Nagoya Math. J., **251** (2023), 734–767
DOI 10.1017/nmj.2023.11

TWISTED SHIFT-INVARIANT SYSTEM IN $L^2(\mathbb{R}^{2N})$

SANTI RANJAN DAS, RABEETHA VELSAMY AND RADHA RAMAKRISHNAN

Abstract. We consider a general twisted shift-invariant system, $V^t(\mathcal{A})$, consisting of twisted translates of countably many generators and study the problem of obtaining a characterization for the system $V^t(\mathcal{A})$ to form a frame sequence or a Riesz sequence. We illustrate our theory with some examples. In addition to these results, we study a dual twisted shift-invariant system and also obtain an orthonormal sequence of twisted translates from a given Riesz sequence of twisted translates.

§1. Introduction

For the past two decades, shift-invariant spaces on $\mathbb{R}^n$ have been studied extensively and applied in various fields such as time–frequency analysis, sampling theory, approximation theory, numerical analysis, and electrical engineering. In the recent past, these spaces have been studied on several locally compact groups. The main problem deals with obtaining characterization theorems for the system of translates to form a frame sequence or a Riesz sequence.

Bownik in [4] characterized shift-invariant spaces on $\mathbb{R}^n$ in terms of range functions and obtained characterization for a system of translates on $\mathbb{R}^n$ to be a frame sequence and a Riesz sequence. Later shift-invariant spaces were studied on locally compact abelian groups in [5], [6], [17] and on a non-abelian compact group in [20].

Currey et al. in [9] characterized a shift-invariant system in terms of range function for SI/Z-type groups. In [16], Iverson studied frames of the form $\{\rho(\xi)f_i\}_{\xi\in K, i\in I}$, where ρ is a representation of a non-abelian compact group K on a Hilbert space H_ρ using operator-valued Zak transform. There are several interesting characterization theorems for frames and Riesz sequences in connection with shift-invariant spaces on various types of Heisenberg groups such as polarized Heisenberg group, standard Heisenberg group, and A. Weil's abstract Heisenberg group. We refer to [1], [2], [19] in this connection. Recently, in [11], characterization theorems for frame sequences and Riesz sequences have been studied for the shift-invariant system in terms of Gramian and dual Gramian, respectively, on the Heisenberg group. In [3], Barbieri et al. investigated the structure of subspaces of a Hilbert space which are invariant under unitary representations of a discrete group. Here, they generalized the concepts of bracket map, fiberization map, dual integrability, and obtained characterization of frames and Riesz bases in a more general setting.

In [18], Radha and Saswata introduced twisted shift-invariant spaces $V^t(\phi)$ in $L^2(\mathbb{R}^{2n})$ and studied the problem of finding a characterization for the system of twisted translates to form a frame sequence or a Riesz sequence. The characterizations were obtained using

Received August 5, 2022. Revised February 21, 2023. Accepted May 4, 2023.
2020 Mathematics subject classification: Primary 42C15; Secondary 43A30.
Keywords: Dual frame, frames, range function, Riesz basis, twisted shift-invariant system, Weyl transform.
Rabeetha Velsamy thanks the University Grants Commission, India for the financial support.

certain "Condition C." On the other hand, Das et al. in [10] introduced twisted $B-$ splines B_n and studied various properties. It was observed that for $n \geq 2$, these splines do not satisfy "Condition C." For $n = 2$, it was shown that the twisted translates of B_n form a Riesz sequence using certain specific numerical computations. But, for $n > 2$, such computations could not be extended. They were also not successful in proving the result for $n = 2$ analytically. Hence, it becomes necessary to obtain a characterization for a system of twisted translates to form a Riesz sequence (also a frame sequence) without using "Condition C." In this paper, as a first step, we attempt to find such a characterization which can help us to verify these results for some simple examples. This is the motivation of this paper.

In this paper, we focus on a twisted shift-invariant system of the form $E^t(\mathcal{A}) := \{T^t_{(k,l)}\phi_s : k,l \in \mathbb{Z}^n, s \in \mathbb{Z}\}$ for a countable collection of functions $\mathcal{A} := \{\phi_s : s \in \mathbb{Z}\}$ in $L^2(\mathbb{R}^{2n})$, where $T^t_{(k,l)}\phi$ denotes the twisted translation of $\phi \in L^2(\mathbb{R}^{2n})$. (There exist different versions of twisted translations depending on the definition of the Schrödinger representation [see, e.g., [12] for a different choice].) We define a class of operators $\{H_l(\xi) : l \in \mathbb{Z}^n, \xi \in \mathbb{T}^n\}$ using the fiber map associated with the system $E^t(\mathcal{A})$. We obtain a characterization for the system $E^t(\mathcal{A})$ to be a frame sequence in terms of the operators $H_l(\xi)$ and their adjoints. We obtain the corresponding Riesz sequence characterization in terms of the Gramian associated with $E^t(\mathcal{A})$. We also prove that the system $E^t(\mathcal{A})$ is a frame sequence (Riesz sequence) if and only if the system $\{\tau\phi_s(\xi + l) : l \in \mathbb{Z}^n, s \in \mathbb{Z}\}$ is a frame sequence (Riesz sequence) for a.e. $\xi \in \mathbb{T}^n$, using the fiber map τ. When $A = \{\phi\}$, we prove that if $E^t(\mathcal{A})$ is a frame sequence, then the weight function $W_\phi(\xi)$ is bounded above and below *a.e.* $\xi \in \mathbb{T}^n \setminus N$, where $N =: \{\xi \in \mathbb{T}^n : \omega_\phi(\xi) = 0\}$. We also prove a similar result when $E^t(\mathcal{A})$ is a Riesz sequence. We illustrate these results with examples. However, the converse need not be true. We provide a counter example.

We then study the problem of dual twisted shift-invariant system. We consider two twisted shift-invariant systems $E^t(\mathcal{A})$ and $E^t(\mathcal{D})$. We assume that they are Bessel sequences. Then we obtain a characterization of $E^t(\mathcal{A})$ and $E^t(\mathcal{D})$ to be dual frames. As a consequence, we show that if ϕ satisfies "Condition C," then as in the classical case, the canonical dual frame is the only dual frame that consists of twisted translations of a single function. In the final part of the paper, we show that if the system of twisted translates $\{T^t_{(k,l)}\phi : k,l \in \mathbb{Z}^n\}$ is a Riesz sequence in $L^2(\mathbb{R}^{2n})$, where $\phi \in L^2(\mathbb{R}^{2n})$ such that it satisfies "Condition C," then there exists a $\phi^\sharp \in V^t(\phi)$ such that $\{T^t_{(k,l)}\phi^\sharp : k,l \in \mathbb{Z}^n\}$ is an orthonormal system in $L^2(\mathbb{R}^{2n})$ and $V^t(\phi) = V^t(\phi^\sharp)$. We illustrate this with an example.

§2. Notation and background

Let $\mathcal{H} \neq 0$ be a separable Hilbert space.

Definition 2.1. A sequence $\{f_k : k \in \mathbb{Z}\}$ in $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist two constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \qquad \forall\, f \in \mathcal{H}. \tag{2.1}$$

If only the right-hand side inequality of (2.1) holds, then $\{f_k : k \in \mathbb{Z}\}$ is called a Bessel sequence. If $A = B$ in (2.1), then $\{f_k : k \in \mathbb{Z}\}$ is called a tight frame. If (2.1) holds with $A = B = 1$, then $\{f_k : k \in \mathbb{Z}\}$ is called a Parseval frame. If $\{f_k : k \in \mathbb{Z}\}$ is a frame for

https://doi.org/10.1017/nmj.2023.11 Published online by Cambridge University Press

$\overline{\mathrm{span}\{f_k : k\in\mathbb{Z}\}}$, then it is called a frame sequence. The frame operator $S:\mathcal{H}\to\mathcal{H}$ associated with a frame $\{f_k : k\in\mathbb{Z}\}$ is defined by

$$Sf := \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle f_k, \qquad \forall\ f\in\mathcal{H}.$$

It can be shown that S is a bounded, invertible, self-adjoint, and positive operator on $\mathcal{H}$. In addition, $\{S^{-1}f_k : k\in\mathbb{Z}\}$ is also a frame with frame operator S^{-1} and frame bounds B^{-1}, A^{-1}.

Definition 2.2. Let $\{f_k : k\in\mathbb{Z}\}$ be a frame for $\mathcal{H}$, and let S be the corresponding frame operator. Then the collection $\{S^{-1}f_k : k\in\mathbb{Z}\}$ is called the canonical dual frame of $\{f_k : k\in\mathbb{Z}\}$.

Definition 2.3. Let $\{f_k : k\in\mathbb{Z}\}$ and $\{g_k : k\in\mathbb{Z}\}$ be two frames for $\mathcal{H}$. Then $\{g_k : k\in\mathbb{Z}\}$ is said to be a dual frame of $\{f_k : k\in\mathbb{Z}\}$ if

$$f = \sum_{k\in\mathbb{Z}} \langle f, g_k\rangle f_k, \qquad \forall\ f\in\mathcal{H}.$$

Theorem 2.4. *Assume that $\{f_k : k\in\mathbb{Z}\}$ and $\{g_k : k\in\mathbb{Z}\}$ are Bessel sequences in $\mathcal{H}$. Then the following are equivalent:*

(i) $f = \sum_{k\in\mathbb{Z}} \langle f, g_k\rangle f_k, \quad \forall\ f\in\mathcal{H}.$

(ii) $f = \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle g_k, \quad \forall\ f\in\mathcal{H}.$

(iii) $\langle f, g\rangle = \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle \langle g_k, g\rangle, \quad \forall\ f, g\in\mathcal{H}.$

When one of the above equivalent conditions is satisfied, $\{f_k : k\in\mathbb{Z}\}$ and $\{g_k : k\in\mathbb{Z}\}$ are dual frames for $\mathcal{H}$. If B denotes an upper frame bound for $\{f_k :\ k\in\mathbb{Z}\}$, then B^{-1} is a lower frame bound for $\{g_k :\ k\in\mathbb{Z}\}$.

Definition 2.5. A sequence $\{f_k : k\in\mathbb{Z}\}$ in $\mathcal{H}$ is called a Riesz basis for $\mathcal{H}$ if there exists a bounded invertible operator U on $\mathcal{H}$ and an orthonormal basis $\{e_k : k\in\mathbb{Z}\}$ of $\mathcal{H}$ such that $U(e_k) = f_k$, $\forall\ k\in\mathbb{Z}$. If $\{f_k : k\in\mathbb{Z}\}$ is a Riesz basis for $\overline{\mathrm{span}\{f_k : k\in\mathbb{Z}\}}$, then it is called a Riesz sequence.

Theorem 2.6. *For a sequence $\{f_k : k\in\mathbb{Z}\}$ in $\mathcal{H}$, the following conditions are equivalent:*

(i) *$\{f_k\}$ is a Riesz basis for $\mathcal{H}$.*

(ii) *$\{f_k\}$ is complete in $\mathcal{H}$, and there exist constants $A, B > 0$ such that for every finite scalar sequence $\{c_k\}$, one has*

$$A\sum_k |c_k|^2 \le \Big\|\sum_k c_k f_k\Big\|^2 \le B\sum_k |c_k|^2.$$

For further study of frames and Riesz bases, we refer to [8].

The Heisenberg group $\mathbb{H}^n$ is a nilpotent Lie group whose underlying manifold is $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ with the group operation defined by $(x,y,t)(u,v,s) = (x+u, y+v, t+s+\frac{1}{2}(u\cdot y - v\cdot x))$ and the Haar measure is the Lebesgue measure $dxdydt$ on $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$. Using the Schrödinger representation π_λ, $\lambda\in\mathbb{R}^*$, given by

$$\pi_\lambda(x,y,t)\phi(\xi) = e^{2\pi i\lambda t} e^{2\pi i\lambda(x\cdot\xi + \frac{1}{2}x\cdot y)}\phi(\xi+y),\ \ \phi\in L^2(\mathbb{R}^n),$$

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we define the group Fourier transform of $f \in L^1(\mathbb{H}^n)$ as

$$\widehat{f}(\lambda) = \int_{\mathbb{H}^n} f(z,t)\pi_\lambda(z,t)\,dzdt, \text{ where } \lambda \in \mathbb{R}^*,$$

which is a bounded operator on $L^2(\mathbb{R}^n)$. In other words, for $\phi \in L^2(\mathbb{R}^n)$, we have

$$\widehat{f}(\lambda)\phi = \int_{\mathbb{H}^n} f(z,t)\pi_\lambda(z,t)\phi\,dzdt,$$

where the integral is a Bochner integral taking values in $L^2(\mathbb{R}^n)$. If f is also in $L^2(\mathbb{H}^n)$, then $\widehat{f}(\lambda)$ is a Hilbert–Schmidt operator. Define

$$f^\lambda(z) = \int_{\mathbb{R}} e^{2\pi i\lambda t} f(z,t)\,dt,$$

which is the inverse Fourier transform of f in the t variable. Then we can write

$$\widehat{f}(\lambda) = \int_{\mathbb{C}^n} f^\lambda(z)\pi_\lambda(z,0)dz.$$

Let $g \in L^1(\mathbb{C}^n)$. Define

$$W_\lambda(g) = \int_{\mathbb{C}^n} g(z)\pi_\lambda(z,0)\,dz.$$

When $\lambda = 1$, it is called the Weyl transform of g, denoted by $W(g)$. This can be explicitly written as

$$W(g)\phi(\xi) = \int_{\mathbb{R}^{2n}} g(x,y)e^{2\pi i(x\cdot\xi + \frac{1}{2}x\cdot y)}\phi(\xi+y)\,dxdy, \ \phi \in L^2(\mathbb{R}^n).$$

The Weyl transform is an integral operator with kernel $K_g(\xi,\eta) = \int_{\mathbb{R}^n} g(x,\eta-\xi)e^{\pi i x\cdot(\xi+\eta)}dx$. If $g \in L^1 \cap L^2(\mathbb{C}^n)$, then $K_g \in L^2(\mathbb{R}^{2n})$, which implies that $W(g)$ is a Hilbert–Schmidt operator whose norm is given by $\|W(g)\|^2_{\mathcal{B}_2} = \|K_g\|^2_{L^2(\mathbb{R}^{2n})}$, where $\mathcal{B}_2$ is the Hilbert space of Hilbert–Schmidt operators on $L^2(\mathbb{R}^n)$ with inner product $(T,S) = \operatorname{tr}(TS^*)$. The Plancherel formula and the inversion formula for the Weyl transform are given by $\|W(g)\|^2_{\mathcal{B}_2} = \|g\|^2_{L^2(\mathbb{C}^n)}$ and $g(w) = \operatorname{tr}(\pi(w)^* W(g))$, $w \in \mathbb{C}^n$, respectively. For a detailed study of analysis on the Heisenberg group, we refer to [14], [21].

DEFINITION 2.7 [18]. Let $\phi \in L^2(\mathbb{R}^{2n})$. For $(k,l) \in \mathbb{Z}^{2n}$, the twisted translation $T^t_{(k,l)}\phi$ of ϕ is defined by

$$T^t_{(k,l)}\phi(x,y) = e^{\pi i(x\cdot l - y\cdot k)}\phi(x-k, y-l), \qquad (x,y) \in \mathbb{R}^{2n}.$$

Using the definition of twisted translation, we have

$$T^t_{(k_1,l_1)}T^t_{(k_2,l_2)} = e^{-\pi i(k_1\cdot l_2 - l_1\cdot k_2)}T^t_{(k_1+k_2,l_1+l_2)}, \qquad \forall\ (k_1,l_1),(k_2,l_2) \in \mathbb{Z}^{2n}. \tag{2.2}$$

REMARK 2.8. Let G be a locally compact group. Let Π be a unitary irreducible representation of G on $\mathcal{H}$. Let V_g denote the matrix coefficient of Π, namely $V_g f(x) = \langle \Pi(x)g, f\rangle$, $f,g \in \mathcal{H}$. Then it is well known that V_g is the intertwining operator between Π and the left regular representation L. More precisely $V_g(\Pi(x)f) = L_x V_g(f)$, $x \in G$ (see [13]). In particular, if G is taken to be the Heisenberg group $\mathbb{H}^n$, π_1, the Schrödinger representation of $\mathbb{H}^n$ on $L^2(\mathbb{R}^n)$, $L_{(x,y,t)}$, the left translation on $\mathbb{H}^n$, then we get the corresponding intertwining operator V_g. However, since the definition of twisted translation $T^t_{(k,l)}$,

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mentioned in this paper, is a projective representation, we obtain the following relation: $|V_g\pi_1(x,y,0)f| = |T^t_{(x,y)}V_g f|$.

