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MÖBIUS INVARIANT FUNCTION SPACES AND DIRICHLET SPACES WITH SUPERHARMONIC WEIGHTS

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Abstract

Let $\mathcal{D}_\mu$ be Dirichlet spaces with superharmonic weights induced by positive Borel measures μ on the open unit disk. Denote by $M(\mathcal{D}_\mu)$ Möbius invariant function spaces generated by $\mathcal{D}_\mu$. In this paper, we investigate the relation among $\mathcal{D}_\mu$, $M(\mathcal{D}_\mu)$ and some Möbius invariant function spaces, such as the space $BMOA$ of analytic functions on the open unit disk with boundary values of bounded mean oscillation and the Dirichlet space. Applying the relation between $BMOA$ and $M(\mathcal{D}_\mu)$, under the assumption that the weight function K is concave, we characterize the function K such that $Q_K = BMOA$. We also describe inner functions in $M(\mathcal{D}_\mu)$ spaces.

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1. Introduction

One of the classical topics in complex analysis is the study of Möbius invariant function spaces in the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$. Möbius invariant function spaces are closely associated with the Möbius group denoted by $\text{Aut}(\mathbb{D})$. The Möbius group consists of all one-to-one analytic functions that map $\mathbb{D}$ onto itself. It is well known that each $\phi \in \text{Aut}(\mathbb{D})$ has the form

$$\phi(z) = e^{i\theta}\sigma_a(z), \qquad \sigma_a(z) = \frac{a-z}{1-\bar{a}z},$$

where θ is real and $a, z \in \mathbb{D}$. Let X be a linear space of analytic functions on $\mathbb{D}$ which is complete in a norm or seminorm $\|.\|_X$. The space X is called Möbius invariant if for each function f in X and each element ϕ in $\text{Aut}(\mathbb{D})$, the composition function $f \circ \phi$ also

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belongs to X and satisfies $\|f \circ \phi\|_X = \|f\|_X$. We refer to Arazy *et al.* [4] for a general exposition on Möbius invariant function spaces.

Denote by $H(\mathbb{D})$ the space of analytic functions in $\mathbb{D}$. Let $(Y, \|.\|_Y)$ be a Banach space of analytic functions in $\mathbb{D}$ containing all constant functions. Following Aleman and Simbotin [3], we denote by $M(Y)$ the Möbius invariant function space generated by Y. Namely, $M(Y)$ is the class of functions $f \in H(\mathbb{D})$ with

$$\|f\|_{M(Y)} = \sup_{\phi \in \mathrm{Aut}(\mathbb{D})} \|f \circ \phi - f(\phi(0))\|_Y < \infty.$$

This construction gives rise to all Möbius invariant Banach spaces on $\mathbb{D}$ (cf. [34, page 1001]).

The study of analytic Hilbert function spaces is also classical. Richter [26] introduced Dirichlet spaces with harmonic weights. Aleman's work [2] initiated the study of Dirichlet spaces with superharmonic weights. These Dirichlet-type spaces are Hilbert spaces. In this paper, we consider a class of Dirichlet spaces $\mathcal{D}_\mu$ with superharmonic weights induced by positive Borel measures μ on $\mathbb{D}$. More precisely, we will study the space $\mathcal{D}_\mu$ consisting of functions $f \in H(\mathbb{D})$ with

$$\int_{\mathbb{D}} |f'(z)|^2 U_\mu(z)\, dA(z) < +\infty,$$

where dA denotes the area measure on $\mathbb{D}$ and

$$U_\mu(z) = \int_{\mathbb{D}} \log\left|\frac{1 - \overline{w}z}{z - w}\right| d\mu(w)$$

is a superharmonic function on $\mathbb{D}$. The $\mathcal{D}_\mu$ spaces are always subsets of the Hardy space H^2 (cf. [2, 17]). Let $d\mu_p(z) = -\Delta(1 - |z|^2)^p\, dA(z)$, where $z \in \mathbb{D}$, $p \in (0, 1)$ and Δ is the Laplace operator. From [1], the space $\mathcal{D}_{\mu_p}$ is equal to the well-studied radial Dirichlet-type spaces $\mathcal{D}_p$ consisting of functions $f \in H(\mathbb{D})$ with

$$\int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^p\, dA(z) < \infty.$$

By [10, Corollary 5.6], there exists a positive Borel measure μ such that $\mathcal{D}_\mu$ is not equal to any generalized radial Dirichlet-type space. It is well known (cf. [5, page 98]) that $U_\mu \not\equiv +\infty$ if and only if

$$\int_{\mathbb{D}} (1 - |z|)\, d\mu(z) < +\infty. \tag{1.1}$$

Thus, throughout this paper, we always assume that μ satisfies the condition (1.1). By (1.1), we get that $\mu(E) < \infty$ for every compact subset E of $\mathbb{D}$. From [10, Lemma 5.1], every $\mathcal{D}_\mu$ space can also be defined as the class of functions $f \in H(\mathbb{D})$ for which

$$\|f\|^2_{\mathcal{D}_\mu} = \int_{\mathbb{D}} |f'(z)|^2 V_\mu(z)\, dA(z) < +\infty,$$

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where

$$V_\mu(z) = \int_{\mathbb{D}} (1 - |\sigma_z(w)|^2)\, d\mu(w).$$

A norm on $\mathcal{D}_\mu$ can be defined by

$$|||f|||^2_{\mathcal{D}_\mu} = |f(0)|^2 + \|f\|^2_{\mathcal{D}_\mu}.$$

In this paper we investigate $M(\mathcal{D}_\mu)$, the Möbius invariant function space generated by the Hilbert function space $\mathcal{D}_\mu$. Namely, $M(\mathcal{D}_\mu)$ consists of functions $f \in H(\mathbb{D})$ with

$$\begin{aligned}\|f\|^2_{M(\mathcal{D}_\mu)} &= \sup_{\phi \in \mathrm{Aut}(\mathbb{D})} \int_{\mathbb{D}} |f'(\phi(z))|^2 |\phi'(z)|^2 V_\mu(z)\, dA(z) \\ &= \sup_{a \in \mathbb{D}, \lambda \in \mathbb{T}} \int_{\mathbb{D}} |f'(w)|^2 V_\mu(\lambda \sigma_a(w))\, dA(w) < \infty,\end{aligned}$$

where $\mathbb{T} := \partial\mathbb{D}$ is the unit circle. A norm on $M(\mathcal{D}_\mu)$ can be defined by $|||f|||^2_{M(\mathcal{D}_\mu)} = |f(0)|^2 + \|f\|^2_{M(\mathcal{D}_\mu)}$. We will see that $M(\mathcal{D}_\mu)$ spaces are associated with several Möbius invariant function spaces such as some special cases of Q_K spaces. For an increasing and right-continuous function $K : (0, \infty) \to [0, \infty)$, let Q_K be the space of all functions $f \in H(\mathbb{D})$ for which

