P) ((\psi - \psi_q) f_{j,p})\| \rightarrow 0 \text{ as } p \rightarrow q.$$

Next we will use the following fact (see [ČŠ14, Lemma 1])

$$(2.4) \quad H_u^* H_v = PM_{\bar{u}}(I - P)M_v \text{ for } u, v \in L^\infty(\Omega),$$

where M_v denotes the multiplication operator by v .

Since \mathbb{D}^n is a product domain and ψ_q is independent of z_n , the fact above, (2.1) and (2.3) with $h_j = H_\varphi^{\mathbb{D}^{n-1}} g_j$ imply that for any j we have

$$\|H_{\psi - \psi_q}^* H_{\psi_q} f_{j,p}\| = |\chi(q)| \left\| P \left((H_\varphi^{\mathbb{D}^{n-1}} g_j)(\bar{\psi} - \bar{\psi}_q) k_p^{\mathbb{D}} \right) \right\| \rightarrow 0 \text{ as } p \rightarrow q$$

and

$$H_{\psi_q}^* H_{\psi_q} f_{j,p} = |\chi(q)|^2 (H_\varphi^{\mathbb{D}^{n-1}*} H_\varphi^{\mathbb{D}^{n-1}} g_j) k_p^{\mathbb{D}}.$$

Then using (2.2) for any j we have

$$(2.5) \quad \|H_{\psi}^* H_{\psi} f_{j,p} - \lambda f_{j,p}\| \rightarrow \left\| |\chi(q)|^2 H_\varphi^{\mathbb{D}^{n-1}*} H_\varphi^{\mathbb{D}^{n-1}} g_j - \lambda g_j \right\|_{L^2(\mathbb{D}^{n-1})} \text{ as } p \rightarrow q.$$

Let $\mu \in \sigma(H_\varphi^{\mathbb{D}^{n-1}*} H_\varphi^{\mathbb{D}^{n-1}})$. Then, by i. in Theorem A, we choose a sequence $\{g_j\} \subset A^2(\mathbb{D}^{n-1})$ such that $\|g_j\| = 1$ for all j and

$$\left\| H_\varphi^{\mathbb{D}^{n-1}*} H_\varphi^{\mathbb{D}^{n-1}} g_j - \mu g_j \right\|_{L^2(\mathbb{D}^{n-1})} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Then for $\lambda = |\chi(q)|^2 \mu$, we have

$$\left\| |\chi(q)|^2 H_\varphi^{\mathbb{D}^{n-1}*} H_\varphi^{\mathbb{D}^{n-1}} g_j - \lambda g_j \right\|_{L^2(\mathbb{D}^{n-1})} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Then using (2.5) and the limit above we choose $p_j \in \mathbb{D}$ such that $p_j \rightarrow q$ as $j \rightarrow \infty$ and

$$\|H_{\psi}^* H_{\psi} f_{j,p_j} - \lambda f_{j,p_j}\| < \left\| |\chi(q)|^2 H_\varphi^{\mathbb{D}^{n-1}*} H_\varphi^{\mathbb{D}^{n-1}} g_j - \lambda g_j \right\|_{L^2(\mathbb{D}^{n-1})} + \frac{1}{j}$$

for all j . Hence

$$\|H_{\psi}^* H_{\psi} f_{j,p_j} - \lambda f_{j,p_j}\| \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Finally, we use ii. in Theorem A to conclude that $\lambda = |\chi(q)|^2 \mu \in \sigma_e(H_{\psi}^* H_{\psi})$ for any $q \in b\mathbb{D}$ and $\mu \in \sigma(H_\varphi^{\mathbb{D}^{n-1}*} H_\varphi^{\mathbb{D}^{n-1}})$. ■

Proof of Corollary 1.2 Without loss of generality let $\mu \in \sigma(H_{\chi_1}^{\mathbb{D}^*} H_{\chi_1}^{\mathbb{D}})$. Then, by Theorem 1.1 for $n = 2$, we have $\mu |\chi_2(q_2)|^2 \in \sigma(H_{\chi_1 \chi_2}^{\mathbb{D}^{2*}} H_{\chi_1 \chi_2}^{\mathbb{D}^2})$. Hence, inductively, we get

$$\mu |\chi_2(q_2) \cdots \chi_n(q_n)|^2 \in \sigma_e(H_{\psi}^* H_{\psi})$$

completing the proof of the corollary. ■

The proof of Theorem 1.3 is similar to the proof of Theorem 1.1. So we will not provide all the details but will highlight the differences and similarities below.

Proof of Theorem 1.3 Let $q \in b\mathbb{D}$ and

$$\lambda_q = \left\| H_{\psi_q}^{\mathbb{D}^{n-1*}} H_{\psi_q}^{\mathbb{D}^{n-1}} \right\|_{L^2(\mathbb{D}^{n-1})} = \left\| H_{\psi_q}^{\mathbb{D}^{n-1}} \right\|_{L^2(\mathbb{D}^{n-1})}^2.$$

Then $\lambda_q \in \sigma(H_{\psi_q}^{\mathbb{D}^{n-1*}} H_{\psi_q}^{\mathbb{D}^{n-1}})$ (see, for instance, [HS96, Theorem 5.14]). By i. in Theorem A there exists $\{g_{j,q}\} \subset A^2(\mathbb{D}^{n-1})$ such that $\|g_{j,q}\| = 1$ and

$$(2.6) \quad \left\| H_{\psi_q}^{\mathbb{D}^{n-1*}} H_{\psi_q}^{\mathbb{D}^{n-1}} g_{j,q} - \lambda_q g_{j,q} \right\|_{L^2(\mathbb{D}^{n-1})} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Let

$$f_{j,q,p}(z', z_n) = g_{j,q}(z') k_p^{\mathbb{D}}(z_n)$$

(again here $k_p^{\mathbb{D}}$ is the normalized kernel for \mathbb{D} centered at p). As in the proof of Theorem 1.1 we have $f_{j,q,p_j} \rightarrow 0$ weakly as $j \rightarrow \infty$ for any $p_j \rightarrow q$. Similar to (2.1), one can show that

$$H_{\psi_q} f_{j,q,p} = (H_{\psi_q}^{\mathbb{D}^{n-1}} g_{j,q}) k_p^{\mathbb{D}}.$$