LEMMA 2.9 [18]. *Let $\phi \in L^2(\mathbb{R}^{2n})$. Then the kernel of the Weyl transform of $T^t_{(k,l)}\phi$ satisfies*

$$K_{T^t_{(k,l)}\phi}(\xi,\eta) = e^{\pi i(2\xi+l)\cdot k}K_\phi(\xi+l,\eta).$$

DEFINITION 2.10 [18]. For $\phi \in L^2(\mathbb{R}^{2n})$ and $\xi \in \mathbb{R}^n$, the function ω_ϕ is defined by

$$\omega_\phi(\xi) := \sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|K_\phi(\xi+m,\eta)|^2\,d\eta.$$

In [18], the characterization theorems such as the system of twisted translates forming a frame sequence or a Riesz sequence are given in terms of the function ω_ϕ.

DEFINITION 2.11 [18]. A function $\phi \in L^2(\mathbb{R}^{2n})$ is said to satisfy "Condition C" if

$$\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}K_\phi(\xi+m,\eta)\overline{K_\phi(\xi+m+l,\eta)}\,d\eta = 0,\ a.e.\ \xi\in\mathbb{T}^n,\ \forall\ l\in\mathbb{Z}^n\setminus\{0\}.$$

§3. Frame sequence in a twisted shift-invariant space

Let $\mathcal{A} := \{\phi_s : s \in \mathbb{Z}\}$ be the countable collection of functions in $L^2(\mathbb{R}^{2n})$ and $E^t(\mathcal{A}) := \{T^t_{(k,l)}\phi_s : k,l \in \mathbb{Z}^n, s \in \mathbb{Z}\}$. We shall denote span$(E^t(\mathcal{A}))$ by $U^t(\mathcal{A})$ and $\overline{U^t(\mathcal{A})}$ by $V^t(\mathcal{A})$. The space $V^t(\mathcal{A})$ is called a twisted shift-invariant space. When $A = \{\phi\}$, it is called a twisted (principal) shift-invariant space. We define $\tau : L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{T}^n, L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n)))$ by

$$\tau f(\xi)(\eta) := \{K_f(\xi+m,\eta)\}_{m\in\mathbb{Z}^n},\ \ \forall\ f\in L^2(\mathbb{R}^{2n}),\ \xi\in\mathbb{T}^n,\ \eta\in\mathbb{R}^n.$$

The map τ is called the fiber map. It can be shown that $L^2(\mathbb{R}^{2n})$ is isometric isomorphic to $L^2(\mathbb{T}^n, L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n)))$ via the fiber map τ. Using Lemma 2.9, the image of $T^t_{(k,l)}f$ under the fiber map τ is given by

$$\tau(T^t_{(k,l)}f)(\xi) = e^{\pi ik\cdot(2\xi+l)}\tau f(\xi+l), \qquad f\in L^2(\mathbb{R}^{2n}),\ k,l\in\mathbb{Z}^n. \tag{3.1}$$

Define $J : \mathbb{T}^n \to \{\text{closed subspaces of } L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))\}$ by $J(\xi) := \overline{\text{span}\{\tau\phi_s(\xi+l) : l\in\mathbb{Z}^n, s\in\mathbb{Z}\}}$. Then J is called a range function. For $\xi\in\mathbb{T}^n$, let $P(\xi)$ denote the orthogonal projection of $L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))$ onto $J(\xi)$. Then the range function J is said to be measurable if $\xi \mapsto \langle P(\xi)\Lambda_1,\Lambda_2\rangle$ is measurable for each $\Lambda_1,\Lambda_2 \in L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))$. The following proposition gives the connection between the image of $V^t(\mathcal{A})$ under the fiber map and the range function.

PROPOSITION 3.1. *For a measurable range function J, define $M_J := \{\Phi \in L^2(\mathbb{T}^n, L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))) : \Phi(\xi) \in J(\xi)$ for a.e. $\xi\in\mathbb{T}^n\}$. Then M_J and $\tau(V^t(\mathcal{A}))$ satisfy the following properties.*

(i) *M_J is a closed subspace of $L^2(\mathbb{T}^n, L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n)))$.*
(ii) *$\tau(V^t(\mathcal{A}))$ is a closed subspace of $L^2(\mathbb{T}^n, L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n)))$. Moreover $\tau(V^t(\mathcal{A}))$ is closed under multiplication by exponentials. In other words, $\Phi\in\tau(V^t(\mathcal{A}))$ implies $e^{2\pi i\langle\cdot,k\rangle}\Phi(\cdot)\in\tau(V^t(\mathcal{A}))$, $\forall\ k\in\mathbb{Z}^n$.*
(iii) *$M_J = \tau(V^t(\mathcal{A}))$.*

The proof follows similar lines as in the proof of Proposition 1.5 in [4].

For $\xi \in \mathbb{T}^n, l \in \mathbb{Z}^n$, define $H_l(\xi) : \ell^2(\mathbb{Z}) \to L^2(\mathbb{R}^n, \ell^2(\mathbb{Z}^n))$ by

$$H_l(\xi)(\{c_s\}) = \sum_{s\in\mathbb{Z}} c_s \tau\phi_s(\xi + l).$$

Suppose $H_l(\xi)$ is a bounded linear operator. Then we find its adjoint using the relation

$$\langle \{c_s\}, H_l(\xi)^*\phi \rangle_{\ell^2(\mathbb{Z})} = \langle H_l(\xi)(\{c_s\}), \phi \rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))},$$

$\forall$ $\{c_s\} \in \ell^2(\mathbb{Z})$ and $\phi \in L^2(\mathbb{R}^n, \ell^2(\mathbb{Z}^n))$. Thus,

$$\begin{aligned}
\langle H_l(\xi)(\{c_s\}_{n\in\mathbb{Z}}), \phi \rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} &= \Big\langle \sum_{s\in\mathbb{Z}} c_s \tau\phi_s(\xi + l), \phi \Big\rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} \\
&= \sum_{s\in\mathbb{Z}} c_s \langle \tau\phi_s(\xi + l), \phi \rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} \\
&= \sum_{s\in\mathbb{Z}} c_s \overline{\langle \phi, \tau\phi_s(\xi + l) \rangle}_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} \\
&= \langle \{c_s\}_{s\in\mathbb{Z}}, \{\langle \phi, \tau\phi_s(\xi + l)\rangle\}_{s\in\mathbb{Z}} \rangle_{\ell^2(\mathbb{Z})}.
\end{aligned}$$

Hence, by the uniqueness of adjoint of an operator, we get

$$H_l(\xi)^*\phi = \{\langle \phi, \tau\phi_s(\xi + l)\rangle\}_{s\in\mathbb{Z}}.$$

Now, we make use of the following result in order to obtain a characterization for the system of twisted translates to form a frame sequence.

LEMMA 3.2. *Let $f \in L^2(\mathbb{R}^{2n})$, and let $H_l(\xi)$ be a bounded linear operator from $\ell^2(\mathbb{Z})$ into $L^2(\mathbb{R}^n, \ell^2(\mathbb{Z}^n))$ for each $l \in \mathbb{Z}^n$, $\xi \in \mathbb{T}^n$. Then $H_l(\xi)^*$ satisfies*

$$\sum_{(k,s)\in\mathbb{Z}^{n+1}} \left|\langle f, T^t_{(k,l)}\phi_s \rangle\right|^2 = \int_{\mathbb{T}^n} \|H_l(\xi)^*\tau f(\xi)\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))}\, d\xi\ , \tag{3.2}$$

for each $l \in \mathbb{Z}^n$.

Proof. Consider

$$\begin{aligned}
\langle f, T^t_{(k,l)}\phi_s \rangle_{L^2(\mathbb{R}^{2n})} &= \langle \tau f, \tau(T^t_{(k,l)}\phi_s) \rangle_{L^2(\mathbb{T}^n, L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n)))} \\
&= \int_{\mathbb{T}^n} \langle \tau f(\xi), \tau(T^t_{(k,l)}\phi_s)(\xi) \rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))}\, d\xi \\
&= \int_{\mathbb{T}^n} \int_{\mathbb{R}^n} \langle \tau f(\xi)(\eta), \tau(T^t_{(k,l)}\phi_s)(\xi)(\eta) \rangle_{\ell^2(\mathbb{Z}^n)} d\eta\, d\xi \\
&= e^{-\pi i k\cdot l} \int_{\mathbb{T}^n} \int_{\mathbb{R}^n} \langle \tau f(\xi)(\eta), \tau\phi_s(\xi + l)(\eta) \rangle_{\ell^2(\mathbb{Z}^n)} e^{-2\pi i k\cdot\xi} d\eta\, d\xi \\
&= e^{-\pi i k\cdot l} \int_{\mathbb{T}^n} \langle \tau f(\xi), \tau\phi_s(\xi + l) \rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} e^{-2\pi i k\cdot\xi}\, d\xi \\
&= e^{-\pi i k\cdot l} \int_{\mathbb{T}^n} F_{l,s}(\xi) e^{-2\pi i k\cdot\xi}\, d\xi \\
&= e^{-\pi i k\cdot l} \widehat{F_{l,s}}(k),
\end{aligned} \tag{3.3}$$

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where $F_{l,s}(\xi) = \langle \tau f(\xi), \tau\phi_s(\xi+l)\rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))}$. Hence, we get

$$\begin{aligned}\sum_{k\in\mathbb{Z}^n} \left|\langle f, T^t_{(k,l)}\phi_s\rangle\right|^2 &= \sum_{k\in\mathbb{Z}^n} |\widehat{F_{l,s}}(k)|^2 \\ &= \|F_{l,s}\|^2_{L^2(\mathbb{T}^n)} \\ &= \int_{\mathbb{T}^n} |F_{l,s}(\xi)|^2\, d\xi, \end{aligned} \tag{3.4}$$

by applying the Plancherel formula for the Fourier series. Now, using the definition of $H_l(\xi)^*$, we obtain

$$\begin{aligned}\sum_{(k,s)\in\mathbb{Z}^{n+1}} \left|\langle f, T^t_{(k,l)}\phi_s\rangle\right|^2 &= \int_{\mathbb{T}^n} \sum_{s\in\mathbb{Z}} \left|\langle \tau f(\xi), \tau\phi_s(\xi+l)\rangle_{L^2(\mathbb{R}^n,l^2(\mathbb{Z}^n))}\right|^2 d\xi \\ &= \int_{\mathbb{T}^n} \|H_l(\xi)^*\tau f(\xi)\|^2\, d\xi,\end{aligned}$$

proving our assertion. ∎

THEOREM 3.3. *The collection $E^t(\mathcal{A})$ is a frame sequence with frame bounds $A, B > 0$ if and only if*

$$AP(\xi) \le \sum_{l\in\mathbb{Z}^n} H_l(\xi)H_l(\xi)^* \le BP(\xi), \ \textit{for a.e. } \xi\in\mathbb{T}^n. \tag{3.5}$$

Proof. Suppose $E^t(\mathcal{A})$ is a frame sequence with frame bounds $A, B > 0$. In other words,

$$A\|f\|^2 \le \sum_{(k,l,s)\in\mathbb{Z}^{2n+1}} \left|\langle f, T^t_{(k,l)}\phi_s\rangle\right|^2 \le B\|f\|^2, \qquad \forall\, f \in V^t(\mathcal{A}). \tag{3.6}$$

Let $\Phi(\xi) \in J(\xi)$, $\xi \in \mathbb{T}^n$. Consider a trigonometric polynomial

$$\psi(\xi) = \sum_{k\in\mathcal{F}} c_k e^{2\pi i k\cdot\xi},$$

where $\mathcal{F}$ is a finite subset of $\mathbb{Z}^n$. Define $f \in L^2(\mathbb{R}^{2n})$ by $\tau f(\xi) = \psi(\xi)\Phi(\xi)$, $\xi\in\mathbb{T}^n$. More explicitly, it can be written as

$$\tau f(\xi) = \sum_{k\in\mathcal{F}} c_k e^{2\pi i k\cdot\xi}\Phi(\xi).$$

By using Proposition 3.1, we can show that $\tau f(\xi)\in J(\xi)$, which implies that $f \in V^t(\mathcal{A})$. Further, we have $H_l(\xi)^*\tau f(\xi) = \psi(\xi)H_l(\xi)^*\Phi(\xi)$. Now, from Lemma 3.2, we get

$$\begin{aligned}\sum_{(k,l,s)\in\mathbb{Z}^{2n+1}} \left|\langle f, T^t_{(k,l)}\phi_s\rangle\right|^2 &= \int_{\mathbb{T}^n} \sum_{l\in\mathbb{Z}^n} \|H_l(\xi)^*\tau f(\xi)\|^2\, d\xi \\ &= \int_{\mathbb{T}^n} \sum_{l\in\mathbb{Z}^n} \|\psi(\xi)H_l(\xi)^*\Phi(\xi)\|^2\, d\xi \\ &= \int_{\mathbb{T}^n} |\psi(\xi)|^2 \sum_{l\in\mathbb{Z}^n} \|H_l(\xi)^*\Phi(\xi)\|^2\, d\xi. \end{aligned}\tag{3.7}$$

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On the other hand,

$$\begin{aligned}\|f\|^2 &= \|\tau f\|^2 \\ &= \int_{\mathbb{T}^n} \|\tau f(\xi)\|^2\, d\xi \\ &= \int_{\mathbb{T}^n} \|\psi(\xi)\Phi(\xi)\|^2\, d\xi \\ &= \int_{\mathbb{T}^n} |\psi(\xi)|^2 \|\Phi(\xi)\|^2\, d\xi. \end{aligned} \tag{3.8}$$

Using (3.7) and (3.8) in (3.6), we get

$$A\int_{\mathbb{T}^n} |\psi(\xi)|^2\|\Phi(\xi)\|^2\, d\xi \le \int_{\mathbb{T}^n} |\psi(\xi)|^2 \sum_{l\in\mathbb{Z}^n} \|H_l(\xi)^*\Phi(\xi)\|^2\, d\xi \le B\int_{\mathbb{T}^n} |\psi(\xi)|^2\|\Phi(\xi)\|^2\, d\xi. \tag{3.9}$$

Since (3.9) holds for any trigonometric polynomial $\psi \in L^2(\mathbb{T}^n)$, we arrive at

$$A\|\Phi(\xi)\|^2 \le \sum_{l\in\mathbb{Z}^n} \|H_l(\xi)^*\Phi(\xi)\|^2 \le B\|\Phi(\xi)\|^2, \text{ for } a.e.\ \xi \in \mathbb{T}^n. \tag{3.10}$$

Now, the above equation together with $\ker\left(\sum_{l\in\mathbb{Z}^n} H_l(\xi)H_l(\xi)^*\right) = J(\xi)^\perp$ leads to (3.5). Conversely, assume that (3.5) holds. Let $f \in V^t(\mathcal{A})$. Then, $\tau f(\xi) \in J(\xi)$ by using Proposition 3.1. Hence, (3.5) reduces to

$$A\|\tau f(\xi)\|^2 \le \sum_{l\in\mathbb{Z}^n} \|H_l(\xi)^*\tau f(\xi)\|^2 \le B\|\tau f(\xi)\|^2, \text{ for } a.e.\ \xi \in \mathbb{T}^n.$$

Taking integral with respect to ξ in the above inequality, we get

$$A\int_{\mathbb{T}^n} \|\tau f(\xi)\|^2 d\xi \le \int_{\mathbb{T}^n} \sum_{l\in\mathbb{Z}^n} \|H_l(\xi)^*\tau f(\xi)\|^2 d\xi \le B\int_{\mathbb{T}^n} \|\tau f(\xi)\|^2 d\xi.$$

Now, using Lemma 3.2 and the isometry of τ in the above inequality, we get

$$A\|f\|^2 \le \sum_{(k,l,s)\in\mathbb{Z}^{2n+1}} \left|\left\langle f, T^t_{(k,l)}\phi_s\right\rangle\right|^2 \le B\|f\|^2.$$

Since $f \in V^t(\mathcal{A})$ is arbitrary, the assertion follows. ∎

Now, we obtain the following corollaries as a consequence of Theorem 3.3.