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K\Big(\log \frac{1}{|\sigma_a(z)|}\Big)\, dA(z) < \infty.$$

From [18], Q_K can also be defined as the set of functions $f \in H(\mathbb{D})$ with

$$\|f\|^2_{Q_K} = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(1 - |\sigma_a(z)|^2)\, dA(z) < \infty.$$

The Q_K spaces are Möbius invariant under the above seminorm. By [18], the theory of Q_K depends only on the behavior of the weight function K near zero. We refer to [18, 19, 28] for more results about Q_K spaces. If $K_0(t) = t \log(e/t)$, $0 < t < 1$, then Q_{K_0} is the analytic version of $Q_1(\mathbb{T})$ space (see [29, 32]). If $K(t) = t^p$, $0 \le p < \infty$, then Q_K coincides with Q_p (see [7, 30, 31]). Clearly, $Q_1 = BMOA$, the set of analytic functions on $\mathbb{D}$ with boundary values of bounded mean oscillation (see [9, 22]). The space Q_0 is equal to the Dirichlet space $\mathcal{D}$. By [6], we see that for all $1 < p < \infty$, Q_p is equal to the Bloch space $\mathcal{B}$ consisting of functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.$$

Rubel and Timoney [27] proved that in some sense the maximal Möbius invariant function space is the Bloch space.

A question mentioned by Wulan in several conferences or workshops is to characterize the weight function K such that $Q_K = BMOA$. See also this question in the recent monograph [28]. The paper is organized as follows. In Section 2, we show that $BMOA = M(\mathcal{D}_\mu)$ if and only if μ is finite. As an application, we answer partially

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the above question. In Section 3, we reveal how different measures μ induce the same space $\mathcal{D}_\mu$. We also study the relation among $\mathcal{D}_\mu$, $M(\mathcal{D}_\mu)$ and the Dirichlet space using Carleson measures. In the last section, we investigate inner functions in $M(\mathcal{D}_\mu)$ with infinite measure μ. We prove that any inner function in $M(\mathcal{D}_\mu)$ must be a Blaschke product. A criterion for Carleson–Newman Blaschke products belonging to $M(\mathcal{D}_\mu)$ is also given.

Throughout this paper, we will write $a \lesssim b$ if there exists a constant C such that $a \le Cb$. Also, the symbol $a \approx b$ means that $a \lesssim b \lesssim a$.

2. The equality between $BMOA$ and Q_K via $M(\mathcal{D}_\mu)$ spaces

In this section, we show that $BMOA = M(\mathcal{D}_\mu)$ if and only if μ is finite. Applying this result, under the assumption that the weight function K is concave, we characterize the function K such that Q_K is equal to $BMOA$.

As usual, denote by H^∞ the space of bounded analytic functions on $\mathbb{D}$. The space H^∞ is Möbius invariant under the following norm:

$$|||f|||_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)|.$$

From Aleman [2, Proposition 3.1, page 83], $H^\infty \subseteq \mathcal{D}_\mu$ if and only if μ is finite. Based on this interesting result, we get the following theorem.

Theorem 2.1. *Let μ be a positive Borel measure on $\mathbb{D}$. Then the following conditions are equivalent.*

(1) $BMOA \subseteq \mathcal{D}_\mu$.
(2) $BMOA = M(\mathcal{D}_\mu)$.
(3) *μ is finite.*

Proof. (3) $\Rightarrow$ (1). Let $f \in BMOA$. Applying the Fubini theorem,

$$\|f\|^2_{\mathcal{D}_\mu} \le \mu(\mathbb{D})\|f\|^2_{BMOA},$$

which implies the desired result.

(1) $\Rightarrow$ (3). Note that $H^\infty \subseteq BMOA$. Together with condition (1), this yields $H^\infty \subseteq \mathcal{D}_\mu$. Thus, μ is finite.

(1) $\Rightarrow$ (2). Since $\mathcal{D}_\mu$ is always a subset of H^2 and $BMOA = M(H^2)$ (see [9]), we have $M(\mathcal{D}_\mu) \subseteq BMOA$. Combining this with condition (1), we obtain that $BMOA = M(\mathcal{D}_\mu)$.

(2) $\Rightarrow$ (1) is true because of $M(\mathcal{D}_\mu) \subseteq \mathcal{D}_\mu$. The proof is complete. □

If the weight function K is concave on $(0, 1)$, then $Q_K \subseteq BMOA$ (cf. [28]). Applying Theorem 2.1, we answer partially the question mentioned in Section 1 as follows.

Theorem 2.2. *Let $K \in C^2(0, 1)$ be increasing and concave on $(0, 1)$ and $\lim_{t\to 0} K(t) = 0$. Then the following conditions are equivalent.*

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(1) $\mathcal{Q}_K = BMOA$.
(2)

$$\int_0^1 [K'(t) - (1-t)K''(t)]\,dt < \infty.$$

PROOF. Note that K is an increasing and concave function on $(0, 1)$ with $\lim_{t\to 0} K(t) = 0$. By [1, page 99], we know that

$$K(1-|z|) = -\frac{1}{2\pi}\int_{\mathbb{D}} \Delta(K(1-|w|))\log\frac{1}{|\sigma_w(z)|}\,dA(w), \quad z \in \mathbb{D},$$

where Δ is the Laplace operator. Set $d\nu(w) = -\Delta(K(1-|w|))\,dA(w)$. Then $\mathcal{Q}_K = M(\mathcal{D}_\nu)$. Hence, $\mathcal{Q}_K = BMOA$ if and only if $BMOA = M(\mathcal{D}_\nu)$. This, together with Theorem 2.1, yields that $\mathcal{Q}_K = BMOA$ if and only if ν is finite. Now we compute $\nu(\mathbb{D})$. Recall that the Laplace operator in polar coordinates is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

We have

$$\begin{aligned}
-\int_{\mathbb{D}} \Delta(K(1-|w|))\,dA(w) &= -\int_0^{2\pi}\int_0^1 \Delta(K(1-r))r\,dr\,d\theta \\
&= -2\pi\int_0^1 [rK''(1-r) - K'(1-r)]\,dr \\
&= 2\pi\int_0^1 [K'(t) - (1-t)K''(t)]\,dt.
\end{aligned}$$