Then $\|H_{\psi_q} f_{j,q,p}\| = \|H_{\psi_q}^{\mathbb{D}^{n-1}} g_{j,q}\|_{L^2(\mathbb{D}^{n-1})}$ and as in (2.2) we have

$$\begin{aligned} & \left\| H_{\psi_q}^* H_{\psi_q} f_{j,q,p} - \lambda_q f_{j,q,p} \right\| - \left\| H_{\psi}^* H_{\psi-\psi_q} f_{j,q,p} + H_{\psi-\psi_q}^* H_{\psi_q} f_{j,q,p} \right\| \\ & \leq \left\| H_{\psi}^* H_{\psi} f_{j,q,p} - \lambda_q f_{j,q,p} \right\| \\ & \leq \left\| H_{\psi_q}^* H_{\psi_q} f_{j,q,p} - \lambda_q f_{j,q,p} \right\| \\ & \quad + \left\| H_{\psi}^* H_{\psi-\psi_q} f_{j,q,p} + H_{\psi-\psi_q}^* H_{\psi_q} f_{j,q,p} \right\|. \end{aligned}$$

As in the proof of Theorem 1.1 one can show that $\|(\psi - \psi_q) f_{j,q,p}\| \rightarrow 0$ as $p \rightarrow q$. This fact implies that

$$\left\| H_{\psi}^* H_{\psi-\psi_q} f_{j,q,p} \right\| = \left\| H_{\psi}^* (I - P) \left((\psi - \psi_q) f_{j,q,p} \right) \right\| \rightarrow 0 \text{ as } p \rightarrow q.$$

Furthermore, since \mathbb{D}^n is a product domain and ψ_q is independent of z_n using (2.3) with

$$h_{j,q} = H_{\psi_q}^{\mathbb{D}^{n-1}} g_{j,q},$$

we have

$$\left\| H_{\psi-\psi_q}^* H_{\psi_q} f_{j,q,p} \right\| = \left\| P \left((H_{\psi_q}^{\mathbb{D}^{n-1}} g_{j,q}) (\overline{\psi} - \overline{\psi_q}) k_p^{\mathbb{D}} \right) \right\| \rightarrow 0 \text{ as } p \rightarrow q.$$

Furthermore, by (2.1) we have

$$H_{\psi_q}^* H_{\psi_q} f_{j,q,p} = (H_{\psi_q}^{\mathbb{D}^{n-1*}} H_{\psi_q}^{\mathbb{D}^{n-1}} g_{j,q}) k_p^{\mathbb{D}}.$$

Then for any fixed j , we have

$$\|H_\psi^* H_\psi f_{j,q,p} - \lambda_q f_{j,q,p}\| \rightarrow \|H_{\psi_q}^{\mathbb{D}^{n-1*}} H_{\psi_q}^{\mathbb{D}^{n-1}} g_{j,q} - \lambda_q g_{j,q}\|_{L^2(\mathbb{D}^{n-1})} \text{ as } p \rightarrow q.$$

However, by (2.6) we can pass to subsequence of $\{g_{j,q}\}$, if necessary, and get

$$\|H_{\psi_q}^{\mathbb{D}^{n-1*}} H_{\psi_q}^{\mathbb{D}^{n-1}} g_{j,q} - \lambda_q g_{j,q}\|_{L^2(\mathbb{D}^{n-1})} < \frac{1}{j}$$

for all j . Next we choose $\{p_j\} \subset \mathbb{D}$ such that $p_j \rightarrow q$ as $j \rightarrow \infty$ and

$$\|H_\psi^* H_\psi f_{j,q,p_j} - \lambda_q f_{j,q,p_j}\| < \frac{2}{j}$$

for all j . Therefore, we have

$$\|H_\psi^* H_\psi f_{j,q,p_j} - \lambda_q f_{j,q,p_j}\| \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Then ii. in Theorem A implies that $\lambda_q \in \sigma_e(H_\psi^* H_\psi)$.

Next we note that $\|H_{\psi_q}^{\mathbb{D}^{n-1*}} H_{\psi_q}^{\mathbb{D}^{n-1}}\|$ depends on q continuously because ψ_q depends on q continuously and

$$\begin{aligned} \|H_{\psi_{q_1}}^{\mathbb{D}^{n-1*}} H_{\psi_{q_1}}^{\mathbb{D}^{n-1}} - H_{\psi_{q_2}}^{\mathbb{D}^{n-1*}} H_{\psi_{q_2}}^{\mathbb{D}^{n-1}}\| &= \|H_{\psi_{q_1} - \psi_{q_2}}^{\mathbb{D}^{n-1*}} H_{\psi_{q_1}}^{\mathbb{D}^{n-1}} + H_{\psi_{q_2}}^{\mathbb{D}^{n-1*}} H_{\psi_{q_1} - \psi_{q_2}}^{\mathbb{D}^{n-1}}\| \\ &\leq \|H_{\psi_{q_1} - \psi_{q_2}}^{\mathbb{D}^{n-1}}\| \left(\|H_{\psi_{q_1}}^{\mathbb{D}^{n-1}}\| + \|H_{\psi_{q_2}}^{\mathbb{D}^{n-1}}\| \right) \\ &\leq \|\psi_{q_1} - \psi_{q_2}\|_{L^\infty} \left(\|H_{\psi_{q_1}}^{\mathbb{D}^{n-1}}\| + \|H_{\psi_{q_2}}^{\mathbb{D}^{n-1}}\| \right). \end{aligned}$$

Then we conclude that $\{\lambda_q : q \in b\mathbb{D}\}$ is connected. Therefore, since we assume that the mapping $q \rightarrow \|H_{\psi_q}^{\mathbb{D}^{n-1}}\|$ is non-constant, we conclude that $\{\lambda_q : q \in b\mathbb{D}\}$ contains an open interval in $(0, \infty)$. That is, $\sigma_e(H_\psi^* H_\psi)$ contains an open interval in $(0, \infty)$. ■

The proof of Theorem 1.3 above implies the following corollary.