COROLLARY 3.4. *The collection $E^t(\mathcal{A})$ is a Parseval frame sequence if and only if*

$$\sum_{l\in\mathbb{Z}^n} H_l(\xi)H_l(\xi)^* = P(\xi), \qquad \text{for } a.e.\ \xi \in \mathbb{T}^n.$$

COROLLARY 3.5. *The collection $E^t(\mathcal{A})$ is a frame sequence with frame bounds A and B if and only if the collection $\{\tau\phi_s(\xi+l) : s \in \mathbb{Z}, l \in \mathbb{Z}^n\}$ is a frame sequence with the same bounds, for a.e. $\xi \in \mathbb{T}^n$.*

Proof. Assume that $E^t(\mathcal{A})$ is a frame sequence with frame bounds A and B. Now, by Theorem 3.3, it is equivalent to

$$A\|\tau f(\xi)\|^2 \leq \sum_{(l,s)\in\mathbb{Z}^{n+1}} |\langle \tau f(\xi), \tau\phi_s(\xi+l)\rangle|^2 \leq B\|\tau f(\xi)\|^2$$

$\forall\ f \in V^t(\mathcal{A})$ and for *a.e.* $\xi \in \mathbb{T}^n$, which proves our assertion. ∎

THEOREM 3.6. *The collection $E^t(\mathcal{A})$ is a frame sequence with frame bounds A and B if and only if*

$$A\|\tau f(\xi)\|^2 \leq \big\langle \tau(Sf)(\xi), \tau f(\xi)\big\rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} \leq B\|\tau f(\xi)\|^2, \tag{3.11}$$

$\forall\ f \in V^t(\mathcal{A})$ *and for a.e.* $\xi \in \mathbb{T}^n$, *where S is the frame operator associated with the system $E^t(\mathcal{A})$.*

Proof. The frame operator S associated with $E^t(\mathcal{A})$ is given by

$$Sf = \sum_{(k,l,s)\in\mathbb{Z}^{2n+1}} \big\langle f, T^t_{(k,l)}\phi_s\big\rangle T^t_{(k,l)}\phi_s, \qquad \forall\ f \in V^t(\mathcal{A}).$$

Using (3.1) and (3.3), we get

$$\begin{aligned}
\tau(Sf)(\xi) &= \sum_{(k,l,s)\in\mathbb{Z}^{2n+1}} \big\langle f, T^t_{(k,l)}\phi_s\big\rangle \tau(T^t_{(k,l)}\phi_s)(\xi) \\
&= \sum_{(k,l,s)\in\mathbb{Z}^{2n+1}} \big\langle f, T^t_{(k,l)}\phi_s\big\rangle e^{\pi ik\cdot(2\xi+l)}\tau\phi_s(\xi+l) \\
&= \sum_{(l,s)\in\mathbb{Z}^{n+1}} \Big(\sum_{k\in\mathbb{Z}^n} \big\langle f, T^t_{(k,l)}\phi_s\big\rangle e^{\pi ik\cdot(2\xi+l)}\Big)\tau\phi_s(\xi+l) \\
&= \sum_{(l,s)\in\mathbb{Z}^{n+1}} \Big(\sum_{k\in\mathbb{Z}^n} \widehat{F}_{l,s}(k) e^{2\pi ik\cdot\xi}\Big)\tau\phi_s(\xi+l) \\
&= \sum_{(l,s)\in\mathbb{Z}^{n+1}} F_{l,s}(\xi)\ \tau\phi_s(\xi+l) \\
&= \sum_{(l,s)\in\mathbb{Z}^{n+1}} \big\langle \tau f(\xi), \tau\phi_s(\xi+l)\big\rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} \tau\phi_s(\xi+l),
\end{aligned}$$

by applying the Fourier series of $F_{l,s}$. Now, using the definition of $H_l(\xi)$ and its adjoint, we have

$$\begin{aligned}
\sum_{l\in\mathbb{Z}^n} H_l(\xi)H_l(\xi)^*\tau f(\xi) &= \sum_{(l,s)\in\mathbb{Z}^{n+1}} \big\langle \tau f(\xi), \tau\phi_s(\xi+l)\big\rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} \tau\phi_s(\xi+l) \\
&= \tau(Sf)(\xi).
\end{aligned} \tag{3.12}$$

By Theorem 3.3, the system $E^t(\mathcal{A})$ is a frame sequence with frame bounds A and B if and only if

$$A\|\tau f(\xi)\|^2 \leq \Big\langle \sum_{l\in\mathbb{Z}^n} H_l(\xi)H_l(\xi)^*\tau f(\xi), \tau f(\xi)\Big\rangle \leq B\|\tau f(\xi)\|^2,$$

for *a.e.* $\xi \in \mathbb{T}^n$. Now, using (3.12) in the above inequality, we arrive at the required result. ∎

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For $\phi \in L^2(\mathbb{R}^{2n})$ and $\xi \in \mathbb{T}^n$, we define

$$W_\phi(\xi) := \frac{1}{\omega_\phi(\xi)} \sum_{l \in \mathbb{Z}^n} \Big| \sum_{m \in \mathbb{Z}^n} \int_{\mathbb{R}^n} K_\phi(\xi+m,\eta)\overline{K_\phi(\xi+m+l,\eta)}\, d\eta \Big|^2, \text{ whenever } \omega_\phi(\xi) \neq 0.$$

THEOREM 3.7. *Let $\mathcal{A} = \{\phi\}$, and let $E^t(\mathcal{A})$ be a frame sequence with frame bounds A and B. Then $A \le W_\phi(\xi) \le B$ for a.e. $\xi \in \mathbb{T}^n \setminus N$, where $N := \{\xi \in \mathbb{T}^n : w_\phi(\xi) = 0\}$.*

Proof. Using Corollary 3.5, we have

$$A\|\Phi(\xi)\|^2 \le \sum_{l \in \mathbb{Z}^n} |\langle \Phi(\xi), \tau\phi(\xi+l)\rangle|^2 \le B\|\Phi(\xi)\|^2,$$

$\forall\ \Phi(\xi) \in J(\xi)$ and for *a.e.* $\xi \in \mathbb{T}^n$. Taking $\Phi(\xi) = \tau\phi(\xi) \in J(\xi)$ in the above inequality, we obtain

$$A\|\tau\phi(\xi)\|^2 \le \sum_{l \in \mathbb{Z}^n} |\langle \tau\phi(\xi), \tau\phi(\xi+l)\rangle|^2 \le B\|\tau\phi(\xi)\|^2,$$

for *a.e.* $\xi \in \mathbb{T}^n$. Now, using the definition of the fiber map τ, we have

$$\begin{aligned} A \sum_{m \in \mathbb{Z}^n} \int_{\mathbb{R}^n} |K_\phi(\xi+m,\eta)|^2 d\eta &\le \sum_{l \in \mathbb{Z}^n} \Big| \sum_{m \in \mathbb{Z}^n} \int_{\mathbb{R}^n} K_\phi(\xi+m,\eta) K_\phi(\xi+m+l,\eta) d\eta \Big|^2 \\ &\le B \sum_{m \in \mathbb{Z}^n} \int_{\mathbb{R}^n} |K_\phi(\xi+m,\eta)|^2 d\eta, \end{aligned}$$

which in turn implies that $A \le W_\phi(\xi) \le B$ for *a.e.* $\xi \in \mathbb{T}^n \setminus N$. ∎

For $l_1, l_2 \in \mathbb{Z}^n$ and $\xi \in \mathbb{T}^n$, using the definition of the fiber map τ, we have

$$\begin{aligned} \langle \tau\phi(\xi+l_1), \tau\phi(\xi+l_2)\rangle_{L^2(\mathbb{R}^n, \ell^2(\mathbb{Z}^n))} &= \int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} K_\phi(\xi+m+l_1,\eta)\overline{K_\phi(\xi+m+l_2,\eta)}\, d\eta \\ &= \int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} K_\phi(\xi+m,\eta)\overline{K_\phi(\xi+m+l_2-l_1,\eta)}\, d\eta. \end{aligned} \tag{3.13}$$

COROLLARY 3.8 [18]. *Let $\mathcal{A} = \{\phi\}$ and ϕ satisfy "Condition C." Then $E^t(\mathcal{A})$ is a frame sequence with frame bounds A and B if and only if*

$$A \le \omega_\phi(\xi) \le B\ , \qquad for\ a.e.\ \xi \in \mathbb{T}^n \setminus N. \tag{3.14}$$

Proof. Let $E^t(\mathcal{A})$ be a frame sequence with frame bounds A and B. Then, by Theorem 3.7, we have $A \le W_\phi(\xi) \le B$, for *a.e.* $\xi \in \mathbb{T}^n \setminus N$. Since ϕ satisfies "Condition C," we get

$$\begin{aligned} W_\phi(\xi) &= \frac{1}{\omega_\phi(\xi)} \Big| \sum_{m \in \mathbb{Z}^n} \int_{\mathbb{R}^n} |K_\phi(\xi+m,\eta)|^2\, d\eta \Big|^2 \\ &= \frac{1}{\omega_\phi(\xi)}\ \omega_\phi^2(\xi) \\ &= \omega_\phi(\xi)\ , \end{aligned} \tag{3.15}$$

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for $\xi \in \mathbb{T}^n \setminus N$, which in turn implies (3.14). Conversely, assume that (3.14) holds. Let $\Phi(\xi) = \sum_{l' \in \mathbb{Z}^n} \alpha_{l'}(\xi)\tau\phi(\xi + l') \in J(\xi)$. Then, using (3.13), we have

$$\begin{aligned}\big\langle \Phi(\xi), \tau\phi(\xi+l)\big\rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} &= \sum_{l' \in \mathbb{Z}^n} \alpha_{l'}(\xi)\big\langle \tau\phi(\xi+l'), \tau\phi(\xi+l)\big\rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} \\ &= \sum_{l' \in \mathbb{Z}^n} \alpha_{l'}(\xi)\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n} K_\phi(\xi+m,\eta)\overline{K_\phi(\xi+m+l-l',\eta)}\,d\eta\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\|\Phi(\xi)\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} &= \big\langle \Phi(\xi), \Phi(\xi)\big\rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} \\ &= \sum_{l_1,l_2\in\mathbb{Z}^n} \alpha_{l_1}(\xi)\overline{\alpha_{l_2}(\xi)}\big\langle \tau\phi(\xi+l_1,\tau\phi(\xi+l_2))\big\rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} \\ &= \sum_{l_1,l_2\in\mathbb{Z}^n} \alpha_{l_1}(\xi)\overline{\alpha_{l_2}(\xi)}\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n} K_\phi(\xi+m,\eta)\overline{K_\phi(\xi+m+l_2-l_1,\eta)}\,d\eta.\end{aligned} \tag{3.17}$$

Since ϕ satisfies "Condition C," (3.16) reduces to

$$\begin{aligned}\big\langle \Phi(\xi), \tau\phi(\xi+l)\big\rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} &= \alpha_l(\xi)\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n} |K_\phi(\xi+m,\eta)|^2\,d\eta \\ &=\alpha_l(\xi)\ \omega_\phi(\xi)\end{aligned} \tag{3.18}$$

and (3.17) reduces to

$$\begin{aligned}\|\Phi(\xi)\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} &= \sum_{l_1\in\mathbb{Z}^n} |\alpha_{l_1}(\xi)|^2\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n} |K_\phi(\xi+m,\eta)|^2\,d\eta \\ &=\omega_\phi(\xi)\sum_{l_1\in\mathbb{Z}^n}|\alpha_{l_1}(\xi)|^2.\end{aligned} \tag{3.19}$$

Now, using (3.18) and (3.19), we obtain

$$\begin{aligned}\sum_{l\in\mathbb{Z}^n}|\big\langle \Phi(\xi), \tau\phi(\xi+l)\big\rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))}|^2 &= \sum_{l\in\mathbb{Z}^n}|\alpha_l(\xi)|^2\ \omega_\phi^2(\xi) \\ &=\omega_\phi(\xi)\Big(\omega_\phi(\xi)\sum_{l\in\mathbb{Z}^n}|\alpha_l(\xi)|^2\Big) \\ &=\omega_\phi(\xi)\ \|\Phi(\xi)\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))}.\end{aligned} \tag{3.20}$$

Now, making use of (3.14), we have

$$A\|\Phi(\xi)\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} \le \omega_\phi(\xi)\ \|\Phi(\xi)\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} \le B\|\Phi(\xi)\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))},$$

for $a.e.\ \xi \in \mathbb{T}^n \setminus N$. Hence, (3.20) gives

$$A\|\Phi(\xi)\|^2_{L^2(\mathbb{R}^n,l^2(\mathbb{Z}^n))} \le \sum_{l\in\mathbb{Z}^n}|\big\langle \Phi(\xi), \tau\phi(\xi+l)\big\rangle_{L^2(\mathbb{R}^n,l^2(\mathbb{Z}^n))}|^2 \le B\|\Phi(\xi)\|^2_{L^2(\mathbb{R}^n,l^2(\mathbb{Z}^n))}, \tag{3.21}$$

for *a.e.* $\xi \in \mathbb{T}^n \setminus N$. Moreover, from (3.19) and (3.20), we have

$$\|\Phi(\xi)\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} = 0 \quad \text{and} \quad \sum_{l\in\mathbb{Z}^n} |\langle \Phi(\xi), \tau\phi(\xi+l)\rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))}|^2 = 0, \quad \forall\, \xi \in N.$$

Thus, (3.21) holds for all $\xi \in N$, all sides being equal to zero. Therefore, using Corollary 3.5, we obtain the required result. ∎

Now, we provide an example of a function $\phi \in L^2(\mathbb{R}^2)$ such that the system $\{T^t_{(k,l)}\phi : k,l \in \mathbb{Z}\}$ forms a frame sequence (i) when ϕ satisfies "Condition C" and (ii) ϕ does not satisfy "Condition C." Before stating the example, we observe the following.

Let $f,g \in L^2(\mathbb{R}^{2n})$ and $l \in \mathbb{Z}^n$. Then using a similar calculation as in (3.3), we get

$$\langle f, T^t_{(k,l)} g\rangle_{L^2(\mathbb{R}^{2n})} = e^{-\pi i k\cdot l}\widehat{F_l}(k)\ ,$$

where $F_l(\xi) = \langle \tau f(\xi), \tau g(\xi+l)\rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))}$. Making use of the Fourier series of F_l, we get

$$F_l(\xi) = \sum_{k\in\mathbb{Z}^n} \langle f, T^t_{(k,l)} g\rangle e^{\pi i k\cdot(2\xi+l)}. \tag{3.22}$$

On the other hand, using the definition of the map τ, we have

$$F_l(\xi) = \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} K_f(\xi+m,\eta)\overline{K_g(\xi+m+l,\eta)}\, d\eta. \tag{3.23}$$

Therefore, from (3.22) and (3.23), we get

$$\sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} K_f(\xi+m,\eta)\overline{K_g(\xi+m+l,\eta)}\, d\eta = \sum_{k\in\mathbb{Z}^n} \langle f, T^t_{(k,l)} g\rangle e^{\pi i k\cdot(2\xi+l)}. \tag{3.24}$$

Let $f = g = \phi$ (say) in (3.24). Then the identity (3.24) can be used to verify "Condition C" for the function ϕ.

EXAMPLE 3.9. Define $\phi := \chi_{[0,2]\times[0,1]}$. For $k,l \in \mathbb{Z}$, consider

$$\begin{aligned} \langle \phi, T^t_{(k,l)}\phi\rangle &= \int_{\mathbb{R}^2} \phi(x,y)\overline{T^t_{(k,l)}\phi(x,y)}\,dydx \\ &= \int_0^2\int_0^1 e^{-\pi i(xl-yk)}\overline{\phi(x-k,y-l)}\,dydx \\ &= \int_{-k}^{2-k}\int_{-l}^{1-l} e^{-\pi i((x+k)l-(y+l)k)}\phi(x,y)\,dydx \\ &= \int_{[-k,2-k]\cap[0,2]}\int_{[-l,1-l]\cap[0,1]} e^{-\pi i(xl-ky)}\,dydx. \end{aligned} \tag{3.25}$$

From the range of the above two integrals, we see that $\langle \phi, T^t_{(k,l)}\phi\rangle \neq 0$ only when $k = -1,0,1$ and $l = 0$. Hence, using (3.24) for $l \neq 0$ and $f = g = \phi$, we have

$$\begin{aligned} \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} K_\phi(\xi+m,\eta)\overline{K_\phi(\xi+m+l,\eta)}\,d\eta &= \sum_{k\in\mathbb{Z}} \langle \phi, T^t_{(k,l)}\phi\rangle e^{\pi i k(2\xi+l)} \\ &= 0, \end{aligned}$$

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for all $\xi \in \mathbb{T}$, showing that ϕ satisfies "Condition C." Further, simplifying (3.25), we get

$$\langle \phi, T^t_{(k,l)}\phi \rangle = \begin{cases} \frac{2}{\pi i}, & (k,l) = (-1,0), \\ 2, & (k,l) = (0,0), \\ -\frac{2}{\pi i}, & (k,l) = (1,0), \\ 0, & otherwise. \end{cases}$$

Using (3.24) for $f = g = \phi$ and $l = 0$, we obtain

$$\begin{aligned} \omega_\phi(\xi) &= \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} |K_\phi(\xi+m,\eta)|^2 \, d\eta \\ &= \sum_{k\in\mathbb{Z}} \langle \phi, T^t_{(k,0)}\phi \rangle e^{2\pi i k\xi} \\ &= \frac{2}{\pi i} e^{-2\pi i\xi} + 2 - \frac{2}{\pi i} e^{2\pi i \xi} \\ &= 2 - \frac{2}{\pi i}(e^{2\pi i\xi} - e^{-2\pi i \xi}) \\ &= 2 - \frac{4}{\pi}\sin(2\pi\xi). \end{aligned}$$

Since $|\frac{4}{\pi}\sin(2\pi\xi)| < \frac{4}{3}$, $\forall\ \xi \in (0,1)$, we get $\frac{2}{3} \le \omega_\phi(\xi) \le \frac{10}{3}$, for $a.e.\ \xi \in \mathbb{T}$. Therefore, using Corollary 3.8, we conclude that the system $\{T^t_{(k,l)}\phi : k,l \in \mathbb{Z}\}$ is a frame sequence with frame bounds $\frac{2}{3}$ and $\frac{10}{3}$.