Thus, $\mathcal{Q}_K = BMOA$ if and only if

$$\int_0^1 [K'(t) - (1-t)K''(t)]\,dt < \infty. \qquad \square$$

Letting $K(t) = \sin t$, we obtain a weight function different from the identity which satisfies the hypotheses and the condition (2) of Theorem 2.2. From the proof of Theorem 2.2, we see that if $K \in C^2(0, 1)$ is an increasing and concave function on $(0, 1)$ with $\lim_{t\to 0} K(t) = 0$, then the space $\mathcal{Q}_K$ is a special case of $M(\mathcal{D}_\mu)$. Thus, $M(\mathcal{D}_\mu)$ spaces also generalize $\mathcal{Q}_p$ spaces for $0 < p \le 1$ and the analytic version of $\mathcal{Q}_1(\mathbb{T})$ space. In the next section, we will show that all nontrivial $M(\mathcal{D}_\mu)$ spaces are between $\mathcal{D}$ and $BMOA$. Comparing with $\mathcal{Q}_K$ and $\mathcal{Q}_p$ spaces, $M(\mathcal{D}_\mu)$ spaces connect $\mathcal{D}$ and $BMOA$ more smoothly. We refer to [11] for a recent investigation of $\mathcal{Q}_p$ spaces and a class of Dirichlet-type spaces $\mathcal{D}_{\mu,p}$ induced by finite positive Borel measures μ on $\mathbb{D}$.

3. The Dirichlet space and $\mathcal{D}_\mu$ and $M(\mathcal{D}_\mu)$ spaces

In this section, we give the precise link between the measures μ and ν such that $\mathcal{D}_\mu = \mathcal{D}_\nu$. We also investigate the relation among $\mathcal{D}_\mu$, $M(\mathcal{D}_\mu)$ and the Dirichlet space via Carleson measures.

The following test functions in $\mathcal{D}_\mu$ were given in [10].

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LEMMA A. *Let μ be a positive Borel measure on $\mathbb{D}$. For every $w \in \mathbb{D}$, let*

$$f_w(z) = \frac{\sigma_w(z)}{\sqrt{V_\mu(w)}} - \frac{\sigma_w(0)}{\sqrt{V_\mu(w)}}, \quad z \in \mathbb{D}.$$

Then

$$\sup_{w\in\mathbb{D}} |||f_w|||_{\mathcal{D}_\mu} < +\infty.$$

We reveal how different measures μ induce the same space $\mathcal{D}_\mu$ as follows.

THEOREM 3.1. *Let μ and ν be positive Borel measures on $\mathbb{D}$. Then $\mathcal{D}_\mu = \mathcal{D}_\nu$ if and only if there exist positive constants C_1 and C_2 such that*

$$C_1 V_\mu(z) \le V_\nu(z) \le C_2 V_\mu(z) \tag{3.1}$$

for all $z \in \mathbb{D}$.

PROOF. Clearly, if the condition (3.1) holds, then $\mathcal{D}_\mu = \mathcal{D}_\nu$. On the other hand, suppose that $\mathcal{D}_\mu = \mathcal{D}_\nu$. By the closed graph theorem,

$$|||f|||_{\mathcal{D}_\nu} \lesssim |||f|||_{\mathcal{D}_\mu}$$

for all $f \in \mathcal{D}_\mu$. For $w \in \mathbb{D}$, define the function f_w as in Lemma A. Combining the above facts and the Fubini theorem,

$$\begin{aligned}
\infty &> \sup_{w\in\mathbb{D}} |||f_w|||^2_{\mathcal{D}_\nu} \\
&\approx \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} |f_w'(z)|^2 V_\nu(z)\, dA(z) \\
&\approx \sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^2}{V_\mu(w)} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|z|^2)(1-|a|^2)}{|1-\overline{a}z|^2|1-\overline{w}z|^4}\, dA(z)\, d\nu(a).
\end{aligned}$$

Let $\Delta(w, 1/2) = \{z \in \mathbb{D} : |\sigma_w(z)| < 1/2\}$ be a pseudo-hyperbolic disk centered at w. It is well known that

$$1 - |w| \approx |1 - \overline{z}w| \approx 1 - |z|$$

for all $z \in \Delta(w, 1/2)$ and the area of $\Delta(w, 1/2)$ is comparable with $(1 - |w|)^2$. Furthermore, by [33, Lemma 4.30],

$$|1 - \overline{a}z| \approx |1 - \overline{a}w|$$

for all $z \in \Delta(w, 1/2)$ and $a \in \mathbb{D}$. Consequently,

$$\begin{aligned}
\infty &> \sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^2}{V_\mu(w)} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|z|^2)(1-|a|^2)}{|1-\overline{a}z|^2|1-\overline{w}z|^4}\, dA(z)\, d\nu(a) \\
&\gtrsim \sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^2}{V_\mu(w)} \int_{\mathbb{D}} \int_{\Delta(w,1/2)} \frac{(1-|z|^2)(1-|a|^2)}{|1-\overline{a}z|^2|1-\overline{w}z|^4}\, dA(z)\, d\nu(a) \\
&\approx \sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^2}{V_\mu(w)} \int_{\mathbb{D}} \int_{\Delta(w,1/2)} \frac{(1-|w|^2)(1-|a|^2)}{|1-\overline{a}w|^2(1-|w|^2)^4}\, dA(z)\, d\nu(a) \\
&\approx \sup_{w\in\mathbb{D}} \frac{V_\nu(w)}{V_\mu(w)}.
\end{aligned}$$

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Similarly,

$$\sup_{w\in\mathbb{D}} \frac{V_\mu(w)}{V_\nu(w)} < \infty.$$

The proof is complete. □

By Theorem 2.1, if μ is finite, then $\mathcal{D}_\mu \neq \mathcal{D}$. In fact, this is also true for infinite measures μ.

PROPOSITION 3.2. *Let μ be a positive Borel measure on $\mathbb{D}$. Then $\mathcal{D}_\mu \neq \mathcal{D}$.*

PROOF. For $w \in \mathbb{D}$, let f_w be the function appearing in Lemma A. Then

$$\|f_w\|_{\mathcal{D}}^2 = \frac{1}{V_\mu(w)} \int_{\mathbb{D}} |\sigma_w'(z)|^2 \, dA(z) = \frac{2\pi}{V_\mu(w)}.$$