Corollary 2.2 *Let $\psi \in C(\overline{\mathbb{D}^2})$ and $\psi_{1,\theta}(\xi) = \psi(e^{i\theta}, \xi)$, $\psi_{2,\theta}(\xi) = \psi(\xi, e^{i\theta})$ for $\xi \in \mathbb{D}$. Then*

$$\left\{ \|H_{\psi_{1,\theta}}^{\mathbb{D}}\|_{L^2(\mathbb{D})}^2 : \theta \in [0, 2\pi] \right\} \cup \left\{ \|H_{\psi_{2,\theta}}^{\mathbb{D}}\|_{L^2(\mathbb{D})}^2 : \theta \in [0, 2\pi] \right\} \subset \sigma_e(H_\psi^* H_\psi).$$

Remark 2.3 We note that Theorem 1.6 shows that the inclusion in Corollary 2.2 is not an equality in general. Indeed, for $\psi(z_1, z_2) = \bar{z}_1 z_2^m$ the quantities $\|H_{\psi_{1,\theta}}^{\mathbb{D}}\|$ and $\|H_{\psi_{2,\theta}}^{\mathbb{D}}\|$ are constant as functions of θ . Therefore, the left-hand side of the inclusion in Corollary 2.2 contains at most two numbers whereas the right hand side, by Theorem 1.6, contains infinitely many numbers.

Proof of Corollary 1.4 If φ is not holomorphic then $H_\varphi^{\mathbb{D}^{n-1}}$ is non-zero operator as $H_\varphi^{\mathbb{D}^{n-1}} 1 \neq 0$. Hence $\|H_\varphi^{\mathbb{D}^{n-1}}\| > 0$. Furthermore, since $|\chi|$ is non-constant on the unit circle, the image of the mapping

$$q \rightarrow \|H_{\psi_q}^{\mathbb{D}^{n-1}}\|_{L^2(\mathbb{D}^{n-1})} = |\chi(q)| \|H_\varphi^{\mathbb{D}^{n-1}}\|_{L^2(\mathbb{D}^{n-1})}$$

contains an open interval. Therefore, by Theorem 1.3, the essential spectrum of $H_\psi^* H_\psi$ contains an open interval. ■

Remark 2.4 One can also prove of Corollary 1.4 using Theorem 1.1 as follows. Since $H_\varphi^{\mathbb{D}^{n-1}}$ is non-zero operator, there exists a positive number $\mu \in \sigma(H_\varphi^{\mathbb{D}^{n-1}*} H_\varphi^{\mathbb{D}^{n-1}})$. Then the image of the mapping $q \rightarrow |\chi(q)|^2 \mu$ contains an open interval as $|\chi|$ is non-constant on the unit circle. Therefore, by Theorem 1.1, the essential spectrum of $H_\psi^* H_\psi$ contains an open interval.

3 Proof of Theorem 1.6

Before we present the proof of Theorem 1.6, we make some elementary computations. Let Ω be a complete Reinhardt domain in \mathbb{C}^n and again $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$. The Bergman kernel function K has the expression

$$K(z, w) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^\alpha \bar{w}^\alpha,$$

where $c_\alpha = 0$ when $\|z^\alpha\|_{L^2(\Omega)} = \infty$ and $c_\alpha = \|z^\alpha\|_{L^2(\Omega)}^{-2}$ otherwise.

A function ψ is called quasi-homogeneous if there exists $f : [0, \infty)^n \rightarrow \mathbb{C}$, $(k_1, \dots, k_n) \in \mathbb{Z}^n$ such that

$$(3.1) \quad \psi(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n}) = f(r_1, \dots, r_n) e^{i(k_1\theta_1 + \dots + k_n\theta_n)}.$$

For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, we will use the notation $|z|$ for $(|z_1|, \dots, |z_n|)$ and write $|z|^\alpha$ for the product $\prod_{j=1}^n |z_j|^{\alpha_j}$. Then (3.1) can also be expressed as:

$$\psi(z) = f(|z|) e^{i\mathbf{k} \cdot \boldsymbol{\theta}},$$

where $\mathbf{k} = (k_1, \dots, k_n)$, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$, and $z_j = |z_j| e^{i\theta_j}$. Now we consider the spectrum of $H_\psi^* H_\psi$ when ψ is bounded and quasi-homogeneous on the complete Reinhardt domain Ω .