Example 3.10. Let $\phi := \chi_{[0,1]\times[0,2]}$. For $k,l \in \mathbb{Z}$,

$$\begin{aligned} \langle \phi, T^t_{(k,l)}\phi \rangle &= \int_{\mathbb{R}^2} \phi(x,y)\overline{T^t_{(k,l)}\phi}\, dydx \\ &= \int_0^1 \int_0^2 e^{-\pi i(xl-yk)} \overline{\phi(x-k,y-l)}\, dydx \\ &= \int_{-k}^{1-k} \int_{-l}^{2-l} e^{-\pi i((x+k)l-(y+l)k)} \phi(x,y)\, dydx \\ &= \int_{x\in[-k,1-k]\cap[0,1]} \int_{y\in[-l,2-l]\cap[0,2]} e^{-\pi i(xl-ky)}\, dydx. \end{aligned}$$

From the range of the above integrals, we see that $\langle \phi, T^t_{(k,l)}\phi \rangle \neq 0$ only when $k = 0$ and $l = -1,0,1$. Computing explicitly, we get

$$\langle \phi, T^t_{(k,l)}\phi \rangle = \begin{cases} \frac{-2}{\pi i}, & (k,l) = (0,-1), \\ 2, & (k,l) = (0,0), \\ \frac{2}{\pi i}, & (k,l) = (0,1), \\ 0, & otherwise. \end{cases} \tag{3.26}$$

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Using (3.24) for $f = g = \phi$ and $l = 1$, we get

$$\begin{aligned}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}} K_\phi(\xi+m,\eta)\overline{K_\phi(\xi+m+1,\eta)}d\eta &= \sum_{k\in\mathbb{Z}}\langle \phi, T^t_{(k,1)}\phi\rangle e^{\pi i k(2\xi+1)}\\ &= \langle \phi, T^t_{(0,1)}\phi\rangle\\ &= \frac{2}{\pi i} \neq 0,\end{aligned}$$

for all $\xi \in \mathbb{T}$, showing that ϕ does not satisfy "Condition C." Now, using Corollary 3.5, we aim to prove that the system $\{T^t_{(k,l)}\phi : k,l \in \mathbb{Z}\}$ forms a frame sequence. Let $\Phi(\xi) = \sum_{l\in\mathbb{Z}} \alpha_l(\xi)\tau\phi(\xi+l) \in J(\xi)$ for some $\{\alpha_l(\xi)\}_{l\in\mathbb{Z}} \in c_{00}(\mathbb{Z})$, for $\xi \in \mathbb{T}$. Then, using (3.13) and (3.24), we get

$$\begin{aligned}\|\Phi(\xi)\|^2_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))} &= \sum_{l_1,l_2\in\mathbb{Z}} \alpha_{l_1}(\xi)\overline{\alpha_{l_2}(\xi)}\langle \tau\phi(\xi+l_1), \tau\phi(\xi+l_2)\rangle_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))}\\ &= \sum_{l_1,l_2\in\mathbb{Z}} \alpha_{l_1}(\xi)\overline{\alpha_{l_2}(\xi)}\int_{\mathbb{R}}\sum_{m\in\mathbb{Z}} K_\phi(\xi+m,\eta)\overline{K_\phi(\xi+m+l_2-l_1,\eta)}\,d\eta\\ &= \sum_{l_1,l_2\in\mathbb{Z}} \alpha_{l_1}(\xi)\overline{\alpha_{l_2}(\xi)}\sum_{k\in\mathbb{Z}}\langle \phi, T^t_{(k,l_2-l_1)}\phi\rangle e^{\pi i k(2\xi+l_2-l_1)}. \qquad (3.27)\end{aligned}$$

Now, applying (3.26) in (3.27), we get

$$\begin{aligned}\|\Phi(\xi)\|^2_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))} &= \sum_{l\in\mathbb{Z}} \alpha_l(\xi)\Big(2\overline{\alpha_l(\xi)} + \frac{2}{\pi i}\overline{\alpha_{l+1}(\xi)} - \frac{2}{\pi i}\overline{\alpha_{l-1}(\xi)}\Big)\\ &= \frac{2}{\pi i}\sum_{l\in\mathbb{Z}} \alpha_l(\xi)\big(\overline{\alpha_{l+1}(\xi)} - \overline{\alpha_{l-1}(\xi)}\big) + 2\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2\\ &= \frac{2}{\pi i}\left(\sum_{l\in\mathbb{Z}} \alpha_l(\xi)\overline{\alpha_{l+1}(\xi)} - \overline{\sum_{l\in\mathbb{Z}} \alpha_l(\xi)\overline{\alpha_{l+1}(\xi)}}\right) + 2\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2\\ &= \frac{2}{\pi i}\left(2i\,\mathrm{Im}\left(\sum_{l\in\mathbb{Z}} \alpha_l(\xi)\overline{\alpha_{l+1}(\xi)}\right)\right) + 2\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2\\ &= \frac{4}{\pi}\,\mathrm{Im}\left(\sum_{l\in\mathbb{Z}} \alpha_l(\xi)\overline{\alpha_{l+1}(\xi)}\right) + 2\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2. \qquad (3.28)\end{aligned}$$

Again using (3.13) and (3.24), we have

$$\begin{aligned}\langle \Phi(\xi), \tau\phi(\xi+l)\rangle_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))} &= \sum_{l_1\in\mathbb{Z}} \alpha_{l_1}(\xi)\langle \tau\phi(\xi+l_1), \tau\phi(\xi+l)\rangle_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))}\\ &= \sum_{l_1\in\mathbb{Z}} \alpha_{l_1}(\xi)\int_{\mathbb{R}}\sum_{m\in\mathbb{Z}} K_\phi(\xi+m,\eta)\overline{K_\phi(\xi+m+l-l_1,\eta)}\,d\eta\\ &= \sum_{l_1\in\mathbb{Z}} \alpha_{l_1}(\xi)\sum_{k\in\mathbb{Z}}\langle \phi, T^t_{(k,l-l_1)}\phi\rangle e^{\pi i k(2\xi+l-l_1)}. \qquad (3.29)\end{aligned}$$

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Now, using (3.26) in (3.29), we get

$$\begin{aligned}\left\langle \Phi(\xi), \tau\phi(\xi+l)\right\rangle_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))} &= 2\alpha_l(\xi) + \frac{2}{\pi i}\alpha_{l-1}(\xi) - \frac{2}{\pi i}\alpha_{l+1}(\xi) \\ &= \frac{2}{\pi i}(\alpha_{l-1}(\xi) - \alpha_{l+1}(\xi)) + 2\alpha_l(\xi).\end{aligned}$$

Hence,

$$\begin{aligned}&\sum_{l\in\mathbb{Z}} |\left\langle \Phi(\xi), \tau\phi(\xi+l)\right\rangle_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))}|^2 \\ &\quad = \sum_{l\in\mathbb{Z}} \left|\frac{2}{\pi i}(\alpha_{l-1}(\xi) - \alpha_{l+1}(\xi)) + 2\alpha_l(\xi)\right|^2 \\ &\quad = \sum_{l\in\mathbb{Z}} \frac{4}{\pi^2}|\alpha_{l-1}(\xi) - \alpha_{l+1}(\xi)|^2 + \sum_{l\in\mathbb{Z}} 4|\alpha_l(\xi)|^2 + \sum_{l\in\mathbb{Z}} 2\,\mathrm{Re}\left(\frac{4}{\pi i}\overline{\alpha_l(\xi)}(\alpha_{l-1}(\xi) - \alpha_{l+1}(\xi))\right) \\ &\quad = \frac{4}{\pi^2}S_1 + 4\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2 + 8S_2, \end{aligned} \tag{3.30}$$

where

$$S_1 := \sum_{l\in\mathbb{Z}} |\alpha_{l-1}(\xi) - \alpha_{l+1}(\xi)|^2 \quad \text{and} \quad S_2 := \sum_{l\in\mathbb{Z}} \mathrm{Re}\big(\frac{1}{\pi i}\overline{\alpha_l(\xi)}(\alpha_{l-1}(\xi) - \alpha_{l+1}(\xi))\big).$$

But

$$\begin{aligned} S_1 &= \sum_{l\in\mathbb{Z}} \big(|\alpha_{l-1}(\xi)|^2 + |\alpha_{l+1}(\xi)|^2 - 2\,\mathrm{Re}(\alpha_{l-1}(\xi)\overline{\alpha_{l+1}(\xi)})\big) \\ &= 2\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2 - 2\,\mathrm{Re}\left(\sum_{l\in\mathbb{Z}}\alpha_{l-1}(\xi)\overline{\alpha_{l+1}(\xi)}\right) \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} S_2 &= \sum_{l\in\mathbb{Z}} \mathrm{Re}\left(-\frac{i}{\pi}\overline{\alpha_l(\xi)}(\alpha_{l-1}(\xi) - \alpha_{l+1}(\xi))\right) \\ &= \sum_{l\in\mathbb{Z}} \frac{-1}{\pi}\,\mathrm{Im}\left(\alpha_l(\xi)(\overline{\alpha_{l-1}(\xi)} - \overline{\alpha_{l+1}(\xi)})\right) \\ &= \frac{-1}{\pi}\,\mathrm{Im}\left(\sum_{l\in\mathbb{Z}}\alpha_l(\xi)\overline{\alpha_{l-1}(\xi)} - \overline{\sum_{l\in\mathbb{Z}}\alpha_l(\xi)\overline{\alpha_{l-1}(\xi)}}\right) \\ &= \frac{-1}{\pi}\,\mathrm{Im}\left(2i\,\mathrm{Im}\left(\sum_{l\in\mathbb{Z}}\alpha_l(\xi)\overline{\alpha_{l-1}(\xi)}\right)\right) \\ &= \frac{-2}{\pi}\,\mathrm{Im}\left(\sum_{l\in\mathbb{Z}}\alpha_l(\xi)\overline{\alpha_{l-1}(\xi)}\right). \end{aligned} \tag{3.32}$$

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Using (3.32) and (3.31) in (3.30), we get

$$\begin{aligned}
&\sum_{l\in\mathbb{Z}}|\langle\Phi(\xi),\tau\phi(\xi+l)\rangle_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))}|^2\\
&=\frac{8}{\pi^2}\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2-\frac{8}{\pi^2}\operatorname{Re}\left(\sum_{l\in\mathbb{Z}}\alpha_{l-1}(\xi)\overline{\alpha_{l+1}(\xi)}\right)+4\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2-\frac{16}{\pi}\operatorname{Im}\left(\sum_{l\in\mathbb{Z}}\alpha_l(\xi)\overline{\alpha_{l-1}(\xi)}\right)\\
&=\left(4+\frac{8}{\pi^2}\right)\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2-\frac{16}{\pi}\operatorname{Im}\left(\overline{\sum_{l\in\mathbb{Z}}\alpha_l(\xi)\overline{\alpha_{l+1}(\xi)}}\right)-\frac{8}{\pi^2}\operatorname{Re}\left(\sum_{l\in\mathbb{Z}}\alpha_{l-1}(\xi)\overline{\alpha_{l+1}(\xi)}\right)\\
&=\left(4+\frac{8}{\pi^2}\right)\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2+\frac{16}{\pi}\operatorname{Im}\left(\sum_{l\in\mathbb{Z}}\alpha_l(\xi)\overline{\alpha_{l+1}(\xi)}\right)-\frac{8}{\pi^2}\operatorname{Re}\left(\sum_{l\in\mathbb{Z}}\alpha_{l-1}(\xi)\overline{\alpha_{l+1}(\xi)}\right).
\end{aligned}\tag{3.33}$$

But

$$\left|\operatorname{Re}\left(\sum_{l\in\mathbb{Z}}\alpha_{l-1}(\xi)\overline{\alpha_{l+1}(\xi)}\right)\right|\le\left|\sum_{l\in\mathbb{Z}}\alpha_{l-1}(\xi)\overline{\alpha_{l+1}(\xi)}\right|\le\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2.$$

Hence,

$$-\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2\le\operatorname{Re}\left(\sum_{l\in\mathbb{Z}}\alpha_{l-1}(\xi)\overline{\alpha_{l+1}(\xi)}\right)\le\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2.\tag{3.34}$$

Now, using (3.34) and (3.28) in (3.33), we get

$$\begin{aligned}
\sum_{l\in\mathbb{Z}}|\langle\Phi(\xi),\tau\phi(\xi+l)\rangle_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))}|^2&\le\left(4+\frac{8}{\pi^2}\right)\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2+\frac{16}{\pi}\operatorname{Im}\left(\sum_{l\in\mathbb{Z}}\alpha_l(\xi)\overline{\alpha_{l+1}(\xi)}\right)+\frac{8}{\pi^2}\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2\\
&=\left(4+\frac{16}{\pi^2}\right)\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2+\frac{16}{\pi}\operatorname{Im}\left(\sum_{l\in\mathbb{Z}}\alpha_l(\xi)\overline{\alpha_{l+1}(\xi)}\right)\\
&<8\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2+\frac{16}{\pi}\operatorname{Im}\left(\sum_{l\in\mathbb{Z}}\alpha_l(\xi)\overline{\alpha_{l+1}(\xi)}\right)\\
&=4\left(2\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2+\frac{4}{\pi}\operatorname{Im}\left(\sum_{l\in\mathbb{Z}}\alpha_l(\xi)\overline{\alpha_{l+1}(\xi)}\right)\right)\\
&=4\|\Phi(\xi)\|^2_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))},
\end{aligned}\tag{3.35}$$

for all $\xi\in(0,1]$ and

$$\begin{aligned}
\sum_{l\in\mathbb{Z}}|\langle\Phi(\xi),\tau\phi(\xi+l)\rangle_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))}|^2&\ge(4+\frac{8}{\pi^2})\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2+\frac{16}{\pi}\operatorname{Im}\left(\sum_{l\in\mathbb{Z}}\alpha_l(\xi)\overline{\alpha_{l+1}(\xi)}\right)-\frac{8}{\pi^2}\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2\\
&=4\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2+\frac{16}{\pi}\operatorname{Im}\left(\sum_{l\in\mathbb{Z}}\alpha_l(\xi)\overline{\alpha_{l+1}(\xi)}\right)\\
&>4\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2+\frac{8}{\pi}\operatorname{Im}\left(\sum_{l\in\mathbb{Z}}\alpha_l(\xi)\overline{\alpha_{l+1}(\xi)}\right)\\
&=2\left(2\sum_{l\in\mathbb{Z}}|\alpha_l(\xi)|^2+\frac{4}{\pi}\operatorname{Im}\left(\sum_{l\in\mathbb{Z}}\alpha_l(\xi)\overline{\alpha_{l+1}(\xi)}\right)\right)\\
&=2\|\Phi(\xi)\|^2_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))},
\end{aligned}\tag{3.36}$$

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for all $\xi \in (0,1]$. Combining (3.35) and (3.36), we get

$$2\|\Phi(\xi)\|^2_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))} \leq \sum_{l\in\mathbb{Z}} |\langle \Phi(\xi), \tau\phi(\xi+l)\rangle_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))}|^2 \leq 4\|\Phi(\xi)\|^2_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))},$$

for all $\xi \in (0,1]$. Then it follows from Corollary 3.5 that $\{T^t_{(k,l)}\phi : k,l \in \mathbb{Z}\}$ is a frame sequence with frame bounds 2 and 4.

§4. Riesz sequence in a twisted shift-invariant space

For $\xi \in \mathbb{T}^n$, define $H(\xi) : \ell^2(\mathbb{Z}^{n+1}) \to L^2(\mathbb{R}^n, \ell^2(\mathbb{Z}^n))$ by

$$H(\xi)(\{c_{l,s}\}) = \sum_{(l,s)\in\mathbb{Z}^{n+1}} c_{l,s}\tau\phi_s(\xi+l).$$

If $H(\xi)$ defines a bounded operator from $\ell^2(\mathbb{Z}^{n+1})$ into $L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))$, then its adjoint $H(\xi)^* : L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n)) \to \ell^2(\mathbb{Z}^{n+1})$ is given by

$$H(\xi)^*\phi = \left\{\langle \phi, \tau\phi_s(\xi+l)\rangle_{L^2(\mathbb{R}^n,l^2(\mathbb{Z}^n))}\right\}_{(l,s)\in\mathbb{Z}^{n+1}}.$$

The Gramian associated with the system $\{\tau\phi_s(\xi+l) : (l,s) \in \mathbb{Z}^{n+1}\}$, denoted by $G(\xi)$, is defined by $G(\xi) = H(\xi)^*H(\xi)$, and the corresponding dual Gramian, denoted by $\widetilde{G}(\xi)$, is defined by $\widetilde{G}(\xi) = H(\xi)H(\xi)^*$.