Since $\lim_{r\to 1} V_\mu(r\zeta) = 0$ for almost every $\zeta \in \mathbb{T}$ (cf. [21, page 94]), we know that

$$\sup_{w\in\mathbb{D}} \|f_w\|_{\mathcal{D}} = \infty.$$

Combining this with Lemma A, we get that $\mathcal{D}_\mu \neq \mathcal{D}$. □

An important tool to study function spaces is Carleson measures. Given an arc I on the unit circle $\mathbb{T}$, the Carleson sector $S(I)$ is given by

$$S(I) = \{r\zeta \in \mathbb{D} : 1 - |I| < r < 1, \zeta \in I\},$$

where $|I|$ is the normalized length of the arc I. A positive Borel measure ν on $\mathbb{D}$ is said to be a Carleson measure if

$$\sup_{I\subseteq\mathbb{T}} \frac{\nu(S(I))}{|I|} < \infty.$$

It is said to be a vanishing Carleson measure if

$$\lim_{|I|\to 0} \frac{\nu(S(I))}{|I|} = 0.$$

It is well known (cf. [15, 21]) that ν is a Carleson measure if and only if

$$\sup_{w\in\mathbb{D}} \int_{\mathbb{D}} |\sigma_w'(z)| \, d\nu(z) < \infty.$$

The measure ν is a vanishing Carleson measure if and only if

$$\lim_{|w|\to 1} \int_{\mathbb{D}} |\sigma_w'(z)| \, d\nu(z) = 0.$$

Let $X \subseteq H(\mathbb{D})$ be a Banach function space. We say that X is trivial if X contains only constant functions. The following theorem establishes a link among $\mathcal{D}$, $\mathcal{D}_\mu$ and $M(\mathcal{D}_\mu)$ spaces via Carleson measures.

THEOREM 3.3. *Let μ be a positive Borel measure on $\mathbb{D}$. Then the following conditions are equivalent.*

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(1) $\mathcal{D} \subseteq M(\mathcal{D}_\mu)$.
(2) $\mathcal{D} \subsetneq \mathcal{D}_\mu$.
(3) $M(\mathcal{D}_\mu)$ *is not trivial.*
(4) $(1-|z|^2)\,d\mu(z)$ *is a Carleson measure on* $\mathbb{D}$.

Proof. (1) ⇒ (2). By condition (1), $M(\mathcal{D}_\mu) \subseteq \mathcal{D}_\mu$ and Proposition 3.2, we get $\mathcal{D} \subsetneq \mathcal{D}_\mu$.

(2) ⇒ (3). Suppose that $\mathcal{D} \subseteq \mathcal{D}_\mu$. Since $\mathcal{D}$ is a Möbius invariant function space,

$$\mathcal{D} = M(\mathcal{D}) \subseteq M(\mathcal{D}_\mu).$$

It follows that $M(\mathcal{D}_\mu)$ is not trivial.

(3) ⇒ (4). Since $M(\mathcal{D}_\mu)$ is not trivial, the identity function $z \in M(\mathcal{D}_\mu)$ (see [4]). Hence,

$$\sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4} V_\mu(z)\,dA(z) < \infty.$$

Using arguments similar to those in the proof of Theorem 3.1, we deduce that for all $w \in \mathbb{D}$,

$$\begin{aligned} 1 &\gtrsim \int_{\mathbb{D}} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4} V_\mu(z)\,dA(z) \\ &\gtrsim \int_{\Delta(w,1/2)} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4} \int_{\mathbb{D}} \frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{z}a|^2}\,d\mu(a)\,dA(z) \\ &\approx \int_{\mathbb{D}} (1-|a|^2)\,d\mu(a) \int_{\Delta(w,1/2)} \frac{1}{|1-\overline{w}a|^2(1-|w|^2)}\,dA(z) \\ &\approx \int_{\mathbb{D}} \frac{(1-|a|^2)(1-|w|^2)}{|1-\overline{w}a|^2}\,d\mu(a). \end{aligned}$$

Thus, $(1-|z|^2)\,d\mu(z)$ is a Carleson measure on $\mathbb{D}$.

(4) ⇒ (1). Suppose that $(1-|z|^2)\,d\mu(z)$ is a Carleson measure on $\mathbb{D}$. Then

$$\sup_{w\in\mathbb{D}} V_\mu(w) = \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|w|^2)(1-|z|^2)}{|1-\overline{z}w|^2}\,d\mu(z) < \infty.$$

Therefore,

$$\int_{\mathbb{D}} |f'(z)|^2 V_\mu(z)\,dA(z) \lesssim \int_{\mathbb{D}} |f'(z)|^2\,dA(z)$$

for every $f \in \mathcal{D}$, which implies that $\mathcal{D} \subseteq \mathcal{D}_\mu$. Again, since $\mathcal{D}$ is a Möbius invariant function space,

$$\mathcal{D} = M(\mathcal{D}) \subseteq M(\mathcal{D}_\mu).$$

The proof is complete. □

Let μ be a positive Borel measure on $\mathbb{D}$. By Theorem 3.3 and the proof of Theorem 2.1, if $M(\mathcal{D}_\mu)$ is not trivial, then

$$\mathcal{D} \subseteq M(\mathcal{D}_\mu) \subseteq BMOA.$$

Furthermore, $M(\mathcal{D}_\mu) = BMOA$ if and only if μ is finite. For the strict inclusion relation between $\mathcal{D}$ and $M(\mathcal{D}_\mu)$, we get the following result.

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Theorem 3.4. *Let μ be a positive Borel measure on $\mathbb{D}$. If $(1-|z|^2)\,d\mu(z)$ is a vanishing Carleson measure on $\mathbb{D}$, then*

$$\mathcal{D} \subsetneq M(\mathcal{D}_\mu).$$

Proof. If $(1-|z|^2)\,d\mu(z)$ is a vanishing Carleson measure on $\mathbb{D}$, from Theorem 3.3, one gets that $\mathcal{D} \subseteq M(\mathcal{D}_\mu)$. Now we adapt an argument from [18, page 1243]. Suppose that $\mathcal{D} = M(\mathcal{D}_\mu)$. Denote by $\mathcal{D}^0$ and $M^0(\mathcal{D}_\mu)$ the spaces of functions g with $g(0) = 0$ in $\mathcal{D}$ and $M(\mathcal{D}_\mu)$, respectively. Then $\mathcal{D}^0 = M^0(\mathcal{D}_\mu)$. From the closed graph theorem, there exists a positive constant C such that

$$\int_{\mathbb{D}} |f'(z)|^2\,dA(z) \le C \sup_{a\in\mathbb{D},\lambda\in\mathbb{T}} \int_{\mathbb{D}} |f'(w)|^2 V_\mu(\lambda\sigma_a(w))\,dA(w) \tag{3.2}$$

for all $f \in M^0(\mathcal{D}_\mu)$. Note that $(1-|z|^2)\,d\mu(z)$ is a vanishing Carleson measure on $\mathbb{D}$. Namely,