By (2.4) we have

$$\begin{aligned} H_\psi^* H_\psi z^\alpha &= P M_{\bar{\psi}} (I - P) M_\psi (z^\alpha) \\ &= P M_{\bar{\psi}} (I - P) f(|z|) |z|^\alpha e^{i(\mathbf{k} + \boldsymbol{\alpha}) \cdot \boldsymbol{\theta}} \\ &= P M_{\bar{\psi}} (z^\alpha \psi - P(|z|^\alpha e^{i(\mathbf{k} + \boldsymbol{\alpha}) \cdot \boldsymbol{\theta}} f(|z|))), \end{aligned}$$

for $\alpha \in \mathbb{N}_0^n$. If the multi-index $\mathbf{k} + \boldsymbol{\alpha} \notin \mathbb{N}_0^n$, then $P(|z|^\alpha e^{i(\mathbf{k} + \boldsymbol{\alpha}) \cdot \boldsymbol{\theta}} f(|z|)) = 0$ which yields that

$$(3.2) \quad H_\psi^* H_\psi z^\alpha = P(z^\alpha |\psi|^2) = \frac{\|z^\alpha \psi\|_{L^2}^2}{\|z^\alpha\|_{L^2}^2} z^\alpha.$$

Otherwise,

$$P(|z|^\alpha e^{i(\mathbf{k} + \boldsymbol{\alpha}) \cdot \boldsymbol{\theta}} f(|z|)) = \int_\Omega K(z, w) |w|^\alpha e^{i(\mathbf{k} + \boldsymbol{\alpha}) \cdot \boldsymbol{\theta}} f(|w|) dV(w)$$

$$\begin{aligned}
 &= \sum_{\beta \in \mathbb{N}_0^n} c_\beta z^\beta \int_{\Omega} \bar{w}^\beta |w|^\alpha e^{i(k+\alpha)\cdot\theta} f(|w|) dV(w) \\
 &= c_{\alpha+k} z^{\alpha+k} \int_{\Omega} \bar{w}^{\alpha+k} |w|^\alpha e^{i(k+\alpha)\cdot\theta} f(|w|) dV(w) \\
 &= \frac{\int_{\Omega} |w|^{2\alpha+k} f(|w|) dV(w)}{\|z^{\alpha+k}\|_{L^2}^2} z^{\mathbf{k}+\alpha},
 \end{aligned}$$

which implies that

$$\begin{aligned}
 H_\psi^* H_\psi z^\alpha &= PM_{\bar{\psi}}(z^\alpha \psi - P(|z|^\alpha e^{i(k+\alpha)\cdot\theta} f(|z|))) \\
 &= PM_{\bar{\psi}}\left(z^\alpha \psi - \frac{\int_{\Omega} |w|^{2\alpha+k} f(|w|) dV(w)}{\|z^{\alpha+k}\|_{L^2}^2} z^{\mathbf{k}+\alpha}\right) \\
 &= P\left(z^\alpha |\psi|^2 - \frac{\int_{\Omega} |w|^{2\alpha+k} f(|w|) dV(w)}{\|z^{\alpha+k}\|_{L^2}^2} |z|^{\mathbf{k}} z^\alpha \overline{f(|z|)}\right) \\
 (3.3) \quad &= z^\alpha \left(\frac{\|z^\alpha \psi\|_{L^2}^2}{\|z^\alpha\|_{L^2}^2} - \frac{|\int_{\Omega} |w|^{2\alpha+k} f(|w|) dV(w)|^2}{\|z^\alpha\|_{L^2}^2 \|z^{\alpha+k}\|_{L^2}^2} \right).
 \end{aligned}$$

Hence the spectrum $\sigma(H_\psi^* H_\psi)$ contains the eigenvalues

$$\frac{\|z^\alpha \psi\|_{L^2}^2}{\|z^\alpha\|_{L^2}^2} - \frac{|\int_{\Omega} |w|^{2\alpha+k} f(|w|) dV(w)|^2}{\|z^\alpha\|_{L^2}^2 \|z^{\alpha+k}\|_{L^2}^2}$$

corresponding to the eigenfunction z^α for $\mathbf{k} + \alpha \in \mathbb{N}_0^n$ and

$$\frac{\|z^\alpha \psi\|_{L^2}^2}{\|z^\alpha\|_{L^2}^2}$$

for $\mathbf{k} + \alpha \notin \mathbb{N}_0^n$. Furthermore, since the eigenfunctions form an orthogonal basis for $A^2(\Omega)$, the closure of this set is the whole spectrum $\sigma(H_\psi^* H_\psi)$ (see by Lemma 3.4 below).

Example 3.1 When $f(|z|) = |z|^{\mathbf{k}}$ for $\mathbf{k} \in \mathbb{N}_0^n$, Equation (3.3) becomes

$$H_\psi^* H_\psi z^\alpha = z^\alpha \left(\frac{\|z^{\alpha+k}\|_{L^2}^2}{\|z^\alpha\|_{L^2}^2} - \frac{\|z^{2\alpha+2\mathbf{k}}\|_{L^1}^2}{\|z^\alpha\|_{L^2}^2 \|z^{\alpha+k}\|_{L^2}^2} \right) = 0.$$

This is not surprising since in this case the symbol $\psi(z) = z^{\mathbf{k}}$ is holomorphic.

Example 3.2 When $\psi(z) = e^{i\mathbf{k}\cdot\theta}$ for $\mathbf{k} \in \mathbb{N}_0^n$, Equation (3.3) becomes

$$H_\psi^* H_\psi z^\alpha = z^\alpha \left(1 - \frac{\|z^{2\alpha+k}\|_{L^1}^2}{\|z^\alpha\|_{L^2}^2 \|z^{\alpha+k}\|_{L^2}^2} \right).$$

Example 3.3 When $\mathbf{k} = \mathbf{0}$, $\psi(z)$ is radial and (3.3) becomes

$$H_\psi^* H_\psi z^\alpha = z^\alpha \left(\frac{\|z^\alpha \psi\|_{L^2}^2}{\|z^\alpha\|_{L^2}^2} - \frac{\|z^{2\alpha} \psi\|_{L^1}^2}{\|z^\alpha\|_{L^2}^4} \right).$$

Lemma 3.4 Let $T : H \rightarrow H$ be a bounded linear map on a separable Hilbert space H . Assume that λ_j is an eigenvalue with a corresponding eigenvector u_j for $j \in \mathbb{N}$ and $\{u_j : j \in \mathbb{N}\}$ forms an orthogonal basis for H . Then $\sigma(T) = \overline{\{\lambda_j : j \in \mathbb{N}\}}$. Furthermore, $\lambda \in \sigma_e(T)$ if and only if it is the limit of a subsequence of $\{\lambda_j\}$.

