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RESTRICTING REPRESENTATIONS OF COMPLETELY SOLVABLE LIE GROUPS

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1. Introduction. We are concerned here with the problem of describing the direct integral decomposition of a unitary representation obtained by restriction from a larger group. This is the dual problem to the more commonly investigated problem of decomposing induced representations. In this paper we work in the context of completely solvable Lie groups—more general than nilpotent, but less general than exponential solvable. Moreover, the groups involved are simply connected. The restriction problem was considered originally in [**2**] and in [**6**] for nilpotent groups. A complete solution was obtained explicitly in terms of the Kirillov orbital parameters (see [**2**, Theorems 4.6 and 4.8] and [**6**, Theorem 4.2]). It is pointed out in [**6**] that it is highly reasonable to expect the Kirillov-Bernat orbital parameters to describe the direct integral decomposition for both induced and restricted representations in the more general context of exponential solvable groups. Such a program is carried out for induced representations in [**7**] (the algebraic or symmetric space cases) and [**8**] (the completely solvable case). Very recently, Fujiwara sent me [**3**] which deals with induced (monomial) representations for arbitrary exponential solvable groups. I now turn my attention to restricted representations.

The basic techniques of the paper are drawn from [**6**] and [**8**]. We start with G simply connected and completely solvable together with an irreducible unitary representaion π of G. We take $H \subset G$ a closed connected subgroup. Our goal is to describe the direct integral decomposition of $\pi|_H$ in terms of the orbital parameters of $\hat{H}$. Exactly as in [**6**] or [**8**], we employ mathematical induction on $\dim G/H$. The argument starts with the case $\dim G/H = 1$. If H is normal—mandatory when G is nilpotent—the argument replicates that of [**6**, Theorem 4.2]. If H is not normal, we draw upon the structure theory developed in [**8**]. Although the invariants governing the situation differ from those in [**8**], it turns out that the number of structural possibilities for a non-normal codimension one restriction amount to—as in the induced representation case—exactly five. These possibilities are enumerated in Theorem 3.2.

We then use the fact that in between a completely solvable group G and a connected subgroup H one can always insert a codimension 1 subgroup G_1 of G. The codimension of H in G_1 is one less than $\dim G/H$. One then employs restriction

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in stages, mathematical induction and the codimension 1 results to obtain a decomposition of $\pi|_H$. The proof that the resulting decomposition agrees with the desired orbital decompostiion takes up most of section four. Basically we are required to demonstrate equality of spectrum, measure and multiplicity in two direct integral decompositions (see (4.2)). Our task is complicated by the fact that there is no analog to [**8**, §4]—that is, to the intermediate monomial case. (Every irreducible representation of a completely solvable group is induced from a character, and so the induced representation argument reduces to that of monomial representations.) Nevertheless, we are able to obtain our main theorem (Theorem 4.1)—a complete description of the direct integral decomposition of an arbitrary restricted representation for completely solvable groups in terms of orbital parameters. As in [**8**], we must do a careful study of the generic dimensions of the varieties which are intersections of G-orbits in $\mathfrak{g}^*$ with the pullback of H-orbits in $\mathfrak{h}^*$ to $\mathfrak{g}^*$ (see Lemmas 4.3 and 4.4).

The other three (short) sections of the paper contain material as follows. Section two contains a precise formulation of the orbital decomposition of a restricted representation (Definition 2.1 and Formula (2.2)). A fundamental lemma in the subject (Lemma 2.4) is recalled, and some notation is established. In section five we relate the results of this paper to those of [**8**]. We indicate what happens when the two operations (induction or restriction) are applied in succession (in either order), and we use that knowledge to get an intrinsic characterization of the five structural possibilities outlined in Theorem 3.2. Finally, section six contains several examples to illustrate the main features of the theorems in sections three and four.

2. Statement of the Main Result. We recall the Kirillov-Bernat orbital parameters (see [**1**], [**4**]). Suppose G is an *exponential solvable* group. That means G is simply connected solvable and its Lie algebra $\mathfrak{g}$ has no purely imaginary eigenvalues. G is called *completely solvable* if it is exponential solvable and every eigenvalue of $\mathfrak{g}$ is real. The symbol $\mathfrak{g}^*$ denotes the real linear dual of $\mathfrak{g}$. G acts on $\mathfrak{g}$ (respectively $\mathfrak{g}^*$) by the adjoint (respectively co-adjoint) action. Then the dual space $\hat{G}$ of equivalence classes of irreducible unitary representations of G is parameterized canonically by the orbit space $\mathfrak{g}^*/G$. More precisely, for $\varphi \in \mathfrak{g}^*$ we may find a real polarization $\mathfrak{b}$ for φ—that is, a subalgebra, $\mathfrak{g}_\varphi \subset \mathfrak{b} \subset \mathfrak{g}$, which is maximal totally istropic for $B_\varphi(X,Y) = \varphi[X,Y]$—that satisfies the Pukanszky condition $B\cdot\varphi = \varphi + \mathfrak{b}^\perp$ $B = \exp\mathfrak{b}$). Then the representation $\pi_\varphi = \mathrm{Ind}_B^G \chi_\varphi$, $\chi_\varphi(\exp X) = e^{i\varphi(X)}, X \in \mathfrak{b}$ is irreducible; its class is independent of the choice of $\mathfrak{b}$; the Kirillov map $\varphi \longrightarrow \pi_\varphi$, $\mathfrak{g}^* \longrightarrow \hat{G}$ is surjective and factors to a bijection $\mathfrak{g}^*/G \longrightarrow \hat{G}$. Given $\pi \in \hat{G}$, we write $\Omega_\pi \in \mathfrak{g}^*/G$ to denote the inverse image of π under the Kirillov map.