$$\lim_{|w|\to 1} V_\mu(w) = 0.$$

Then there exists a constant $s \in (0,1)$ such that

$$V_\mu(w) \le \frac{1}{2C}$$

for all $w \in \mathbb{D}$ with $s \le |w| < 1$. Combining this with (3.2),

$$\begin{aligned}\int_{\mathbb{D}} |f'(z)|^2\,dA(z) \le C &\sup_{a\in\mathbb{D},\lambda\in\mathbb{T}} \int_{\Delta(a,s)} |f'(w)|^2 V_\mu(\lambda\sigma_a(w))\,dA(w)\\ &+ C \sup_{a\in\mathbb{D},\lambda\in\mathbb{T}} \int_{\mathbb{D}\setminus\Delta(a,s)} |f'(w)|^2 V_\mu(\lambda\sigma_a(w))\,dA(w)\\ \le C &\sup_{a\in\mathbb{D},\lambda\in\mathbb{T}} \int_{\Delta(a,s)} |f'(w)|^2 V_\mu(\lambda\sigma_a(w))\,dA(w)\\ &+ \frac{1}{2}\int_{\mathbb{D}} |f'(w)|^2\,dA(w),\end{aligned}$$

where

$$\Delta(a,s) = \{w \in \mathbb{D} : |\sigma_a(w)| < s\}.$$

Consequently,

$$\int_{\mathbb{D}} |f'(z)|^2\,dA(z) \le 2C \sup_{a\in\mathbb{D},\lambda\in\mathbb{T}} \int_{\Delta(a,s)} |f'(w)|^2 V_\mu(\lambda\sigma_a(w))\,dA(w)$$

for all $f \in M^0(\mathcal{D}_\mu)$. Since $(1-|z|^2)\,d\mu(z)$ is also a Carleson measure, V_μ is a bounded function. Hence,

$$\int_{\mathbb{D}} |f'(z)|^2\,dA(z) \lesssim \sup_{a\in\mathbb{D}} \int_{\Delta(a,s)} |f'(w)|^2\,dA(w).$$

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From [8, Theorem 1], for any $s \in (0,1)$,

$$\sup_{a\in\mathbb{D}} \int_{\Delta(a,s)} |f'(w)|^2\, dA(w) \approx \|f\|_{\mathcal{B}}^2.$$

Thus,

$$\int_{\mathbb{D}} |f'(z)|^2\, dA(z) \lesssim \|f\|_{\mathcal{B}}^2$$

for all $f \in M^0(\mathcal{D}_\mu)$. Let $h \in \mathcal{B}$ with $h(0) = 0$. For $0 < r < 1$, set $h_r(z) = h(rz)$, $z \in \mathbb{D}$. Clearly, $\|h_r\|_{\mathcal{B}} \le \|h\|_{\mathcal{B}}$. Since $h_r \in M^0(\mathcal{D}_\mu)$,

$$\int_{\mathbb{D}} |rh'(rz)|^2\, dA(z) \lesssim \|h_r\|_{\mathcal{B}}^2 \lesssim \|h\|_{\mathcal{B}}^2.$$

Combining this with the Fatou lemma, one gets that $h \in \mathcal{D}$. Thus, $\mathcal{D} = \mathcal{B}$, which contradicts the fact that $\mathcal{D} \subsetneq \mathcal{B}$. Thus, $\mathcal{D} \subsetneq M(\mathcal{D}_\mu)$. □

Note that $M(\mathcal{D}_\mu)$ spaces are always subsets of $BMOA$. Checking the proof of the above theorem, we can get the following result. Let μ and ν be positive Borel measures on $\mathbb{D}$. If

$$\lim_{|w|\to 1} \frac{V_\mu(w)}{V_\nu(w)} = 0,$$

then $M(\mathcal{D}_\nu) \subsetneq M(\mathcal{D}_\mu)$. We leave the details to the interested reader.

4. Inner functions in $M(\mathcal{D}_\mu)$ spaces

A bounded analytic function I on $\mathbb{D}$ is called an inner function if $|I(\zeta)| = 1$ for almost every $\zeta \in \mathbb{T}$. A sequence $\{a_k\}_{k=1}^{\infty} \subseteq \mathbb{D}$ is said to be a Blaschke sequence if

$$\sum_{k=1}^{\infty} (1 - |a_k|) < \infty.$$

The above condition implies the convergence of the corresponding Blaschke product B, defined as

$$B(z) = \prod_{k=1}^{\infty} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a_k} z}.$$

It is well known (cf. [15]) that any inner function I can be represented as a product of a constant $\gamma \in \mathbb{T}$, a Blaschke product and a singular inner function

$$S_\nu(z) = \exp\bigg(\int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta}\, d\nu(\zeta)\bigg),$$

where ν is a positive singular Borel measure on $\mathbb{T}$.

We will need some definitions concerning an important class of sequences and Blaschke products. A sequence $\{a_k\}_{k=1}^{\infty} \subseteq \mathbb{D}$ is called an interpolating sequence if there exists a positive constant δ such that

$$\inf_k \prod_{j\neq k} \varrho(a_j, a_k) \ge \delta.$$

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Here $\varrho(a_j, a_k) = |\sigma_{a_j}(a_k)|$ denotes the pseudo-hyperbolic metric in $\mathbb{D}$. The Blaschke product corresponding to an interpolating sequence is called an interpolating Blaschke product. A Blaschke product is called a Carleson–Newman Blaschke product if it can be expressed as a product of finitely many interpolating Blaschke products. It is well known (cf. [24]) that a Blaschke product corresponding to a sequence $\{a_k\}_{k=1}^{\infty}$ is a Carleson–Newman Blaschke product if and only if $\sum_{k=1}^{\infty}(1-|a_k|^2)\delta_{a_k}$ is a Carleson measure. Here δ_{a_k} is the unit point mass measure at $a_k \in \mathbb{D}$. The relation between interpolating sequences and Carleson measures originally comes from Carleson's famous works [13, 14] studying the interpolation problem and the corona theorem for H^∞. We refer the reader to [12, 16, 23–25] for more information about Carleson–Newman Blaschke products.

In this section, we will consider the problem of characterizing when a given inner function is contained in a given Möbius invariant function space $M(\mathcal{D}_\mu)$. It follows from Theorem 2.1 that, if μ is a finite positive Borel measure on $\mathbb{D}$, then the set of all inner functions is contained in $M(\mathcal{D}_\mu)$. From now on we will focus our study on the spaces $M(\mathcal{D}_\mu)$ corresponding to infinite measures μ on $\mathbb{D}$. Let μ be an infinite positive Borel measure on $\mathbb{D}$. In Theorem 4.4, we will show that, in that case, $M(\mathcal{D}_\mu)$ does not contain singular inner functions and we will characterize the set of Carleson–Newman Blaschke products contained in $M(\mathcal{D}_\mu)$. Let CNM denote the set of Möbius invariant function spaces X satisfying the following property:

if B is a Blaschke product belonging to X,
then B is a Carleson–Newman Blaschke product.