Proof. Since $\sigma(T)$ is a compact set containing $\{\lambda_j : j \in \mathbb{N}\}$, we have $\overline{\{\lambda_j : j \in \mathbb{N}\}} \subset \sigma(T)$. To prove the converse, assume that $\lambda \notin \overline{\{\lambda_j : j \in \mathbb{N}\}}$. Then we define $Su_j = \frac{1}{\lambda_j - \lambda} u_j$ for all j . Then S is a bounded operator, as $\{(\lambda_j - \lambda)^{-1}\}$ is a bounded sequence, and it is the inverse of $T - \lambda I$, as

$$(T - \lambda I)Su_j = S(T - \lambda I)u_j = u_j,$$

for all j . Then, $\lambda \notin \sigma(T)$. That is, $\sigma(T) \subset \overline{\{\lambda_j : j \in \mathbb{N}\}}$.

We recall that $\sigma_e(T) = \sigma(T) \setminus \sigma_d(T)$ where $\sigma_d(T)$ is composed of isolated eigenvalues with finite multiplicity. To prove the last statement, let $\lambda \in \sigma_e(T)$. Then $\lambda \in \sigma(T)$ is not an isolated eigenvalue with finite multiplicity. That is, either λ is not isolated in $\sigma(T)$ or it is an eigenvalue with infinite multiplicity. Either way, there exists a subsequence of $\{\lambda_j\}$ converging to λ . Conversely, if $\lambda \in \sigma(T)$ is the limit of a subsequence $\{\lambda_{j_k}\}$ of $\{\lambda_j\}$, then either $\lambda = \lambda_{j_k}$ for infinitely many k s (hence, it is an eigenvalue with infinite multiplicity) or λ is not isolated in $\sigma(T)$. Again, either way $\lambda \in \sigma_e(T)$. ■

Finally, we present the proof of Theorem 1.6.

Proof of Theorem 1.6 Note that $\psi(z) = z^n \bar{z}^m = |z|^{n+m} e^{i(n-m)\theta}$. By (3.2) and (3.3), it follows that

$$(3.4) \quad H_\psi^* H_\psi z^\alpha = \begin{cases} z^\alpha \frac{\|z^{\alpha+n}\bar{z}^m\|_{L^2}^2}{\|z^\alpha\|_{L^2}^2} & \alpha + n - m \notin \mathbb{N}_0^n \\ z^\alpha \left(\frac{\|z^{\alpha+n}\bar{z}^m\|_{L^2}^2}{\|z^\alpha\|_{L^2}^2} - \frac{\|z^{2\alpha+2n-m}\bar{z}^m\|_{L^1}^2}{\|z^\alpha\|_{L^2}^2 \|z^{\alpha+n-m}\|_{L^2}^2} \right) & \alpha + n - m \in \mathbb{N}_0^n. \end{cases}$$

We note that for multi-index $\beta \in \mathbb{N}_0^n$ we have,

$$\|z^\beta\|_{L^2}^2 = \|z^{2\beta}\|_{L^1} = \frac{\pi^n}{\prod_{k=1}^n (\beta_k + 1)}.$$

Substituting this into (3.4) yields that

$$H_\psi^* H_\psi z^\alpha = \begin{cases} z^\alpha \prod_{k=1}^n \frac{\alpha_k + 1}{\alpha_k + n_k + m_k + 1} & \alpha + n - m \notin \mathbb{N}_0^n \\ z^\alpha \left(\prod_{k=1}^n \frac{\alpha_k + 1}{\alpha_k + n_k + m_k + 1} - \prod_{k=1}^n \frac{(\alpha_k + 1)(\alpha_k + n_k - m_k + 1)}{(\alpha_k + n_k + 1)^2} \right) & \alpha + n - m \in \mathbb{N}_0^n. \end{cases}$$

Hence

$$H_\psi^* H_\psi z^\alpha = \lambda_{n,m,\alpha,B_n} z^\alpha,$$

for $\alpha \in \mathbb{N}_0^n$ where λ_{n,m,α,B_n} is defined before Theorem 1.6 and $B_n = \{1, 2, 3, \dots, n\}$. That is, z^α is an eigenvector corresponding to the eigenvalue λ_{n,m,α,B_n} . Furthermore, since $\{z^\alpha : \alpha \in \mathbb{N}_0^n\}$ forms an orthogonal basis for $A^2(\mathbb{D}^n)$, Lemma 3.4 implies that

$$\sigma(H_\psi^* H_\psi) = \overline{\{\lambda_{n,m,\alpha,B_n} : \alpha \in \mathbb{N}_0^n\}}.$$

We note that in case $\mathbf{m} = 0$ the operator $H_\psi^* H_\psi z^\alpha$ is the zero operator and hence 0 is the only eigenvalue with infinite multiplicity. For the rest of the proof, we will assume that $\mathbf{m} \neq 0$.

Lemma 3.4 implies that each element in the spectrum is either an eigenvalue or a limit of a sequence of eigenvalues. To describe the spectrum outside of the point spectrum, we assume that $\lambda \in \sigma(H_\psi^* H_\psi)$ such that

$$\lambda = \lim_{j \rightarrow \infty} \lambda_{\mathbf{n}, \mathbf{m}, \alpha(j), B_n},$$

for a sequence $\{\alpha(j)\} \subset \mathbb{N}_0^n$. Next we will show that either $\lambda = 0$ or $\lambda = \lambda_{\mathbf{n}, \mathbf{m}, \alpha, B}$ for some $\emptyset \neq B \subset B_n$ and $\alpha \in \mathbb{N}_0^n$. We assume that $\lambda \neq 0$.