THEOREM 4.1. *The family $E^t(\mathcal{A})$ is a Riesz sequence with Riesz bounds A and B if and only if*

$$AI \leq G(\xi) \leq BI, \quad \textit{for a.e. } \xi \in \mathbb{T}^n, \tag{4.1}$$

where I is the identity operator on $\ell^2(\mathbb{Z}^{n+1})$.

Proof. Let $E^t(\mathcal{A})$ be a Riesz sequence with Riesz bounds A and B. Then,

$$A\|\{c_{k,l,s}\}\|^2_{\ell^2(\mathbb{Z}^{2n+1})} \leq \|\sum_{k,(l,s)\in\mathcal{F}\times\mathcal{F}'} c_{k,l,s}T^t_{(k,l)}\phi_s\|^2_{L^2(\mathbb{R}^{2n})} \leq B\|c_{k,l,s}\|^2_{\ell^2(\mathbb{Z}^{2n+1})}, \tag{4.2}$$

$\forall\ \{c_{k,l,s}\}_{(k,(l,s))\in\mathcal{F}\times\mathcal{F}'} \in c_{00}(\mathbb{Z}^{2n+1})$, where $\mathcal{F}$ and $\mathcal{F}'$ are finite subsets of $\mathbb{Z}^n$ and $\mathbb{Z}^{n+1}$, respectively. By isometry of τ and (3.1), we have

$$\begin{aligned}
\Big\|\sum_{(k,(l,s))\in\mathcal{F}\times\mathcal{F}'} c_{k,l,s}T^t_{(k,l)}\phi_s\Big\|^2_{L^2(\mathbb{R}^{2n})} &= \Big\|\sum_{(k,(l,s))\in\mathcal{F}\times\mathcal{F}'} c_{k,l,s}\tau(T^t_{(k,l)}\phi_s)\Big\|^2_{L^2(\mathbb{T}^n,L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n)))}\\
&= \int_{\mathbb{T}^n}\Big\|\sum_{(k,(l,s))\in\mathcal{F}\times\mathcal{F}'} c_{k,l,s}\tau(T^t_{(k,l)}\phi_s)(\xi)\Big\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))}d\xi\\
&= \int_{\mathbb{T}^n}\Big\|\sum_{(k,(l,s))\in\mathcal{F}\times\mathcal{F}'} c_{k,l,s}e^{\pi ik\cdot(2\xi+l)}\tau\phi_s(\xi+l)\Big\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))}d\xi\\
&= \int_{\mathbb{T}^n}\Big\|\sum_{(l,s)\in\mathcal{F}'} p_{l,s}(\xi)\tau\phi_s(\xi+l)\Big\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))}d\xi, \qquad (4.3)
\end{aligned}$$

https://doi.org/10.1017/nmj.2023.11 Published online by Cambridge University Press

where $p_{l,s}(\xi)=\sum_{k\in\mathcal{F}} c_{k,l,s}e^{\pi ik\cdot(2\xi+l)}$ is a trigonometric polynomial in $L^2(\mathbb{T}^n)$. Now, for $k'\in\mathbb{Z}^n$,

$$\begin{aligned}\widehat{p_{l,s}}(k') &= \int_{\mathbb{T}^n} p_{l,s}(\xi)e^{-2\pi ik'\cdot\xi}d\xi \\ &= \int_{\mathbb{T}^n}\left(\sum_{k\in\mathcal{F}} c_{k,l,s}e^{\pi ik\cdot(2\xi+l)}\right)e^{-2\pi ik'\cdot\xi}d\xi \\ &= \sum_{k\in\mathcal{F}} c_{k,l,s}e^{\pi ik\cdot l}\int_{\mathbb{T}^n} e^{-2\pi i(k'-k)\cdot\xi}d\xi \\ &= \sum_{k\in\mathcal{F}} c_{k,l,s}e^{\pi ik\cdot l}\delta_{k,k'} \\ &= c_{k',l,s}e^{\pi ik'\cdot l}. \end{aligned} \tag{4.4}$$

Hence, we get

$$\begin{aligned}\sum_{(k,(l,s))\in\mathcal{F}\times\mathcal{F}'} |c_{k,l,s}|^2 &= \sum_{k,(l,s)\in\mathcal{F}\times\mathcal{F}'} |\widehat{p_{l,s}}(k)|^2 \\ &= \sum_{(l,s)\in\mathcal{F}'}\left(\sum_{k\in\mathcal{F}}|\widehat{p_{l,s}}(k)|^2\right) \\ &= \sum_{(l,s)\in\mathcal{F}'} \|p_{l,s}\|^2_{L^2(\mathbb{T}^n)} \\ &= \sum_{(l,s)\in\mathcal{F}'}\int_{\mathbb{T}^n}|p_{l,s}(\xi)|^2\,d\xi, \end{aligned} \tag{4.5}$$

by applying the Plancherel formula for the Fourier series of $p_{l,s}$. Using (4.3) and (4.5) in (4.2), we get

$$A\sum_{(l,s)\in\mathcal{F}'}\int_{\mathbb{T}^n}|p_{l,s}(\xi)|^2\,d\xi \le \int_{\mathbb{T}^n}\left\|\sum_{(l,s)\in\mathcal{F}'} p_{l,s}(\xi)\tau\phi_s(\xi+l)\right\|^2 d\xi \le B\sum_{(l,s)\in\mathcal{F}'}\int_{\mathbb{T}^n}|p_{l,s}(\xi)|^2\,d\xi. \tag{4.6}$$

Consider a family $\mathcal{C}:=\{m_{l,s}\in L^\infty(\mathbb{T}^n):(l,s)\in\mathbb{Z}^{n+1}\}$. Then, by Lusin's theorem, there exists a sequence of trigonometric polynomials $\{p^{(i)}_{l,s}\}_{i\in\mathbb{N}}$ such that $\|p^{(i)}_{l,s}\|_{L^\infty(\mathbb{T}^n)}\le\|m_{l,s}\|_{L^\infty(\mathbb{T}^n)}$, for all $i\in\mathbb{N}$, and $p^i_{l,s}(\xi)\to m_{l,s}(\xi)$ as $i\to\infty$ for *a.e.* $\xi\in\mathbb{T}^n$. Now, using (4.6), we have

$$A\sum_{(l,s)\in\mathcal{F}'}\int_{\mathbb{T}^n}|p^{(i)}_{l,s}(\xi)|^2\,d\xi \le \int_{\mathbb{T}^n}\left\|\sum_{(l,s)\in\mathcal{F}'} p^{(i)}_{l,s}(\xi)\tau\phi_s(\xi+l)\right\|^2 d\xi \le B\sum_{(l,s)\in\mathcal{F}'}\int_{\mathbb{T}^n}|p^{(i)}_{l,s}(\xi)|^2\,d\xi,$$

$\forall\, i\in\mathbb{N}$. Taking limit as $i\to\infty$ and then applying Lebesgue dominated convergence theorem in the above inequality, we obtain

$$A\sum_{(l,s)\in\mathcal{F}'}\int_{\mathbb{T}^n}|m_{l,s}(\xi)|^2\,d\xi \le \int_{\mathbb{T}^n}\left\|\sum_{(l,s)\in\mathcal{F}'} m_{l,s}(\xi)\tau\phi_s(\xi+l)\right\|^2 d\xi \le B\sum_{(l,s)\in\mathcal{F}'}\int_{\mathbb{T}^n}|m_{l,s}(\xi)|^2\,d\xi. \tag{4.7}$$

Now, (4.1) is equivalent to

$$A \sum_{(l,s)\in\mathcal{E}} |c_{l,s}|^2 \le \left\| \sum_{(l,s)\in\mathcal{E}} c_{l,s}\tau\phi_s(\xi+l)\right\|^2 \le B \sum_{(l,s)\in\mathcal{E}} |c_{l,s}|^2 \tag{4.8}$$

for $a.e.$ $\xi \in \mathbb{T}^n$ and for every finite set $\mathcal{E} \subset \mathbb{Z}^{n+1}$. Thus, we aim to prove (4.8). Suppose (4.8) were not true. Then there would exist a Lebesgue measurable set $D \subset \mathbb{T}^n$ with positive Lebesgue measure such that at least one of the following inequalities holds:

$$\left\| \sum_{(l,s)\in\mathcal{E}} c'_{l,s}\tau\phi_s(\xi+l)\right\|^2 > B \sum_{(l,s)\in\mathcal{E}} |c'_{l,s}|^2, \quad \forall\, \xi \in D, \tag{4.9}$$

$$\left\| \sum_{(l,s)\in\mathcal{E}} c'_{l,s}\tau\phi_s(\xi+l)\right\|^2 < A \sum_{(l,s)\in\mathcal{E}} |c'_{l,s}|^2, \quad \forall\, \xi \in D. \tag{4.10}$$

Assume that (4.9) holds. Define $m'_{l,s}(\xi) := \frac{c'_{l,s}}{\sqrt{\mu(D)}}\chi_D(\xi)$, where $\mu(D)$ denotes the Lebesgue measure of D. Then $m'_{l,s} \in \mathcal{C}$ and

$$\begin{aligned}
\int_{\mathbb{T}^n} \left\| \sum_{(l,s)\in\mathcal{E}} m'_{l,s}(\xi)\tau\phi_s(\xi+l)\right\|^2 d\xi &= \int_D \left\| \sum_{(l,s)\in\mathcal{E}} \frac{c'_{l,s}}{\sqrt{\mu(D)}}\tau\phi_s(\xi+l)\right\|^2 d\xi \\
&= \frac{1}{\mu(D)} \int_D \left\| \sum_{(l,s)\in\mathcal{E}} c'_{l,s}\tau\phi_s(\xi+l)\right\|^2 d\xi \\
&> \frac{1}{\mu(D)} B \sum_{(l,s)\in\mathcal{E}} |c'_{l,s}|^2 \int_D d\xi \\
&= B \sum_{(l,s)\in\mathcal{E}} |c'_{l,s}|^2,
\end{aligned}$$

which is a contradiction to (4.7). Similarly, if we assume (4.10), we will get a contradiction to (4.7) again. Therefore, we obtain (4.1).

Conversely, assume that (4.1) holds. We need to prove (4.2). So let $\{c_{k,l,s}\}_{(k,(l,s))\in\mathcal{F}\times\mathcal{F}'} \in c_{00}(\mathbb{Z}^{2n+1})$. For $\xi \in \mathbb{T}^n$, we define

$$p_{l,s}(\xi) = \sum_{k\in\mathcal{F}} c_{k,l,s} e^{\pi i k\cdot(2\xi+l)}.$$

Using (4.1), we have

$$A \sum_{(l,s)\in\mathcal{E}} |p_{l,s}(\xi)|^2 \le \left\| \sum_{(l,s)\in\mathcal{E}} p_{l,s}(\xi)\tau\phi_s(\xi+l)\right\|^2 \le B \sum_{(l,s)\in\mathcal{E}} |p_{l,s}(\xi)|^2,$$

for $a.e.$ $\xi \in \mathbb{T}^n$. Now, integrating with respect to ξ over $\mathbb{T}^n$, we obtain

$$A \sum_{(l,s)\in\mathcal{E}} \int_{\mathbb{T}^n} |p_{l,s}(\xi)|^2 \le \int_{\mathbb{T}^n} \left\| \sum_{(l,s)\in\mathcal{E}} p_{l,s}(\xi)\tau\phi_s(\xi+l)\right\|^2 \le B \sum_{(l,s)\in\mathcal{E}} \int_{\mathbb{T}^n} |p_{l,s}(\xi)|^2,$$

which together with (4.3) and (4.5) gives the required result. ∎

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COROLLARY 4.2. *The family* $E^t(\mathcal{A})$ *is a Riesz sequence with bounds* A *and* B *if and only if the family* $\{\tau\phi_s(\xi+l):(l,s)\in\mathbb{Z}^{n+1}\}$ *is a Riesz sequence with the same bounds, for a.e.* $\xi\in\mathbb{T}^n$.

Proof. Let the family $E^t(\mathcal{A})$ be a Riesz sequence with bounds A and B. Then, by Theorem 4.1, it is equivalent to (4.1), which in turn is equivalent to

$$A\sum_{(l,s)\in\mathbb{Z}^{n+1}}|c_{l,s}|^2\le\left\|\sum_{(l,s)\in\mathbb{Z}^{n+1}}c_{l,s}\tau\phi_s(\xi+l)\right\|^2\le B\sum_{(l,s)\in\mathbb{Z}^{n+1}}|c_{l,s}|^2,$$

$\forall\ \{c_{l,s}\}\in\ell^2(\mathbb{Z}^{n+1})$ and for *a.e.* $\xi\in\mathbb{T}^n$, proving our assertion. ∎

COROLLARY 4.3. *Let* $\mathcal{A}=\{\phi\}$, *and let* $E^t(\mathcal{A})$ *be a Riesz sequence with bounds* A *and* B. *Then* $A\le W_\phi(\xi)\le B$, *for a.e.* $\xi\in\mathbb{T}^n$.

Proof. Since the system $E^t(\mathcal{A})$ is a Riesz sequence with bounds A and B, it is a frame sequence with the same bounds. Thus, by Theorem 3.7, we have $A\le W_\phi(\xi)\le B$, for *a.e.* $\xi\in\mathbb{T}^n\setminus N$. Now, in order to obtain the required result, it is enough to show that $\mu(N)=0$. By Corollary 4.2, we have

$$A\|\{c_{l,s}\}\|^2_{\ell^2(\mathbb{Z}^{n+1})}\le\left\|\sum_{(l,s)\in\mathbb{Z}^{n+1}}c_{l,s}\tau\phi(\xi+l)\right\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))},$$

$\forall\ \{c_{l,s}\}\in\ell^2(\mathbb{Z}^{n+1})$ and for *a.e.* $\xi\in\mathbb{T}^n$. In particular, by choosing $c_{l,s}=\delta_{((l,s),(0,0))}$, $\forall\ (l,s)\in\mathbb{Z}^{n+1}$, in the above inequality, we get $0<A\le\|\tau\phi(\xi)\|^2$, for *a.e.* $\xi\in\mathbb{T}^n$. Hence, $w_\phi(\xi)=\|\tau\phi(\xi)\|^2>0$, for *a.e.* $\xi\in\mathbb{T}^n$, showing that $\mu(N)=0$. ∎

But the converse need not be true. We illustrate it with a counter example.