Some examples of spaces contained in CNM are (cf. [19, 20, 32]) the well-known Q_p spaces for $0 < p < 1$, some Q_K spaces and the analytic version of $Q_1(\mathbb{T})$ space. In Corollary 4.5, we give a complete characterization of the inner functions in the spaces $M(\mathcal{D}_\mu) \in CNM$.

Lemma 4.1. *Let μ be an infinite positive Borel measure on $\mathbb{D}$ and let I be an inner function. Then $I \in M(\mathcal{D}_\mu)$ if and only if*

$$\sup_{\phi\in \mathrm{Aut}(\mathbb{D})} \int_{\mathbb{D}} (1-|I\circ\phi(w)|^2)\, d\mu(w) < \infty.$$

Proof. It is well known that (cf. [5, pages 105–106]) for $f \in H^2$ and $w \in \mathbb{D}$,

$$\frac{2}{\pi}\int_{\mathbb{D}} \log\left|\frac{1-\overline{w}z}{z-w}\right| |f'(z)|^2\, dA(z) = \frac{1}{2\pi}\int_{\mathbb{T}} |f(\zeta)|^2 \frac{1-|w|^2}{|\zeta-w|^2}|d\zeta| - |f(w)|^2. \tag{4.1}$$

By the above formula and the Fubini theorem, we see that $I \in M(\mathcal{D}_\mu)$ if and only if

$$\sup_{\phi\in \mathrm{Aut}(\mathbb{D})} \int_{\mathbb{D}} (1-|I\circ\phi(w)|^2)\, d\mu(w) < \infty. \qquad \square$$

Lemma 4.2. *Let μ be an infinite positive Borel measure on $\mathbb{D}$. Let $I = \prod_{j=1}^{n} I_j$, where all I_j are inner functions. Then $I \in M(\mathcal{D}_\mu)$ if and only if $I_j \in M(\mathcal{D}_\mu)$ for $j = 1, 2, \ldots, n$.*

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Proof. Since $|I(z)| < 1$ for $z \in \mathbb{D}$, the conclusion follows from Lemma 4.1 and the inequalities

$$1 - |I_j(z)|^2 \le 1 - |I(z)|^2 \le \sum_{k=1}^{n} (1 - |I_k(z)|^2), \quad z \in \mathbb{D}, \ j = 1, 2, \dots, n. \qquad \square$$

Lemma 4.3. *Let μ be an infinite positive Borel measure on $\mathbb{D}$. Let ν be a positive singular Borel measure on $\mathbb{T}$. Then $S_\nu \notin M(\mathcal{D}_\mu)$.*

Proof. We will consider three cases.

(i) Suppose that $\nu = t\delta_1$, $t > 0$. Then $S_\nu(z) = \exp(-t(1+z)/(1-z))$ and $|S_\nu(z)| = \exp(-t(1-|z|^2)/(|1-z|^2))$, $z \in \mathbb{D}$. Fix $c > 0$. We denote by D_c the horodisk

$$D_c = \left\{ z \in \mathbb{D} : \frac{1 - |z|^2}{|1 - z|^2} > c \right\},$$

which is a disk tangent to the unit circle at 1 (see, for example, [21, page 73]). Note that $|S_\nu| \le e^{-tc}$ on D_c. For every $a \in \mathbb{D}$, let $\mu_a = \mu \circ \sigma_a$. Then, by formula (4.1) and the Fubini theorem, it is easy to see that for every $a \in \mathbb{D}$,

$$\begin{aligned} \|S_\nu \circ \sigma_a\|^2_{\mathcal{D}_\mu} \approx \int_{\mathbb{D}} |(S_\nu \circ \sigma_a)'(z)|^2 U_\mu(z)\, dA(z) &\approx \int_{\mathbb{D}} (1 - |S_\nu(\sigma_a(z))|^2)\, d\mu(z) \\ &\gtrsim \int_{\sigma_a(D_c)} (1 - |S_\nu(\sigma_a(z))|^2)\, d\mu(z) \\ &\approx \int_{D_c} (1 - |S_\nu(z)|^2)\, d\mu_a(z) \\ &\gtrsim (1 - e^{-2tc})\mu(\sigma_a(D_c)). \end{aligned} \tag{4.2}$$

For every $r \in (-1, 1)$, let $\phi_r(z) = -\sigma_r(z)$, $z \in \mathbb{D}$. Note that, if s is the point where ∂D_c intersects the interval $(-1, 1)$, then ϕ_r maps D_c to the disk having diameter the interval $(-(r-s)/(1-rs), 1)$; in particular, $\phi_r(D_c) \nearrow \mathbb{D}$ as $r \to 1$. Therefore, from the inequality (4.2),

$$\lim_{r\to 1} \|S_\nu \circ \phi_r\|^2_{\mathcal{D}_\mu} \gtrsim \lim_{r\to 1} (1 - e^{-2tc})\mu(\phi_r(D_c)) \approx (1 - e^{-2tc})\mu(\mathbb{D}) = +\infty$$

and $S_\nu \notin M(\mathcal{D}_\mu)$. Similarly, we show that $S_{t\delta_\zeta} \notin M(\mathcal{D}_\mu)$ for every $\zeta \in \mathbb{T}$.

(ii) Suppose that ν has an atom at the point $\zeta \in \mathbb{T}$ and let $t = \nu(\{\zeta\}) > 0$. Then $|S_\nu| \le |S_{t\delta_\zeta}|$ on $\mathbb{D}$. Since $S_{t\delta_\zeta} \notin M(\mathcal{D}_\mu)$,

$$\begin{aligned} \sup_{a\in\mathbb{D}} \|S_\nu \circ \sigma_a\|^2_{\mathcal{D}_\mu} &\approx \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (1 - |S_\nu(\sigma_a(z))|^2)\, d\mu(z) \\ &\gtrsim \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (1 - |S_{t\delta_\zeta}(\sigma_a(z))|^2)\, d\mu(z) \\ &= +\infty. \end{aligned}$$

Therefore, $S_\nu \notin M(\mathcal{D}_\mu)$.