We note that $z^{\alpha(j)}$ is an eigenvector corresponding to the eigenvalue $\lambda_{\mathbf{n}, \mathbf{m}, \alpha(j), B_n}$. Let us assume that

$$\lambda_{\mathbf{n}, \mathbf{m}, \alpha(j), B_n} = \prod_{k=1}^n \frac{\alpha_k(j) + 1}{\alpha_k(j) + n_k + m_k + 1}.$$

For each j there exists $k \in B_n$ such that $\alpha_k(j) < m_k - n_k$. Then there exists k such that $\alpha_k(j) < m_k - n_k$ for infinitely many js . That is, the k th sequence $\{\alpha_k(j)\}_{j=1}^\infty$ has a subsequence bounded by $m_k - n_k$. We pass to that subsequence of $\{\alpha(j)\}$ and still call it $\{\alpha(j)\}$.

Next we construct B as follows. Let $k_1 \in B_n$ be the smallest integer so that $\{\alpha_{k_1}(j)\}_{j=1}^\infty$ has a bounded subsequence. Then we pass onto a subsequence, still calling it $\{\alpha(j)\}$, so that $\{\alpha_{k_1}(j)\}_{j=1}^\infty$ is a constant sequence. If $\alpha_k(j) \rightarrow \infty$ as $j \rightarrow \infty$ for all $k \geq k_1 + 1$ then we stop here and $B = \{k_1\}$. Otherwise, we choose $k_2 \geq k_1 + 1$ to be the smallest integer so that $\{\alpha_{k_2}(j)\}_{j=1}^\infty$ has a bounded subsequence. Now we pass onto a subsequence, again calling it $\{\alpha(j)\}$, so that $\{\alpha_{k_2}(j)\}_{j=1}^\infty$ is a constant sequence. After finitely many steps we obtain $B = \{k_1, \dots, k_p\}$ and a subsequence $\{\alpha(j)\}$ so that $\{\alpha_k(j)\}$ is a constant sequence for every $k \in B$ and $\alpha_k(j) \rightarrow \infty$ as $j \rightarrow \infty$ for $k \notin B$.

Hence for this subsequence of $\{\alpha(j)\}$, taking limit as $j \rightarrow \infty$, we have

$$\begin{aligned} \lambda &= \lim_{j \rightarrow \infty} \lambda_{\mathbf{n}, \mathbf{m}, \alpha(j), B_n} = \lim_{j \rightarrow \infty} \prod_{k=1}^n \frac{\alpha_k(j) + 1}{\alpha_k(j) + n_k + m_k + 1} \\ &= \prod_{k \in B} \frac{\alpha_k + 1}{\alpha_k + n_k + m_k + 1} = \lambda_{\mathbf{n}, \mathbf{m}, \alpha, B}, \end{aligned}$$

where $\alpha \in \mathbb{N}_0^n$ such that α_k is arbitrary for $k \notin B$ and $\alpha_k = \alpha_k(j)$ for $k \in B$. We note that $\alpha_k < m_k - n_k$ for some $k \in B$.

Similarly, if $\lim_{j \rightarrow \infty} \lambda_{\mathbf{n}, \mathbf{m}, \alpha(j), B_n} = \lambda$ where

$$\lambda_{\mathbf{n}, \mathbf{m}, \alpha(j), B_n} = \left(\prod_{k=1}^n \frac{\alpha_k(j) + 1}{\alpha_k(j) + n_k + m_k + 1} - \prod_{k=1}^n \frac{(\alpha_k(j) + 1)(\alpha_k(j) + n_k - m_k + 1)}{(\alpha_k(j) + n_k + 1)^2} \right)$$

then

$$\lambda = \lim_{j \rightarrow \infty} \lambda_{\mathbf{n}, \mathbf{m}, \alpha(j), B_n} = \lambda_{\mathbf{n}, \mathbf{m}, \alpha, B},$$

and $\alpha_k \geq m_k - n_k$ for all $k \in B$. Hence we have shown that

$$\sigma(H_\Psi^* H_\Psi) \subset \{0\} \cup \{\lambda_{n,m,\alpha,B} : \alpha \in \mathbb{N}_0^n, \emptyset \neq B \subset B_n\}.$$

Next we prove the converse of the inclusion above. First, we choose $\alpha_k(j) = j$ for all k . Then it is easy to see that $\lambda_{n,m,\alpha(j),B_n} \rightarrow 0$ as $j \rightarrow \infty$. Hence $0 \in \sigma(H_\Psi^* H_\Psi)$. Secondly, let us assume that $\emptyset \neq B \subset B_n$ and $\alpha_k < m_k - n_k$ for some $k \in B$. We choose $\alpha_k(j) = j$ for $k \notin B$ and $\alpha_k(j) = \alpha_k$ for $k \in B$. Then

$$\lambda_{n,m,\alpha(j),B_n} \rightarrow \lambda_{n,m,\alpha,B} \in \sigma(H_\Psi^* H_\Psi) \text{ as } j \rightarrow \infty.$$

Similarly, if $\alpha_k \geq m_k - n_k$ for all $k \in B$ we choose $\{\alpha(j)\}$ as above again and conclude that $\lambda_{n,m,\alpha,B} \in \sigma(H_\Psi^* H_\Psi)$. Therefore,

$$\sigma(H_\Psi^* H_\Psi) = \{0\} \cup \{\lambda_{n,m,\alpha,B} : \alpha \in \mathbb{N}_0^n, \emptyset \neq B \subset B_n\}.$$

We finish the proof by proving the claims about multiplicities. Let us assume that $n_k + m_k \geq 1$ for all $k \in B_n$ and λ is a non-zero eigenvalue of infinite multiplicity. Then there exists a sequence of multi-indices $\{\alpha(j)\}$ such that $\lambda_{n,m,\alpha(j),B_n} = \lambda$ for all j . After passing to a subsequence, we may assume that $\lim_{j \rightarrow \infty} \alpha_k(j)$ exists allowing infinity as the limit for all k .