EXAMPLE 4.4. We define $\phi\in L^2(\mathbb{R}^2)$ by $K_\phi(\xi,\eta)=\chi_{[0,2)\times[0,1)}(\xi,\eta),\ (\xi,\eta)\in\mathbb{R}^2$. Then, for $\xi,\eta\in[0,1)$, we have

$$K_\phi(\xi+m,\eta)=\begin{cases}1, & m=0,1,\\ 0, & otherwise,\end{cases}\qquad\text{and}\qquad K_\phi(\xi+m+1,\eta)=\begin{cases}1, & m=-1,0,\\ 0, & otherwise.\end{cases}$$

Hence, we get

$$\begin{aligned}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}K_\phi(m+\xi,\eta)\overline{K_\phi(m+\xi+1,\eta)}d\eta&=\sum_{m\in\mathbb{Z}}\int_0^1K_\phi(m+\xi,\eta)K_\phi(m+\xi+1,\eta)d\eta\\&=1,\qquad\forall\ \xi\in[0,1),\end{aligned}$$

showing that ϕ does not satisfy "Condition C." Now,

$$\begin{aligned}\omega_\phi(\xi)&=\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}|K_\phi(m+\xi,\eta)|^2\,d\eta\\&=\sum_{m\in\mathbb{Z}}\int_0^1|K_\phi(m+\xi,\eta)|^2\,d\eta\\&=2,\qquad\forall\ \xi\in[0,1).\end{aligned}\tag{4.11}$$

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Hence, for $\xi \in [0,1)$, we get

$$\begin{aligned}
W_\phi(\xi) =& \frac{1}{\omega_\phi(\xi)} \sum_{l\in\mathbb{Z}} \Big| \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} K_\phi(m+\xi,\eta)\overline{K_\phi(\xi+m+l,\eta)}\,d\eta \Big|^2 \\
=& \frac{1}{2} \sum_{l\in\mathbb{Z}} \Big| \sum_{m\in\mathbb{Z}} \int_0^1 K_\phi(m+\xi,\eta)\overline{K_\phi(\xi+m+l,\eta)}\,d\eta \Big|^2 \\
=& \frac{1}{2} \sum_{l\in\mathbb{Z}} \Big| \int_0^1 K_\phi(\xi+l,\eta)+K_\phi(\xi+1+l,\eta)\,d\eta \Big|^2 \\
=& \frac{1}{2} \sum_{l=-1,0,1} \Big| \int_0^1 K_\phi(\xi+l,\eta)+K_\phi(\xi+1+l,\eta)\,d\eta \Big|^2 \\
=& \frac{1}{2}\times 6 \\
=& 3,
\end{aligned} \tag{4.12}$$

showing that the weight function is bounded above and below for *a.e.* $\xi \in [0,1)$. Now, we shall make use of Corollary 4.2 in order to prove that the system $\{T^t_{(k,l)}\phi : k,l \in \mathbb{Z}\}$ does not form a Riesz sequence. For $\{c_l\} \in \ell^2(\mathbb{Z})$ and $\xi \in [0,1)$, we have

$$\begin{aligned}
\Big\| \sum_l c_l \tau\phi(\xi+l) \Big\|^2_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))} =& \sum_{l,l'} c_l\overline{c_{l'}} \langle \tau\phi(\xi+l), \tau\phi(\xi+l')\rangle_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))} \\
=& \sum_{l,l'} c_l\overline{c_{l'}} \Big(\sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} K_\phi(m+\xi,\eta)\overline{K_\phi(m+\xi+l'-l,\eta)}\,d\eta \Big) \\
=& \sum_{l,l'} c_l\overline{c_{l'}} \Big(\sum_{m\in\mathbb{Z}} \int_0^1 K_\phi(m+\xi,\eta)\overline{K_\phi(m+\xi+l'-l,\eta)}\,d\eta \Big) \\
=& \sum_{l,l'} c_l\overline{c_{l'}} \Big(\int_0^1 K_\phi(\xi+l'-l,\eta)+K_\phi(1+\xi+l'-l,\eta)\,d\eta \Big) \\
=& \sum_l c_l\overline{c_l} + \sum_l c_l\overline{c_{l+1}} + \sum_l c_l\overline{c_{l-1}} + \sum_l c_l\overline{c_l} \\
=& 2\sum_l |c_l|^2 + 2\,\mathrm{Re}\Big(\sum_l c_l\overline{c_{l+1}} \Big).
\end{aligned} \tag{4.13}$$

For each $n \in \mathbb{N}$, we define $X^{(n)} \in \ell^2(\mathbb{Z})$ by

$$X^{(n)}_l = \begin{cases} \frac{(-1)^l}{\sqrt{2n}}, & l = 1,2,\ldots,2n, \\ 0, & \textit{otherwise}. \end{cases}$$

It is easy to see that $\|X^{(n)}\|_{\ell^2(\mathbb{Z})} = 1$, $\forall\, n \in \mathbb{N}$. For each $n \in \mathbb{N}$, by taking $c_l = X^{(n)}_l$ in (4.13), we obtain

$$\begin{aligned}
\Big\| \sum_l X^{(n)}_l \tau\phi(\xi+l) \Big\|^2_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))} &= 2\sum_l |X^{(n)}_l|^2 + 2\,\mathrm{Re}\Big(\sum_l X^{(n)}_l \overline{X^{(n)}_{l+1}} \Big) \\
&= 2 + 2\,\mathrm{Re}\Big(\sum_l X^{(n)}_l \overline{X^{(n)}_{l+1}} \Big).
\end{aligned} \tag{4.14}$$

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But

$$\begin{aligned}\sum_{l\in\mathbb{Z}} X_l^{(n)}\overline{X_{l+1}^{(n)}} &= \sum_{l=1}^{2n-1}\frac{(-1)^l}{\sqrt{2n}}\times\frac{(-1)^{l+1}}{\sqrt{2n}}\\ &= -\frac{1}{2n}\sum_{l=1}^{2n-1}(1)\\ &= -\frac{1}{2n}(2n-1)\\ &= \frac{1}{2n}-1.\end{aligned}$$

Hence, (4.14) reduces to

$$\Big\|\sum_{l} X_l^{(n)}\tau\phi(\xi+l)\Big\|^2_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))} = 2+2\Big(\frac{1}{2n}-1\Big) = \frac{1}{n} = \frac{1}{n}\|X^{(n)}\|^2_{\ell^2(\mathbb{Z})},\ \forall\ n\in\mathbb{N}.$$

Since the above equation holds for all $n\in\mathbb{N}$, using Corollary 4.2, we deduce that the system $\{T^t_{(k,l)}\phi : k,l\in\mathbb{Z}\}$ does not satisfy the lower Riesz condition.

COROLLARY 4.5 [18]. *Let $\mathcal{A}=\{\phi\}$ and ϕ satisfy "Condition C." Then $E^t(\phi)$ is a Riesz sequence with bounds A and B if and only if*

$$A\le w_\phi(\xi)\le B,\ \textit{for a.e. } \xi\in\mathbb{T}^n. \tag{4.15}$$

Proof. Let the system $E^t(\mathcal{A})$ be a Riesz sequence with bounds A and B. Then, from Corollary 4.3, we have $A\le W_\phi(\xi)\le B$, for *a.e.* $\xi\in\mathbb{T}^n$. Since ϕ satisfies "Condition C," by (3.15), we have $W_\phi(\xi)=w_\phi(\xi)$, for *a.e.* $\xi\in\mathbb{T}^n$, which proves (4.15).

Conversely, assume that (4.15) holds. Since ϕ satisfies "Condition C," using the similar steps in obtaining (3.19), we can get

$$\Big\|\sum_{l\in\mathbb{Z}^n} c_l\tau\phi(\xi+l)\Big\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} = \omega_\phi(\xi)\sum_{l\in\mathbb{Z}^n}|c_l|^2,$$

$\forall\ \{c_l\}\in c_{00}(\mathbb{Z}^n)$ and $\xi\in\mathbb{T}^n$. Now, applying (4.15) in the above equation, we have

$$A\|\{c_l\}\|^2_{\ell^2(\mathbb{Z}^n)} \le \Big\|\sum_{l\in\mathbb{Z}^n} c_l\tau\phi(\xi+l)\Big\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} \le B\|\{c_l\}\|^2_{\ell^2(\mathbb{Z}^n)},$$

$\forall\ \{c_l\}\in c_{00}(\mathbb{Z}^n)$ and for *a.e.* $\xi\in\mathbb{T}^n$. Hence, the system $\{\tau\phi(\xi+l) : (l\in\mathbb{Z}^n\}$ is a Riesz sequence with bounds A and B, for *a.e.* $\xi\in\mathbb{T}^n$. Therefore, using Corollary 4.2, we obtain the required result. ∎

Now, we provide an example of a function $\phi\in L^2(\mathbb{R}^2)$ which does not satisfy "Condition C" and the system $\{T^t_{(k,l)}\phi : k,l\in\mathbb{Z}\}$ is a Riesz sequence.

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EXAMPLE 4.6. We define $\phi \in L^2(\mathbb{R}^2)$ by

$$K_\phi(\xi,\eta) = 2\chi_{[0,1)\times[0,1)}(\xi,\eta) + \chi_{[1,2)\times[0,1)}(\xi,\eta), \quad (\xi,\eta) \in \mathbb{R}^2.$$

For $\xi,\eta \in [0,1)$, using the definition of K_ϕ, we observe that

$$K_\phi(\xi+m,\eta) = \begin{cases} 2, & m=0, \\ 1, & m=1, \\ 0, & otherwise, \end{cases} \qquad \text{and} \qquad K_\phi(\xi+m+1,\eta) = \begin{cases} 2, & m=-1, \\ 1, & m=0, \\ 0, & otherwise. \end{cases}$$

Hence, for $\xi,\eta \in [0,1)$, we get

$$K_\phi(\xi+m,\eta)\overline{K_\phi(\xi+m+1,\eta)} = \begin{cases} 2, & m=0, \\ 0, & otherwise, \end{cases}$$

showing that

$$\sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} K_\phi(\xi+m,\eta)\overline{K_\phi(\xi+m+1,\eta)} d\eta = \sum_{m\in\mathbb{Z}} \int_0^1 K_\phi(\xi+m,\eta) K_\phi(\xi+m+1,\eta) d\eta.$$
$$=2.$$

Thus, ϕ does not satisfy "Condition C." We shall make use of Corollary 4.2 in order to show that the system $\{T^t_{(k,l)}\phi : k,l \in \mathbb{Z}\}$ is a Riesz sequence. For $\{c_l\} \in \ell^2(\mathbb{Z}^n)$, using (3.13), we have

$$\begin{aligned}\Big\|\sum_l c_l \tau\phi(\xi+l)\Big\|^2_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))} &= \sum_{l,l'} c_l \overline{c_{l'}} \big\langle \tau\phi(\xi+l), \tau\phi(\xi+l')\big\rangle_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))} \\ &= \sum_{l,l'} c_l \overline{c_{l'}} \Big(\sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} K_\phi(\xi+m,\eta)\overline{K_\phi(\xi+m+l'-l,\eta)}\, d\eta \Big) \\ &= \sum_{l,l'} c_l \overline{c_{l'}} \Big(\sum_{m\in\mathbb{Z}} \int_0^1 K_\phi(\xi+m,\eta)\overline{K_\phi(\xi+m+l'-l,\eta)}\, d\eta \Big).\end{aligned}$$

The possible values of m and $l'-l$ that can survive in the sum are $m=0$ and $l'-l=-1,0,1$. Therefore, for $\xi \in [0,1)$, we get

$$\begin{aligned}\Big\|\sum_l c_l \tau\phi(\xi+l)\Big\|^2_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))} &= 5\sum_l |c_l|^2 + 2\sum_l \overline{c_{l+1}} c_l + 2\sum_l \overline{c_{l-1}} c_l \\ &= 5\sum_l |c_l|^2 + 2\sum_l \overline{c_{l+1}} c_l + 2\overline{\sum_l \overline{c_{l+1}} c_l} \\ &= 5\sum_l |c_l|^2 + 4\,\mathrm{Re}\Big(\sum_l \overline{c_{l+1}} c_l\Big).\end{aligned}$$

Now, using $\big|\mathrm{Re}\big(\sum_l \overline{c_{l+1}} c_l\big)\big| \le \sum_l |c_l|^2$, we can show that

$$\sum_l |c_l|^2 \le \Big\|\sum_l c_l \tau\phi(\xi+l)\Big\|^2_{L^2(\mathbb{R},\ell^2(\mathbb{Z}))} \le 9\sum_l |c_l|^2, \qquad \forall\, \xi \in [0,1),$$

proving our assertion.

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§5. Dual twisted shift-invariant system

Let $\mathcal{D} = \{\varphi_s : s \in \mathbb{Z}\}$ be a collection of functions in $L^2(\mathbb{R}^{2n})$. As we defined $E^t(\mathcal{A})$, $U^t(\mathcal{A})$, $V^t(\mathcal{A})$, and $H_l(\xi)$ for the collection $\mathcal{A}$, we define $E^t(\mathcal{D})$, $U^t(\mathcal{D})$, $V^t(\mathcal{D})$, and $T_l(\xi)$, respectively, for the collection $\mathcal{D}$.

PROPOSITION 5.1. *Let $E^t(\mathcal{A})$ and $E^t(\mathcal{D})$ be two Bessel sequences. Then, for $\phi, \varphi \in L^2(\mathbb{R}^{2n})$, we have the following identity:*

$$\sum_{(k,s)\in\mathbb{Z}^{n+1}} \langle \phi, T^t_{(k,l)}\phi_s\rangle \langle T^t_{(k,l)}\varphi_s, \varphi\rangle = \int_{\mathbb{T}^n} \langle H_l(\xi)^*\tau\phi(\xi), T_l(\xi)^*\tau\varphi(\xi)\rangle_{\ell^2(\mathbb{Z})}\, d\xi, \tag{5.1}$$

for each $l \in \mathbb{Z}^n$.

Proof. For $\phi \in L^2(\mathbb{R}^{2n})$, using (3.1), we have

$$\begin{aligned}\langle \phi, T^t_{(k,l)}\phi_s\rangle_{L^2(\mathbb{R}^{2n})} &= \langle \tau\phi, \tau T^t_{(k,l)}\phi_s\rangle_{L^2(\mathbb{T}^n, L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n)))} \\ &= \int_{\mathbb{T}^n} e^{-\pi i k\cdot(2\xi+l)} \langle \tau\phi(\xi), \tau\phi_s(\xi+l)\rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))}\, d\xi \\ &= e^{-\pi i k\cdot l}\widehat{F}_{l,s}(k), \end{aligned}\tag{5.2}$$

where $F_{l,s}(\xi) = \langle \tau\phi(\xi), \tau\phi_s(\xi+l)\rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))}$. Similarly, for $\varphi \in L^2(\mathbb{R}^{2n})$, we can show that

$$\langle \varphi, T^t_{(k,l)}\varphi_s\rangle = e^{-\pi i k\cdot l}\widehat{G}_{l,s}(k), \tag{5.3}$$

where $G_{l,s}(\xi) = \langle \tau\varphi(\xi), \tau\varphi_s(\xi+l)\rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))}$. We notice that $H_l(\xi)^*\tau\phi(\xi) = \{F_{l,s}(\xi)\}_{s\in\mathbb{Z}}$ and $T_l(\xi)^*\tau\varphi(\xi) = \{G_{l,s}(\xi)\}_{s\in\mathbb{Z}}$. Now, making use of (5.2) and (5.3), we obtain

$$\begin{aligned}\sum_{(k,s)\in\mathbb{Z}^{n+1}} \langle \phi, T^t_{(k,l)}\phi_s\rangle \langle T^t_{(k,l)}\varphi_s, \varphi\rangle &= \sum_{s\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}^n} \widehat{F}_{l,s}(k)\overline{\widehat{G}_{l,s}(k)}\Big) \\ &= \sum_{s\in\mathbb{Z}} \langle \widehat{F}_{l,s}, \widehat{G}_{l,s}\rangle_{\ell^2(\mathbb{Z}^n)} \\ &= \sum_{s\in\mathbb{Z}} \langle F_{l,s}, G_{l,s}\rangle_{L^2(\mathbb{T}^n)} \\ &= \sum_{s\in\mathbb{Z}} \int_{\mathbb{T}^n} F_{l,s}(\xi)\overline{G_{l,s}(\xi)}\, d\xi \\ &= \int_{\mathbb{T}^n} \langle \{F_{l,s}(\xi)\}_{s\in\mathbb{Z}}, \{G_{l,s}(\xi)\}_{s\in\mathbb{Z}}\rangle_{\ell^2(\mathbb{Z})}\, d\xi \\ &= \int_{\mathbb{T}^n} \langle H_l(\xi)^*\tau\phi(\xi), T_l(\xi)^*\tau\varphi(\xi)\rangle_{\ell^2(\mathbb{Z})}\, d\xi, \end{aligned}$$

for each $l \in \mathbb{Z}^n$, proving (5.1). ∎

THEOREM 5.2. *Let $E^t(\mathcal{A})$ and $E^t(\mathcal{D})$ be two Bessel sequences. Then $E^t(\mathcal{A})$ and $E^t(\mathcal{D})$ are dual frames if and only if*

$$\sum_{l\in\mathbb{Z}^n} T_l(\xi)H_l(\xi)^* = I, \qquad \text{for a.e. } \xi \in \mathbb{T}^n, \tag{5.4}$$

where I is the identity operator on $L^2(\mathbb{R}^n, \ell^2(\mathbb{Z}^n))$.