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(iii) Suppose that ν has no atoms. Note that, by assumption, $\mu(\mathbb{D}) = \infty$. We will show that there exists $\xi_0 \in \mathbb{T}$ such that

$$\mu(D(\xi_0, \delta) \cap \mathbb{D}) = \infty \tag{4.3}$$

for every $\delta > 0$. Here $D(\xi_0, \delta)$ is the Euclidean disk with center ξ_0 and radius δ. Otherwise, by the compactness of $\mathbb{T}$, there would exist $\zeta_1, \ldots, \zeta_n \in \mathbb{T}$ and $\delta_0 > 0$ such that $\mu(D(\zeta_i, \delta_0) \cap \mathbb{D}) < \infty$, $i = 1, \ldots, n$, and $\mathbb{T} \subset \bigcup_{i=1}^n D(\zeta_i, \delta_0)$. Let $r > 0$ be such that $\mathbb{D} \setminus r\mathbb{D} \subseteq \bigcup_{i=1}^n D(\zeta_i, \delta_0)$. Then $\mu(\mathbb{D} \setminus r\mathbb{D}) < \infty$. Since $\mu(\overline{r\mathbb{D}}) < \infty$, we get $\mu(\mathbb{D}) < \infty$. This is a contradiction.

Since ν has no atoms, $\mathrm{supp}(\nu) \setminus \{\xi_0\} \neq \emptyset$. Let $\xi_1 \in \mathrm{supp}(\nu) \setminus \{\xi_0\}$. We can assume that $\xi_1 = 1$, since, otherwise, we can compose μ and ν with the rotation $z \mapsto z/\xi_1$. Fix $c > 0$ and let D_c and ϕ_r, $r \in (0, 1)$, be as above. Note that for every $\zeta \in \mathbb{T}$, there exists a unique $\eta \in \partial D_c$ such that $\lim_{r\to1} \phi_r(\eta) = \zeta$. Let η_0 be the point in ∂D_c such that $\lim_{r\to1} \phi_r(\eta_0) = \xi_0$ and let $\epsilon = |1 - \eta_0|/2$. Then there exists $\delta_0 > 0$ such that $D(\xi_0, \delta_0) \cap \phi_r(D_c \setminus D(1, \epsilon))$, $r \in (0, 1)$, is an increasing family of sets and

$$D(\xi_0, \delta_0) \subset \bigcup_{r\in(0,1)} \phi_r(D_c \setminus D(1, \epsilon)). \tag{4.4}$$

From (4.3) and (4.4),

$$\lim_{r\to1} \mu(\phi_r(D_c \setminus D(1, \epsilon))) = +\infty. \tag{4.5}$$

Let $I_\epsilon = \{\zeta \in \mathbb{T} : |1 - \zeta| < \epsilon\}$. Since $1 \in \mathrm{supp}(\nu)$, $\nu(I_\epsilon) > 0$. Then, for every $z \in D_c \setminus D(1, \epsilon)$ and for every $\zeta \in I_\epsilon$, we have $|1 - \zeta| < \epsilon \le |1 - z|$, so

$$|\zeta - z| \le |1 - \zeta| + |1 - z| \le 2|1 - z|$$

and

$$\frac{1 - |z|^2}{|\zeta - z|^2} \ge \frac{1 - |z|^2}{4|1 - z|^2} \ge \frac{c}{4}. \tag{4.6}$$

From the inequality (4.6), we obtain that for every $z \in D_c \setminus D(1, \epsilon)$,

$$-\log|S_\nu(z)| = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2}\, d\nu(\zeta) \ge \int_{I_\epsilon} \frac{1 - |z|^2}{|\zeta - z|^2}\, d\nu(\zeta) \ge \frac{c}{4}\nu(I_\epsilon) > 0. \tag{4.7}$$

Therefore, from (4.5) and (4.7),

$$\begin{aligned}
\lim_{r\to1} \|S_\nu \circ \phi_r\|^2_{\mathcal{D}_\mu} &\approx \lim_{r\to1} \int_{\mathbb{D}} (1 - |S_\nu(\phi_r(z))|^2)\, d\mu(z) \\
&\gtrsim \lim_{r\to1} \int_{\phi_r(D_c\setminus D(1,\epsilon))} (1 - |S_\nu(\phi_r(z))|^2)\, d\mu(z) \\
&\approx \lim_{r\to1} \int_{D_c\setminus D(1,\epsilon)} (1 - |S_\nu(z)|^2)\, d(\mu \circ \phi_r)(z) \\
&\gtrsim (1 - e^{-c\nu(I_\epsilon)/4}) \lim_{r\to1} \mu(\phi_r(D_c \setminus D(1, \epsilon))) \\
&= +\infty,
\end{aligned}$$

and hence $S_\nu \notin M(\mathcal{D}_\mu)$. □

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Now we state the main result of this section, as follows.

THEOREM 4.4. *Let μ be an infinite positive Borel measure on $\mathbb{D}$. Then the following statements are true.*

(1) *If an inner function I belongs to $M(\mathcal{D}_\mu)$, then I must be a Blaschke product.*
(2) *Let B be a Carleson–Newman Blaschke product with zeros $\{a_k\}_{k=1}^{\infty}$. Then $B \in M(\mathcal{D}_\mu)$ if and only if*

$$\sup_{\phi\in \mathrm{Aut}(\mathbb{D})} \sum_{k=1}^{\infty} \int_{\mathbb{D}} (1 - |\sigma_{a_k}(\phi(w))|^2)\, d\mu(w) < \infty. \tag{4.8}$$

PROOF. (1) Let I be an inner function belonging to $M(\mathcal{D}_\mu)$. Note that I can be represented as a product of a constant $\gamma \in \mathbb{T}$, a Blaschke product and a singular inner function. Applying Lemmas 4.2 and 4.3, we obtain that I must be a Blaschke product.

(2) First, we assume that condition (4.8) holds. From the following elementary inequality:

$$1 - \prod_{k=1}^{\infty} x_k \le \sum_{k=1}^{\infty} (1 - x_k), \quad x_k \in (0, 1],$$

one gets here

$$1 - |B(z)| \le \sum_{k=1}^{\infty} (1 - |\sigma_{a_k}(z)|^2), \quad z \in \mathbb{D}.$$

Consequently,

$$1 - |B(\phi(z))| \le \sum_{k=1}^{\infty} (1 - |\sigma_{a_k}(\phi(z))|^2)$$

for any $\phi \in \mathrm{Aut}(\mathbb{D})$ and $z \in \mathbb{D}$. Combining this with the Fubini theorem,

$$\sup_{\phi\in \mathrm{Aut}(\mathbb{D})} \int_{\mathbb{D}} (1 - |B(\phi(z))|)\, d\mu(z) \le \sup_{\phi\in \mathrm{Aut}(\mathbb{D})} \sum_{k=1}^{\infty} \int_{\mathbb{D}} (1 - |\sigma_{a_k}(\phi(z))|^2)\, d\mu(z).$$

By the above inequality, condition (4.8) and Lemma 4.1, we get $B \in M(\mathcal{D}_\mu)$.