If $B = \emptyset$ then $\lambda = 0$. For the rest of the proof, we assume that $B \neq \emptyset$ and, by passing to a subsequence if necessary, $\alpha_k(j) = \alpha_k$ for $k \in B$ and $\alpha_k(j) \rightarrow \infty$ for $k \notin B$. In case $\alpha_k(j) < m_k - n_k$ for some $k \in B$, it follows that

$$\lambda_{n,m,\alpha(j),B_n} < \prod_{k \in B} \frac{\alpha_k(j) + 1}{\alpha_k(j) + n_k + m_k + 1} = \prod_{k \in B} \frac{\alpha_k + 1}{\alpha_k + n_k + m_k + 1} = \lambda,$$

which is a contradiction with $\lambda_{n,m,\alpha(j),B_n} = \lambda$ for all j .

On the other hand, if $\alpha_k(j) \geq m_k - n_k$ for all $k = 1, 2, \dots, n$, the equality $\lambda_{n,m,\alpha(j),B_n} = \lambda$ for all j implies that

$$\begin{aligned} \lambda &= \prod_{k \in B} \frac{\alpha_k(j) + 1}{\alpha_k(j) + n_k + m_k + 1} - \prod_{k \in B} \frac{(\alpha_k(j) + 1)(\alpha_k(j) + n_k - m_k + 1)}{(\alpha_k(j) + n_k + 1)^2} \\ (3.5) \quad &= \prod_{k=1}^n \frac{\alpha_k(j) + 1}{\alpha_k(j) + n_k + m_k + 1} - \prod_{k=1}^n \frac{(\alpha_k(j) + 1)(\alpha_k(j) + n_k - m_k + 1)}{(\alpha_k(j) + n_k + 1)^2}. \end{aligned}$$

Next we will prove that this is impossible for all j . Let us set

$$\begin{aligned} a &= \prod_{k \in B} \frac{\alpha_k(j) + 1}{\alpha_k(j) + n_k + m_k + 1}, \\ b &= \prod_{k \in B} \frac{(\alpha_k(j) + 1)(\alpha_k(j) + n_k - m_k + 1)}{(\alpha_k(j) + n_k + 1)^2}. \end{aligned}$$

Then (3.5) becomes

$$\begin{aligned} (3.6) \quad a - b - a \prod_{k \notin B} \frac{\alpha_k(j) + 1}{\alpha_k(j) + n_k + m_k + 1} \\ + b \prod_{k \notin B} \frac{(\alpha_k(j) + 1)(\alpha_k(j) + n_k - m_k + 1)}{(\alpha_k(j) + n_k + 1)^2} = 0. \end{aligned}$$

Rearranging this equation, we obtain a polynomial equation

$$\begin{aligned} &(a - b) \prod_{k \notin B} (\alpha_k(j) + n_k + 1)^2 (\alpha_k(j) + n_k + m_k + 1) \\ &\quad - a \prod_{k \notin B} (\alpha_k(j) + n_k + 1)^2 (\alpha_k(j) + 1) \\ &\quad + b \prod_{k \notin B} (\alpha_k(j) + 1) (\alpha_k(j) + n_k - m_k + 1) (\alpha_k(j) + n_k + m_k + 1) = 0. \end{aligned}$$

Rewriting the equality above, we get

$$(3.7) \quad \begin{aligned} &(a - b) \prod_{k \notin B} (\alpha_k(j) + n_k + 1)^2 \left(\prod_{k \notin B} (\alpha_k(j) + n_k + m_k + 1) - \prod_{k \notin B} (\alpha_k(j) + 1) \right) \\ &\quad - b \prod_{k \notin B} (\alpha_k(j) + 1) m_k^2 = 0. \end{aligned}$$

If $m_k \geq 1$ for some $k \in B$ then $a - b > 0$ and

$$\prod_{k \notin B} (\alpha_k(j) + n_k + m_k + 1) - \prod_{k \notin B} (\alpha_k(j) + 1) \geq 1.$$

Furthermore, $\prod_{k \notin B} (\alpha_k(j) + n_k + 1)^2$ dominates $\prod_{k \notin B} (\alpha_k(j) + 1) m_k^2$ as $j \rightarrow \infty$. Hence, the left-hand side of (3.7) converges to ∞ as $j \rightarrow \infty$, reaching a contradiction.

On the other hand, if $m_k = 0$ for all $k \in B$ and since $\mathbf{m} \neq 0$, we have $m_k \geq 1$ for some $k \notin B$. Then $a - b = 0$ and (3.6) implies

$$- \prod_{k \notin B} \frac{\alpha_k(j) + 1}{\alpha_k(j) + n_k + m_k + 1} + \prod_{k \notin B} \frac{(\alpha_k(j) + 1)(\alpha_k(j) + n_k - m_k + 1)}{(\alpha_k(j) + n_k + 1)^2} = 0.$$

However, the left-hand side of the equation above is equal to a positive multiple of the following expression

$$- \prod_{k \notin B} (\alpha_k(j) + n_k + 1)^2 + \prod_{k \notin B} ((\alpha_k(j) + n_k + 1)^2 - m_k^2)$$

which is negative, reaching a contradiction again. Therefore, we showed that, if $m_k + n_k \geq 1$ for all $k \in B_n$ then each eigenvalue is of finite multiplicity.