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Proof. Assume that $E^t(\mathcal{A})$ and $E^t(\mathcal{D})$ are dual frames. Then, by Theorem 2.4, we have

$$\sum_{(k,l,s)\in\mathbb{Z}^{2n+1}} \langle \phi, T^t_{(k,l)}\phi_s\rangle \langle T^t_{(k,l)}\varphi_s, \varphi\rangle = \langle \phi, \varphi\rangle \tag{5.5}$$

for all $\phi,\varphi \in L^2(\mathbb{R}^{2n})$. In particular, by taking $\phi=\varphi=\psi$ in (5.5), we get

$$\sum_{(k,l,s)\in\mathbb{Z}^{2n+1}} \langle \psi, T^t_{(k,l)}\phi_s\rangle \langle T^t_{(k,l)}\varphi_s, \psi\rangle = \|\psi\|^2, \qquad \forall\ \psi \in L^2(\mathbb{R}^{2n}). \tag{5.6}$$

By using Proposition 5.1, we can rewrite (5.6) as

$$\int_{\mathbb{T}^n} \sum_{l\in\mathbb{Z}^n} \langle H_l(\xi)^*\tau\psi(\xi), T_l(\xi)^*\tau\psi(\xi)\rangle_{\ell^2(\mathbb{Z})}\, d\xi = \|\psi\|^2, \qquad \forall\ \psi \in L^2(\mathbb{R}^{2n}). \tag{5.7}$$

Now, we aim to prove (5.4), which is equivalent to

$$\sum_{l\in\mathbb{Z}^n} \langle H_l(\xi)^*\Phi, T_l(\xi)^*\Phi\rangle = \|\Phi\|^2, \text{ for } a.e.\ \xi\in\mathbb{T}^n, \quad \forall\ \Phi \in L^2(\mathbb{R}^n, \ell^2(\mathbb{Z}^n)). \tag{5.8}$$

Let $\Phi \in L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))$. Then, we can write $\Phi(\eta) = \{C^\eta_m\}_{m\in\mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n)$, for $a.e.$ $\eta\in\mathbb{R}^n$. For $p\in L^2(\mathbb{T}^n)$, we define $\psi \in L^2(\mathbb{R}^{2n})$ by $K_\psi(\xi,\eta) = p(\xi-m)C^\eta_m$, where m is that unique element in $\mathbb{Z}^n$ for which $\xi - m \in \mathbb{T}^n$. Then, for $\xi\in\mathbb{T}^n$, we have $K_\psi(\xi+m,\eta) = p(\xi)C^\eta_m$, $\forall\ m\in\mathbb{Z}^n$. Hence, for $\xi\in\mathbb{T}^n, \eta\in\mathbb{R}^n$, we get

$$\begin{aligned}\tau\psi(\xi)(\eta) &= \{K_\psi(\xi+m,\eta)\}_{m\in\mathbb{Z}^n}\\ &= p(\xi)\{C^\eta_m\}_{m\in\mathbb{Z}^n}\\ &= p(\xi)\Phi(\eta),\end{aligned}$$

showing that $\tau\psi(\xi) = p(\xi)\Phi$. Consequently, we get

$$\begin{aligned}\|\psi\|^2_{L^2(\mathbb{R}^{2n})} &= \|\tau\psi\|^2_{L^2(\mathbb{T}^n, L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n)))}\\ &= \int_{\mathbb{T}^n} \|\tau\psi(\xi)\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))}\, d\xi\\ &= \int_{\mathbb{T}^n} |p(\xi)|^2 \|\Phi\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))}\, d\xi\end{aligned} \tag{5.9}$$

and

$$\int_{\mathbb{T}^n} \sum_{l\in\mathbb{Z}^n} \langle H_l(\xi)^*\tau\psi(\xi), T_l(\xi)^*\tau\psi(\xi)\rangle = \int_{\mathbb{T}^n} |p(\xi)|^2 \sum_{l\in\mathbb{Z}^n} \langle H_l(\xi)^*\Phi, T_l(\xi)^*\Phi\rangle\, d\xi. \tag{5.10}$$

By making use of (5.9) and (5.10) in (5.7), we obtain

$$\int_{\mathbb{T}^n} |p(\xi)|^2 \sum_{l\in\mathbb{Z}^n} \langle H_l(\xi)^*\Phi, T_l(\xi)^*\Phi\rangle\, d\xi = \int_{\mathbb{T}^n} |p(\xi)|^2 \|\Phi\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))}\, d\xi.$$

Since this is true for all $p\in L^2(\mathbb{T}^n)$, we arrive at (5.8).

Conversely, assume that (5.4) holds. Then, for $\phi,\varphi\in L^2(\mathbb{R}^{2n})$ using Proposition 5.1, we have

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$$\begin{aligned}\sum_{(k,l,s)\in\mathbb{Z}^{2n+1}} \big\langle \phi, T^t_{(k,l)}\phi_s\big\rangle\big\langle T^t_{(k,l)}\varphi_s,\varphi\big\rangle &= \int_{\mathbb{T}^n} \Big\langle \sum_{l\in\mathbb{Z}^n} T_l(\xi)H_l(\xi)^*\tau\phi(\xi),\tau\varphi(\xi)\Big\rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))}\,d\xi\\ &= \int_{\mathbb{T}^n}\big\langle \tau\phi(\xi),\tau\varphi(\xi)\big\rangle_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))}\,d\xi\\ &= \langle\tau\phi,\tau\varphi\rangle_{L^2(\mathbb{T}^n,L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n)))}\\ &= \langle\phi,\varphi\rangle_{L^2(\mathbb{R}^{2n})}.\end{aligned}$$

Then the result follows by using Theorem 2.4. □

Now, let $\mathcal{A}=\{\phi\}$, and let $E^t(\mathcal{A})$ be a frame sequence. In other words, the system $E^t(\mathcal{A})$ is a frame for

$$V^t(\mathcal{A})=\overline{\operatorname{span}\{T^t_{(k,l)}\phi : k,l\in\mathbb{Z}^n\}}.$$

Let $\mathcal{D}=\{\widetilde{\phi}\}$, for some $\widetilde{\phi}\in L^2(\mathbb{R}^{2n})$. Then, the system $E^t(\mathcal{D})$ is said to be an oblique dual of $E^t(\mathcal{A})$ if

$$f=\sum_{k,l\in\mathbb{Z}^n}\langle f,T^t_{(k,l)}\widetilde{\phi}\rangle T^t_{(k,l)}\phi,\qquad \forall\, f\in V^t(\mathcal{A}).$$

Then we have the following theorem.

THEOREM 5.3. *Let $E^t(\mathcal{A})$ and $E^t(\mathcal{D})$ be two Bessel sequences. Then $E^t(\mathcal{D})$ is an oblique dual of $E^t(\mathcal{A})$ if and only if*

$$\sum_{l\in\mathbb{Z}^n} H_l(\xi)T_l(\xi)^*=I_\xi,\ \ for\ a.e.\ \xi\in\mathbb{T}^n, \tag{5.11}$$

where I_ξ is the identity operator on $J(\xi)$.

The proof follows similar lines as that of Theorem 5.2.

In addition, if we assume that ϕ satisfies "Condition C," then as in the classical case (Corollary 7.4.2 in [7]), we can show that the canonical dual frame is the only dual frame that consists of twisted translations of a single function. Toward this end, we make use of the following results.

PROPOSITION 5.4 [18]. *Let $\phi\in L^2(\mathbb{R}^{2n})$ be such that ϕ satisfies "Condition C." Then the map $f\mapsto r$ initially defined on $U^t(\mathcal{A})$ can be extended to an isometric isomorphism of $V^t(\phi)$ onto $L^2(\mathbb{T}^n,\ell^2(\mathbb{Z}^n);\omega_\phi)$.*

THEOREM 5.5 [18]. *Let $\phi\in L^2(\mathbb{R}^{2n})$ and $a,b>0$. Suppose $\{T^t_{(bk,al)}\phi : (k,l)\in\mathbb{Z}^{2n}\}$ is a frame for $L^2(\mathbb{R}^{2n})$ with frame operator S. Then the canonical dual frame also has the same structure and is given by $T^t_{(bk,al)}S^{-1}\phi$.*

COROLLARY 5.6. *Let $\phi\in L^2(\mathbb{R}^{2n})$ be such that ϕ satisfies "Condition C." Let $\{T^t_{(k,l)}\phi : k,l\in\mathbb{Z}^n\}$ be a frame sequence. Then there exists unique $\tilde{\phi}\in V^t(\phi)$ such that $f=\sum_{k,l\in\mathbb{Z}^n}\langle f,T^t_{(k,l)}\tilde{\phi}\rangle T^t_{(k,l)}\phi\ \ \forall f\in V^t(\phi)$ with $\widetilde{\phi}=S^{-1}\phi$*

Proof. Let S be the frame operator associated with $\{T^t_{(k,l)}\phi : k,l\in\mathbb{Z}^n\}$. Then, for all $f\in V^t(\phi)$, we have $f=\sum_{k,l\in\mathbb{Z}^n}\langle f,T^t_{(k,l)}S^{-1}\phi\rangle T^t_{(k,l)}\phi$, by Theorem 5.5. Let $\widetilde{\phi}\in V^t(\phi)$. We have $\widetilde{\phi}=\sum_{k,l\in\mathbb{Z}^n}\alpha_{k,l}T^t_{(k,l)}\phi$, for some $\{\alpha_{k,l}\}\in\ell^2(\mathbb{Z}^{2n})$. For $\xi\in\mathbb{T}^n$, using (3.1), we get

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$$\begin{aligned}\tau\tilde{\phi}(\xi) &= \sum_{k,l\in\mathbb{Z}^n} \alpha_{k,l}\tau(T^t_{(k,l)}\phi)(\xi)\\ &= \sum_{k,l\in\mathbb{Z}^n} \alpha_{k,l}e^{\pi ik\cdot(2\xi+l)}\tau\phi(\xi+l)\\ &= \sum_{l\in\mathbb{Z}^n} r_l(\xi)\tau\phi(\xi+l),\end{aligned} \tag{5.12}$$

where $r_l(\xi) = \sum_{k\in\mathbb{Z}^n} \alpha_{k,l}e^{\pi ik\cdot(2\xi+l)}$. Since ϕ satisfies "Condition C," using the similar steps as in obtaining (3.19), we can show that

$$\|\tau\widetilde{\phi}(\xi)\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))} = \omega_\phi(\xi)\sum_{l\in\mathbb{Z}^n} |r_l(\xi)|^2.$$

Hence, using isometry of the fiber map τ, we get

$$\begin{aligned}\|\widetilde{\phi}\|^2_{L^2(\mathbb{R}^{2n})} &= \int_{\mathbb{T}^n} \|\tau\widetilde{\phi}(\xi)\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))}\,d\xi\\ &= \int_{\mathbb{T}^n} \omega_\phi(\xi)\sum_{l\in\mathbb{Z}^n}|r_l(\xi)|^2\,d\xi\\ &= \|r\|^2_{L^2(\mathbb{T}^n,\ell^2(\mathbb{Z}^n);\omega_\phi)},\end{aligned} \tag{5.13}$$

where $r(\xi) = \{r_l(\xi)\}_{l\in\mathbb{Z}^n}$, $\xi\in\mathbb{T}^n$. Further, by Theorem 5.3, we have

$$\begin{aligned}\tau\phi(\xi) &= \sum_{l\in\mathbb{Z}^n} H_l(\xi)T_l(\xi)^*\tau\phi(\xi)\\ &= \sum_{l\in\mathbb{Z}^n}\langle\tau\phi(\xi),\tau\tilde{\phi}(\xi+l)\rangle\tau\phi(\xi+l)\\ &= \sum_{l\in\mathbb{Z}^n}\left\langle\tau\phi(\xi),\sum_{l'\in\mathbb{Z}^n} r_{l'}(\xi)\tau\phi(\xi+l+l')\right\rangle\tau\phi(\xi+l)\\ &= \sum_{l,l'\in\mathbb{Z}^n}\overline{r_{l'}(\xi)}\langle\tau\phi(\xi),\tau\phi(\xi+l+l')\rangle\tau\phi(\xi+l),\end{aligned} \tag{5.14}$$

for $a.e.$ $\xi\in\mathbb{T}^n$. Since ϕ satisfies "Condition C," using (3.13), we get

$$\begin{aligned}\langle\tau\phi(\xi),\tau\phi(\xi+l+l')\rangle &= \int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n} K_\phi(\xi+m,\eta)\overline{K_\phi(\xi+m+l+l',\eta)}\,d\eta\\ &= 0,\end{aligned}$$

whenever $l+l'\neq 0$. Hence, (5.14) reduces to

$$\begin{aligned}\tau\phi(\xi) &= \sum_{l\in\mathbb{Z}^n}\overline{r_{-l}(\xi)}\omega_\phi(\xi)\tau\phi(\xi+l)\\ &= \sum_{l\in\mathbb{Z}^n} r'_{-l}(\xi)\tau\phi(\xi+l),\end{aligned}$$

where

$$r'_{-l}(\xi) = \overline{r_{-l}(\xi)}\omega_\phi(\xi). \tag{5.15}$$

Hence,

$$\begin{aligned}\|\phi\|^2_{L^2(\mathbb{C}^n)} &= \int_{\mathbb{T}^n} \|\tau\phi(\xi)\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}^n))}\,d\xi \\ &= \int_{\mathbb{T}^n} \omega_\phi(\xi)\sum_{l\in\mathbb{Z}^n} |r'_l(\xi)|^2\,d\xi \\ &= \|r'\|^2_{L^2(\mathbb{T}^n,\ell^2(\mathbb{Z}^n);\omega_\phi)},\end{aligned}$$

where $r'(\xi) = \{r'_l(\xi)\}_{l\in\mathbb{Z}^n}$. Using (5.15), we notice that

$$r'(\xi) = \left\{\overline{r_l(\xi)}\omega_\phi(\xi)\right\}_{l\in\mathbb{Z}^n} = \omega_\phi(\xi)\overline{r(\xi)}.$$

Thus, by using Proposition 5.4, $r' \in L^2(\mathbb{T}^n,\ell^2(\mathbb{Z}^n);\omega_\phi)$ is uniquely determined. Hence, $r \in L^2(\mathbb{T}^n,\ell^2(\mathbb{Z}^n);\omega_\phi)$ can be uniquely determined via (5.15). Therefore, by Proposition 5.4, we obtain the uniqueness of $\widetilde{\phi}$ from (5.13). □

§6. Orthogonalization in a twisted shift-invariant space

It is well known that if $\{T_k\phi : k \in \mathbb{Z}\}$ is a Riesz sequence in $L^2(\mathbb{R})$, then there exists $\phi^\sharp \in V(\phi)$ such that $\{T_k\phi^\sharp : k \in \mathbb{Z}\}$ is an orthonormal system for $V(\phi)$. This is a useful criterion in wavelet literature (see Lemma 1.8 in [15]). We wish to prove a similar result for the system of twisted translates on $L^2(\mathbb{R}^{2n})$. Toward this end, we have the following result.

THEOREM 6.1 [18]. *Let $\phi \in L^2(\mathbb{R}^{2n})$. Then $\{T^t_{(k,l)}\phi : (k,l) \in \mathbb{Z}^{2n}\}$ is an orthonormal system in $L^2(\mathbb{R}^{2n})$ if and only if $\omega_\phi(\xi) = 1$ for a.e. $\xi \in \mathbb{T}^n$ and ϕ satisfies "Condition C."*

It is clear from the statement of Theorem 6.1 that it is mandatory to assume "Condition C," when we talk about the orthonormality of the system of twisted translates. With this comment, we are ready to state our result.