On the other hand, let $B \in M(\mathcal{D}_\mu)$. Then

$$\begin{aligned} \log |B(z)|^2 &= \sum_{k=1}^{\infty} \log\Big(1 - \frac{(1 - |a_k|^2)(1 - |z|^2)}{|1 - \overline{a_k} z|^2}\Big) \\ &\le -\sum_{k=1}^{\infty} \frac{(1 - |a_k|^2)(1 - |z|^2)}{|1 - \overline{a_k} z|^2} \\ &= -\sum_{k=1}^{\infty} (1 - |\sigma_{a_k}(z)|^2) \end{aligned}$$

for any $z \in \mathbb{D}$. Consequently,

$$1 - |B(z)|^2 \ge 1 - \exp\Big(-\sum_{k=1}^{\infty} (1 - |\sigma_{a_k}(z)|^2)\Big), \quad z \in \mathbb{D}.$$

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Note that B is a Carleson–Newman Blaschke product. By [24], $\sum_{k=1}^{\infty}(1-|a_k|^2)\delta_{a_k}$ is a Carleson measure. Namely,

$$M =: \sup_{z\in\mathbb{D}} \sum_k (1-|\sigma_{a_k}(z)|^2) < \infty.$$

Bear in mind that

$$\frac{1-e^{-t}}{t} \approx 1, \quad 0<t<M.$$

Therefore,

$$1-|B(z)|^2 \gtrsim \sum_{k=1}^{\infty}(1-|\sigma_{a_k}(z)|^2)$$

for all $z\in\mathbb{D}$. Combining this with Lemma 4.1,

$$\sup_{\phi\in\mathrm{Aut}(\mathbb{D})} \sum_{k=1}^{\infty} \int_{\mathbb{D}} (1-|\sigma_{a_k}(\phi(w))|^2)\,d\mu(w) \lesssim \sup_{\phi\in\mathrm{Aut}(\mathbb{D})} \int_{\mathbb{D}} (1-|B\circ\phi(w)|^2)\,d\mu(w) < \infty.$$

The proof is complete. □

The following result is a direct consequence of Theorem 4.4.

Corollary 4.5. *Let I be an inner function and let μ be an infinite positive Borel measure on $\mathbb{D}$ such that $M(\mathcal{D}_\mu)\in CNM$. Then $I\in M(\mathcal{D}_\mu)$ if and only if I is a Carleson–Newman Blaschke product with zeros $\{a_k\}_{k=1}^{\infty}$ satisfying*

$$\sup_{\phi\in\mathrm{Aut}(\mathbb{D})} \sum_{k=1}^{\infty} \int_{\mathbb{D}} (1-|\sigma_{a_k}(\phi(w))|^2)\,d\mu(w) < \infty.$$

Clearly, the condition of μ in Theorem 4.4 is best possible. Applying Theorem 4.4, we can characterize inner functions in some function spaces. For example, let $K_1(t) = t(\log e^2/t)^2$, $0<t<1$. By [18, Theorem 2.6], Q_{K_1} is located strictly between $\bigcup_{0<p<1} Q_p$ and the space of the analytic version of $Q_1(\mathbb{T})$. The characterization of inner functions in Q_{K_1} was not studied in previous papers. Using Theorem 4.4, we obtain a complete characterization of inner functions in Q_{K_1} as follows.

Corollary 4.6. *Let I be an inner function. Then $I\in Q_{K_1}$ if and only if I is a Blaschke product with zeros $\{a_k\}_{k=1}^{\infty}$ satisfying*

$$\sup_{a\in\mathbb{D}} \sum_{k=1}^{\infty} \int_{\mathbb{D}} (1-|\sigma_{a_k}(\sigma_a(z))|^2)[K_1'(1-|z|)-|z|K_1''(1-|z|)]\,dA(z) < \infty.$$

Proof. It is easy to check that K_1 is increasing and concave on $(0,1)$ with $\lim_{t\to 0^+} K(t)=0$. Thus, $Q_{K_1}=M(\mathcal{D}_{\mu_1})$, where $d\mu_1(w) = -\Delta(K_1(1-|w|))\,dA(w)$, $w\in\mathbb{D}$. By [18, Theorem 2.6], $Q_{K_1}\subsetneqq BMOA$; hence, μ_1 is an infinite measure, a fact which can also be proved via a direct computation. Clearly, Q_{K_1} is a subset of the space of the analytic version of $Q_1(\mathbb{T})$. By [32, page 1100], the space of the analytic version of $Q_1(\mathbb{T})$

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belongs to CNM; therefore, $\mathcal{Q}_{K_1} \in CNM$. This, together with Corollary 4.5, yields that an inner function I belongs to $\mathcal{Q}_{K_1}$ if and only if I is a Blaschke product with zeros $\{a_k\}_{k=1}^{\infty}$ satisfying

$$\sup_{\phi \in \mathrm{Aut}(\mathbb{D})} \sum_{k=1}^{\infty} \int_{\mathbb{D}} (1 - |\sigma_{a_k}(\phi(w))|^2)\, d\mu_1(w) < \infty. \tag{4.9}$$

Note that $K_1(1-|z|)$ is a radial function. By the change of variables, we compute the above integral as follows.

$$\begin{aligned}
&\sup_{\phi \in \mathrm{Aut}(\mathbb{D})} \sum_{k=1}^{\infty} \int_{\mathbb{D}} (1 - |\sigma_{a_k}(\phi(w))|^2)\, d\mu_1(w) \\
&\quad \approx \sup_{\phi \in \mathrm{Aut}(\mathbb{D})} \sum_{k=1}^{\infty} \int_0^1 \int_0^{2\pi} (1 - |\sigma_{a_k}(\phi(re^{i\theta}))|^2)[K_1'(1-r) - rK_1''(1-r)]\, d\theta\, dr \\
&\quad \approx \sup_{\phi \in \mathrm{Aut}(\mathbb{D})} \sum_{k=1}^{\infty} \int_{\mathbb{D}} (1 - |\sigma_{a_k}(z)|^2)[K_1'(1 - |\phi^{-1}(z)|) - |\phi^{-1}(z)| K_1''(1 - |\phi^{-1}(z)|)] \\
&\qquad \times |(\phi^{-1})'(z)|^2\, dA(z) \\
&\quad \approx \sup_{a \in \mathbb{D}} \sum_{k=1}^{\infty} \int_{\mathbb{D}} (1 - |\sigma_{a_k}(\sigma_a(z))|^2)[K_1'(1-|z|) - |z| K_1''(1-|z|)]\, dA(z).
\end{aligned}$$

Combining the above computation with condition (4.9), we get the desired result. □

Finally, we pose two natural questions as follows. Is it true that $M(\mathcal{D}_\mu) \in CNM$ for every infinite positive Borel measure μ on $\mathbb{D}$? If the answer is negative, how can we characterize the measures μ such that $M(\mathcal{D}_\mu) \in CNM$?

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