Finally, let us define $B_0 = \{k \in B_n : m_k = n_k = 0\}$. Now we will show that all of the eigenvalues have infinite multiplicities when $B_0 \neq \emptyset$. We note that since $\mathbf{m} \neq 0$ we know that $B_0 \subsetneq B_n$.

Assume that $\lambda_{\mathbf{n}, \mathbf{m}, \alpha(j), B_n}$ is an eigenvalue. Then either $\alpha_k < m_k - n_k$ for some k or $\alpha_k \geq m_k - n_k$ for all k . In the first case, we have $\alpha_k < m_k - n_k$ for some $k \notin B_0$. Then

$$\lambda_{\mathbf{n}, \mathbf{m}, \alpha(j), B_n} = \prod_{k=1}^n \frac{\alpha_k(j) + 1}{\alpha_k(j) + n_k + m_k + 1} = \prod_{k \notin B_0} \frac{\alpha_k + 1}{\alpha_k + n_k + m_k + 1} = \lambda_{\mathbf{n}, \mathbf{m}, \alpha, B_n},$$

for $\alpha_k(j) = j$ for $k \in B_0$ and $\alpha_k(j) = \alpha_k$ for $k \notin B_0$. Similarly, if $\alpha_k \geq m_k - n_k$ for all k we make the same choice $\alpha_k(j) = j$ for $k \in B_0$ and $\alpha_k(j) = \alpha_k$ for $k \notin B_0$ and one can see that

$$\lambda_{\mathbf{n}, \mathbf{m}, \alpha(j), B_n} = \prod_{k=1}^n \frac{\alpha_k(j) + 1}{\alpha_k(j) + n_k + m_k + 1} - \prod_{k=1}^n \frac{(\alpha_k(j) + 1)(\alpha_k(j) + n_k - m_k + 1)}{(\alpha_k(j) + n_k + 1)^2}$$

$$\begin{aligned}
&= \prod_{k \notin B_0} \frac{\alpha_k + 1}{\alpha_k + n_k + m_k + 1} - \prod_{k \notin B_0} \frac{(\alpha_k + 1)(\alpha_k + n_k - m_k + 1)}{(\alpha_k + n_k + 1)^2} \\
&= \lambda_{\mathbf{n}, \mathbf{m}, \alpha, B_n}.
\end{aligned}$$

Therefore, in either case the eigenvalues have infinite multiplicities. \blacksquare

By the proof of Theorem 1.6, we can also characterize the essential spectrum of Hankel operators with monomial symbols.

Corollary 3.5 *Let $\psi(z) = z^n \bar{z}^m$ for some $\mathbf{m}, \mathbf{n} \in \mathbb{N}_0^n$. If $m_k + n_k = 0$ for some $k \in B_n$, then the essential spectrum of $H_\psi^* H_\psi$ on $A^2(\mathbb{D}^n)$*

$$\sigma_e(H_\psi^* H_\psi) = \sigma(H_\psi^* H_\psi) = \{0\} \cup \{\lambda_{\mathbf{n}, \mathbf{m}, \alpha, B} : \alpha \in \mathbb{N}_0^n, \theta \neq B \subset B_n\}.$$

On the other hand, if $n_k + m_k \geq 1$ for all $k \in B_n$ and $m_k \geq 1$ for some $k \in B_n$, then the essential spectrum of $H_\psi^ H_\psi$ on $A^2(\mathbb{D}^n)$*

$$\sigma_e(H_\psi^* H_\psi) = \{0\} \cup \{\lambda_{\mathbf{n}, \mathbf{m}, \alpha, B} : \alpha \in \mathbb{N}_0^n, \theta \neq B \subset B_n \text{ and } B \neq B_n\}.$$

Proof. First let us assume that $B_0 = \{k \in B_n : m_k + n_k = 0\} \neq \theta$. Then all of the eigenvalues have infinite multiplicity. So the discrete spectrum is empty and the essential spectrum is identical to the spectrum.

Next we assume that $B_0 = \theta$ and $m_k \geq 1$ for some $k \in B_n$. Then 0 is in the essential spectrum because $0 = \lim_{j \rightarrow \infty} \lambda_{\mathbf{n}, \mathbf{m}, \alpha(j), B_n}$ for $\alpha_k(j) = j$ for all $k \in B_n$. Next one can see that for any $\alpha \in \mathbb{N}_0^n$, the value $\lambda_{\mathbf{n}, \mathbf{m}, \alpha, B}$ is in the essential spectrum for $\theta \neq B \subset B_n$ and $B \neq B_n$ as follows. Let $\{\alpha(j)\}$ be a sequence in \mathbb{N}_0^n with $\alpha_k(j) = \alpha_k$ for $k \in B$ and $\alpha_k(j) = j$ for $k \notin B$. Then as shown in the proof of Theorem 1.6, the constant $\lambda_{\mathbf{n}, \mathbf{m}, \alpha(j), B_n}$ is an eigenvalue and $\lambda_{\mathbf{n}, \mathbf{m}, \alpha, B} = \lim_{j \rightarrow \infty} \lambda_{\mathbf{n}, \mathbf{m}, \alpha(j), B_n}$.

On the other hand, Lemma 3.4 implies that every λ in the essential spectrum is the limit of a sequence of eigenvalues. By passing to subsequences argument as in Theorem 1.6 and the fact that B_0 is empty, we can see that such a λ will be of form $\lambda_{\mathbf{n}, \mathbf{m}, \alpha, B}$ for some $\alpha \in \mathbb{N}_0^n$ and some proper subset B of B_n . \blacksquare

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