THEOREM 6.2. *Let $\phi \in L^2(\mathbb{R}^{2n})$ such that ϕ satisfies "Condition C." Let $E^t(\phi)$ be a Riesz sequence. Then there exists $\phi^\sharp \in L^2(\mathbb{R}^{2n})$ such that the system $E^t(\phi^\sharp)$ is an orthonormal system in $L^2(\mathbb{R}^{2n})$ with $V^t(\phi) = V^t(\phi^\sharp)$.*

Proof. Since $E^t(\phi)$ is a Riesz sequence, by Corollary 4.5, there exists $A, B > 0$ such that $A \le \omega_\phi(\xi) \le B$, for *a.e.* $\xi \in \mathbb{T}^n$. Hence, using the periodicity of ω_ϕ, we have $A \le \omega_\phi(\xi) \le B$, for *a.e.* $\xi \in \mathbb{R}^n$. We define

$$K(\xi,\eta) := \frac{K_\phi(\xi,\eta)}{\sqrt{\omega_\phi(\xi)}}, \tag{6.1}$$

for $\eta \in \mathbb{R}^n$ and for *a.e.* $\xi \in \mathbb{R}^n$. Now,

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |K(\xi,\eta)|^2\,d\xi d\eta &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\frac{K_\phi(\xi,\eta)}{\sqrt{\omega_\phi(\xi)}}\right|^2 d\xi d\eta \\ &\le \frac{1}{A}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |K_\phi(\xi,\eta)|^2\,d\xi d\eta \\ &= \frac{1}{A}\|K_\phi\|^2_{L^2(\mathbb{R}^{2n})} \\ &= \frac{1}{A}\|\phi\|^2_{L^2(\mathbb{R}^{2n})} \\ &< \infty,\end{aligned}$$

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proving that $K \in L^2(\mathbb{R}^{2n})$. Define $\Theta : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$(\Theta f)(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy, \qquad \forall\, f \in L^2(\mathbb{R}^n).$$

Here, Θ is an integral operator with kernel K in $L^2(\mathbb{R}^{2n})$. Thus, $\Theta \in \mathcal{B}_2$. Since the Weyl transform $W : L^2(\mathbb{R}^{2n}) \to \mathcal{B}_2$ is an isometric isomorphism, there exists unique $\phi^\sharp \in L^2(\mathbb{R}^{2n})$ such that $W\phi^\sharp = \Theta$. Hence, we obtain

$$\begin{aligned} K_{\phi^\sharp}(\xi,\eta) &= K(\xi,\eta) \\ &= \frac{K_\phi(\xi,\eta)}{\sqrt{\omega_\phi(\xi)}}. \end{aligned} \tag{6.2}$$

Since ϕ satisfies "Condition C," for $l \in \mathbb{Z}^n \setminus \{0\}$, we have

$$\begin{aligned} \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} K_{\phi^\sharp}(\xi+m,\eta)\overline{K_{\phi^\sharp}(\xi+m+l,\eta)}\, d\eta &= \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} \frac{K_\phi(\xi+m,\eta)}{\sqrt{\omega_\phi(\xi+m)}} \frac{\overline{K_\phi(\xi+m+l,\eta)}}{\sqrt{\omega_\phi(\xi+m+l)}}\, d\eta \\ &= \frac{1}{\omega_\phi(\xi)} \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} K_\phi(\xi+m,\eta)\overline{K_\phi(\xi+m+l,\eta)}\, d\eta \\ &= 0, \end{aligned}$$

for $a.e.$ $\xi \in \mathbb{T}^n$, showing that $\phi^\sharp$ satisfies "Condition C." Moreover,

$$\begin{aligned} \omega_{\phi^\sharp}(\xi) &= \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |K_{\phi^\sharp}(\xi+m,\eta)|^2 d\eta \\ &= \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} \left| \frac{K_\phi(\xi+m,\eta)}{\sqrt{\omega_\phi(\xi+m)}} \right|^2 d\eta \\ &= \frac{1}{\omega_\phi(\xi)} \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |K_\phi(\xi+m,\eta)|^2 d\eta \\ &= 1, \end{aligned}$$

for $a.e.$ $\xi \in \mathbb{T}^n$. Hence, by Theorem 6.1, the system $E^t(\phi^\sharp)$ is an orthonormal system. Now, it remains to prove that $V^t(\phi) = V^t(\phi^\sharp)$. Let $\sqrt{\omega_\phi(\xi)} = \sum_{k\in\mathbb{Z}^n} a_k e^{2\pi i k\cdot\xi}$ be the Fourier series of $\sqrt{\omega_\phi(\xi)}$. Then, by using (6.2) and Lemma 2.9, we get

$$\begin{aligned} K_\phi(\xi,\eta) &= \sqrt{\omega_\phi(\xi,\eta)} K_{\phi^\sharp}(\xi,\eta) \\ &= \sum_{k\in\mathbb{Z}^n} a_k e^{2\pi i k\cdot\xi} K_{\phi^\sharp}(\xi,\eta) \\ &= \sum_{k\in\mathbb{Z}^n} a_k K_{T^t_{(k,0)}\phi^\sharp}(\xi,\eta) \\ &= \sum_{k\in\mathbb{Z}^n} a_k \int_{\mathbb{R}^n} T^t_{(k,0)}\phi^\sharp(x,\eta-\xi) e^{\pi i x\cdot(\xi+\eta)}\, dx \\ &= \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}^n} a_k T^t_{(k,0)}\phi^\sharp(x,\eta-\xi) e^{\pi i x\cdot(\xi+\eta)}\, dx \\ &= K_{\sum_{k\in\mathbb{Z}^n} a_k T^t_{(k,0)}\phi^\sharp}(\xi,\eta), \end{aligned}$$

which, by the uniqueness of the kernel, implies that $\phi = \sum_{k\in\mathbb{Z}^n} a_k T^t_{(k,0)}\phi^\sharp$. Thus, using (2.2), we get

$$\begin{aligned} T^t_{(k',l')}\phi(x,y) &= \sum_{k\in\mathbb{Z}^n} a_k T^t_{(k',l')} T^t_{(k,0)}\phi^\sharp(x,y) \\ &= \sum_{k\in\mathbb{Z}^n} a_k e^{\pi i l'\cdot k} T^t_{((k'+k),l')}\phi^\sharp(x,y), \end{aligned}$$

showing that $V^t(\phi) \subset V^t(\phi^\sharp)$. Similarly, we can prove $V^t(\phi^\sharp) \subset V^t(\phi)$, by using the Fourier series of $\frac{1}{\sqrt{\omega_\phi}}$ in (6.2). □

Now, we illustrate this theorem with an example. Recall that for $f,g \in L^2(\mathbb{R}^{2n})$, we have (3.24).

EXAMPLE 6.3. Let $\phi(x,y) = \sum_{m\in\mathbb{Z}} \frac{1}{10^{|m|}} \chi_{[m,m+1)\times[0,1)}(x,y), \qquad (x,y) \in \mathbb{R}^2$. Then, for $k,l \in \mathbb{Z}$, we have

$$\begin{aligned} \langle \phi, T^t_{(k,l)}\phi\rangle &= \int_{\mathbb{R}^2} \phi(x,y)\overline{T^t_{(k,l)}\phi(x,y)}\,dydx \\ &= \int_0^1 \int_{\mathbb{R}} \phi(x,y) e^{-\pi i(xl-yk)}\phi(x-k,y-l)\,dxdy. \end{aligned} \tag{6.3}$$

It is easy to see that $\operatorname{supp}\phi = \mathbb{R}\times[0,1]$ and $\operatorname{supp} T^t_{(k,l)}\phi = \mathbb{R}\times[l,l+1]$. Hence, $\langle \phi, T^t_{(k,l)}\phi\rangle = 0$, whenever $l \neq 0$. Consequently, using (3.24), we obtain

$$\begin{aligned} \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} K_\phi(\xi+m,\eta)\overline{K_\phi(\xi+m+l,\eta)}\,d\eta &= \sum_{k\in\mathbb{Z}^n} \langle \phi, T^t_{(k,l)}\phi\rangle e^{\pi i k(2\xi+l)} \\ &= 0, \qquad \forall\, l \neq 0, \end{aligned}$$

showing that ϕ satisfies "Condition C." Now, we make use of Corollary (4.5) to show that the system $E^t(\phi)$ forms a Riesz sequence. From the definition of ϕ, we get

$$\begin{aligned} \|\phi\|^2_{L^2(\mathbb{R}^{2n})} &= \int_{\mathbb{R}^2} |\phi(x,y)|^2 dxdy \\ &= \int_{\mathbb{R}}\int_0^1 |\phi(x,y)|^2 dydx \\ &= \sum_{m\in\mathbb{Z}} \int_m^{m+1} |\phi(x,y)|^2 dx \\ &= \sum_{m\in\mathbb{Z}} \int_m^{m+1} \frac{1}{100^{|m|}} dx \\ &= \sum_{m\in\mathbb{Z}} \frac{1}{100^{|m|}} \\ &= \frac{101}{99}, \end{aligned}$$

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and, for $k \neq 0$ and $l = 0$ in (6.3), we get

$$\begin{aligned}\left\langle \phi, T^t_{(k,0)}\phi \right\rangle &= \int_0^1 \left(\sum_{m\in\mathbb{Z}} \int_m^{m+1} \phi(x,y)\phi(x-k,y)\,dx \right) e^{\pi iyk}\,dy \\ &= \int_0^1 \left(\sum_{m\in\mathbb{Z}} \int_m^{m+1} \frac{1}{10^{|m|}} \phi(x-k,y)\,dx \right) e^{\pi iyk}\,dy \\ &= \int_0^1 \left(\sum_{m\in\mathbb{Z}} \int_{m-k}^{m+1-k} \frac{1}{10^{|m|}} \phi(x,y)\,dx \right) e^{\pi iyk}\,dy \\ &= \int_0^1 \left(\sum_{m\in\mathbb{Z}} \int_{m-k}^{m+1-k} \frac{1}{10^{|m|}} \frac{1}{10^{|m-k|}}\,dx \right) e^{\pi iyk}\,dy \\ &= \left(\sum_{m\in\mathbb{Z}} \frac{1}{10^{|m|}} \frac{1}{10^{|m-k|}} \right) \left(\int_0^1 e^{\pi iyk}\,dy \right). \end{aligned} \tag{6.4}$$

But

$$\int_0^1 e^{\pi iyk}\,dy = \begin{cases} 0, & \text{if } k \text{ is even,} \\ -\frac{2}{\pi ik}, & \text{if } k \text{ is odd.} \end{cases}$$

Hence, (6.4) gives

$$\left\langle \phi, T^t_{(k,0)}\phi \right\rangle = \begin{cases} 0, & \text{if } k \text{ is even,} \\ -\frac{2}{\pi ik} \sum_{m\in\mathbb{Z}} \frac{1}{10^{|m|}} \frac{1}{10^{|m-k|}}, & \text{if } k \text{ is odd.} \end{cases} \tag{6.5}$$

Now, we aim to compute $\sum_{m\in\mathbb{Z}} \frac{1}{10^{|m|}} \frac{1}{10^{|m-k|}}$ explicitly for odd k.

First, we assume that k is an odd positive integer. Then

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \frac{1}{10^{|m|}} \frac{1}{10^{|m-k|}} &= \sum_{m=0}^{\infty} \frac{1}{10^m} \frac{1}{10^{|m-k|}} + \sum_{m=-\infty}^{-1} \frac{1}{10^{-m}} \frac{1}{10^{|m-k|}} \\ &= \sum_{m=0}^{k} \frac{1}{10^m} \frac{1}{10^{k-m}} + \sum_{m=k+1}^{\infty} \frac{1}{10^m} \frac{1}{10^{m-k}} + \sum_{m=1}^{\infty} \frac{1}{10^m} \frac{1}{10^{|-m-k|}} \\ &= \sum_{m=0}^{k} \frac{1}{10^k} + 2\sum_{m=1}^{\infty} \frac{1}{10^m} \frac{1}{10^{m+k}} \\ &= \frac{k+1}{10^k} + \frac{2}{10^k} \sum_{m=1}^{\infty} \frac{1}{100^m} \\ &= \frac{k}{10^k} + \frac{101}{99} \frac{1}{10^k}. \end{aligned} \tag{6.6}$$

Similarly, if k is a negative odd integer, we can show that

$$\sum_{m\in\mathbb{Z}} \frac{1}{10^{|m|}} \frac{1}{10^{|m-k|}} = \frac{-k}{10^{-k}} + \frac{101}{99} \frac{1}{10^{-k}}. \tag{6.7}$$

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Using (6.6) and (6.7) in (6.5), we get

$$\left\langle \phi, T^t_{(k,0)}\phi \right\rangle = \begin{cases} 0, & \text{if } k \in 2\mathbb{Z}\setminus\{0\}, \\ -\frac{2}{\pi i}\left(\frac{1}{10^k} + \frac{101}{99}\frac{1}{k10^k}\right), & \text{if } k \in 2\mathbb{N}-1, \\ \frac{2}{\pi i}\left(\frac{1}{10^{-k}} + \frac{101}{99}\frac{1}{(-k)10^{-k}}\right), & \text{if } k \in 1-2\mathbb{N}. \end{cases}$$

Now, using (3.24), we have

$$\begin{aligned} \omega_\phi(\xi) &= \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} |K_\phi(\xi+m,\eta)|^2\, d\eta \\ &= \sum_{k\in\mathbb{Z}} \left\langle \phi, T^t_{(k,0)}\phi \right\rangle e^{2\pi i k\xi} \\ &= \|\phi\|^2 + \sum_{k\in 2\mathbb{N}-1} \left\langle \phi, T^t_{(k,0)}\phi \right\rangle e^{2\pi i k\xi} + \sum_{k\in 1-2\mathbb{N}} \left\langle \phi, T^t_{(k,0)}\phi \right\rangle e^{2\pi i k\xi} \\ &= \frac{101}{99} - \frac{2}{\pi i}\sum_{k\in 2\mathbb{N}-1}\left(\frac{1}{10^k} + \frac{101}{99}\frac{1}{k10^k}\right)e^{2\pi i k\xi} + \frac{2}{\pi i}\sum_{k\in 1-2\mathbb{N}}\left(\frac{1}{10^{-k}} + \frac{101}{99}\frac{1}{(-k)10^{-k}}\right)e^{2\pi i k\xi} \\ &= \frac{101}{99} - \frac{2}{\pi i}\sum_{k\in 2\mathbb{N}-1}\left(\frac{1}{10^k} + \frac{101}{99}\frac{1}{k10^k}\right)e^{2\pi i k\xi} + \frac{2}{\pi i}\sum_{k\in 2\mathbb{N}-1}\left(\frac{1}{10^k} + \frac{101}{99}\frac{1}{k10^k}\right)e^{-2\pi i k\xi} \\ &= \frac{101}{99} - \frac{2}{\pi i}\sum_{k\in 2\mathbb{N}-1}\left(\frac{1}{10^k} + \frac{101}{99}\frac{1}{k10^k}\right)\left(e^{2\pi i k\xi} - e^{-2\pi i k\xi}\right) \\ &= \frac{101}{99} - \frac{4}{\pi}\sum_{k\in 2\mathbb{N}-1}\left(\frac{1}{10^k} + \frac{101}{99}\frac{1}{k10^k}\right)\sin(2\pi k\xi) \\ &= \frac{101}{99} - R(\xi), \end{aligned} \tag{6.8}$$

where

$$R(\xi) := \frac{4}{\pi}\sum_{k\in 2\mathbb{N}-1}\left(\frac{1}{10^k} + \frac{101}{99}\frac{1}{k10^k}\right)\sin(2\pi k\xi).$$

Now, we have

$$\begin{aligned} |R(\xi)| &\le \frac{4}{\pi}\sum_{k\in 2\mathbb{N}-1}\left(\frac{1}{10^k} + \frac{101}{99}\frac{1}{k10^k}\right) \\ &= \frac{4}{\pi}\left[\sum_{k\in 2\mathbb{N}-1}\frac{1}{10^k} + \frac{101}{99}\sum_{k\in 2\mathbb{N}-1}\frac{1}{k10^k}\right], \qquad \forall\ \xi\in[0,1). \end{aligned} \tag{6.9}$$

Decomposing

$$\sum_{k=1}^{\infty}\frac{1}{10^k} = \sum_{k\in 2\mathbb{N}-1}\frac{1}{10^k} + \sum_{k=1}^{\infty}\frac{1}{10^{2k}} \qquad \text{and} \qquad \sum_{k=1}^{\infty}\frac{1}{k10^k} = \sum_{k\in 2\mathbb{N}-1}\frac{1}{k10^k} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k10^{2k}},$$

we can show that

$$\sum_{k\in 2\mathbb{N}-1}\frac{1}{10^k} = \frac{10}{99} \qquad \text{and} \qquad \sum_{k\in 2\mathbb{N}-1}\frac{1}{k10^k} = -\ln\left(\frac{9}{10}\right) + \frac{1}{2}\ln\left(\frac{99}{100}\right).$$

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Hence, from (6.9), we obtain

$$|R(\xi)| \le \frac{4}{\pi}\left[\frac{10}{99}+\frac{101}{99}\left(-\ln\left(\frac{9}{10}\right)+\frac{1}{2}\ln\left(\frac{99}{100}\right)\right)\right] < 0.26, \qquad \forall\, \xi \in [0,1).$$

Thus, from (6.8), we get

$$\omega_\phi(\xi) < \frac{101}{99} + 0.26 < 1.3, \qquad \forall\, \xi \in [0,1),$$

and

$$\omega_\phi(\xi) > \frac{101}{99} - 0.26 > 0.7, \qquad \forall\, \xi \in [0,1).$$

Therefore, by Corollary 4.5, the system $E^t(\phi)$ is a Riesz sequence with bounds 0.7 and 1.3. Now, by Theorem 5.1, there exists $\phi^\sharp \in L^2(\mathbb{R}^2)$, given by

$$K_{\phi^\sharp}(\xi,\eta) = \frac{K_\phi(\xi,\eta)}{\sqrt{\omega_\phi(\xi)}}, \qquad (\xi,\eta) \in \mathbb{R}^2, \tag{6.10}$$

such that the system $E^t(\phi^\sharp)$ is an orthonormal system in $L^2(\mathbb{R}^2)$ with $V^t(\phi) = V^t(\phi^\sharp)$.

Acknowledgment. We would like to thank the referee for meticulously reading the manuscript and giving us the valuable suggestions.

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Santi Ranjan Das
School of Mathematical Sciences
NISER Bhubaneswar
Odisha, India
santiranjandas100@gmail.com

Rabeetha Velsamy
Department of Mathematics
Indian Institute of Technology
Madras, India
rabeethavelsamy@gmail.com

Radha Ramakrishnan
Department of Mathematics
Indian Institute of Technology
Madras, India
radharam@iitm.ac.in

https://doi.org/10.1017/nmj.2023.11 Published online by Cambridge University Press