All of the preceding is valid for any exponential solvable group, but we shall only deal with completely solvable groups in this paper. Now suppose G is completely solvable, $H \subset G$ is a (closed) connected subgroup. We adopt the terminology of [**6**, Definition 2.1] or [**8**, Definition 2.1] (see also **6**, Theorem 4.2]).

DEFINITION 2.1. For $\pi \in \hat{G}$ we say that the restriction $\pi|_H$ obeys the *orbital spectrum formula* if

$$(2.1) \qquad \pi|_H = \int_{p(\Omega_\pi)/H}^{\oplus} n_\pi^\psi \ \nu_\psi \ d\lambda_{G,H}^\pi (\psi),$$

where $p: \mathfrak{g}^* \longrightarrow \mathfrak{h}^*$ is the canonical projection, $\lambda_{G,H}^\pi$ is the push-forward of the canonical invariant measure (class) on Ω_π (under $\mathfrak{g}^* \longrightarrow \mathfrak{h}^*/H$), and

$$n_\pi^\psi = \#H\text{-orbits on } \Omega_\pi \cap p^{-1}(H \cdot \psi).$$

The main result of the paper is the following. [1]

THEOREM 2.2. *Let G be completely solvable, $H \subset G$ closed and connected, $\pi \in \hat{G}$. Then the restricted representation $\pi|_H$ satisfies the orbital spectrum formula.*

Both Definition 2.1 and Theorem 2.2 are the precise analog for restrictions of the correspondingly numbered results in [**8**] for induced representations.

Now any exponential solvable group—in particular any completely solvable group—is type I [**10**]. Therefore the unitary representation $\pi|_H$ has a direct integral decomposition

$$\pi|_H = \int_{\mathfrak{S}_\pi}^{\oplus} n_\pi(\nu)\nu \ d\lambda_\pi(\nu),$$

where the measure class $[\lambda_\pi]$ is uniquely determined; the multiplicity function $n_\pi(\nu)$ is uniquely determined ($[\lambda_\pi]$-a.e.); and the spectrum $\mathfrak{S}_\pi$—meaning any subset of $\hat{G}$ in which λ_π is concentrated—is also determined ($[\lambda_\pi]$-a.e.). To prove Theorem 2.2 we must verify that the triple $(\lambda_{G,H}^\pi, \ n_\pi^\nu, \ p(\Omega_\pi)/H)$ constitutes these ingredients for the restricted representation $\pi|_H$. As in the case of induced representations [**8**], the scheme of the proof is modelled after [**6**]. Namely, the argument is by induction on $\dim G/H$. In the case $\dim G/H = 1$, matters are complicated by the fact that—unlike nilpotent groups—codimension 1 subgroups need not be normal. As with induced representations, there are five distinct structural

[1] The referee has suggested that I mention that much less precise statements of Theorem 2.2 were given by I. K. Busyatskaya (Func. Anal. & Appl. **7** (1973), 79–80) and I. M. Shchepochkina (ibid. **11** (1977), 93–94). Neither of these theses was ever published.

possibilities for the restricted representation when $\dim G/H = 1$ and G is completely solvable. These cases are examined in detail in the next section. To handle $\dim G/H > 1$, we place between H and G a connected subgroup G_1 of codimension 1 in G (always possible when G is completely solvable). We then employ restriction in stages $\pi|_H = (\pi|_{G_1})|_H$. The first restriction obeys the orbital spectrum formula by the codimension 1 case; the second obeys it because of the induction assumption. In section 4 we show how to combine these facts to obtain the orbital spectrum formula for $\pi|_H$. We note that there is no analog here of the monomial step in the induced representation argument (see [**8**, Sections 4, 5]).

We close this section by citing two known results and establishing some notation.

THEOREM 2.3. *Theorem 2.2 is true if H is normal.*

This is proven in [**6**, Theorem 6.2].

LEMMA 2.4. *Let $N \subset G$ be normal and connected, $\varphi \in \mathfrak{g}^*$, $\theta = \varphi|_{\mathfrak{n}} \in \mathfrak{n}^*$, $\gamma = \gamma_\theta \in \hat{N}$. The Lie algebra of the stability group G_γ is $\mathfrak{g}_\gamma = \mathfrak{g}_\theta + \mathfrak{n}$. Then*

$$N_\theta \cdot \varphi = \varphi + \mathfrak{g}_\gamma^\perp.$$

See [**9**, Lemma 2] or [**5**, p. 271].

NOTATION. Whenever $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ we write $p_{\mathfrak{g},\mathfrak{h}}\colon \mathfrak{g}^* \longrightarrow \mathfrak{h}^*$ for the canonical projection $p_{\mathfrak{g},\mathfrak{h}}(\varphi) = \varphi|_{\mathfrak{h}}$, $\varphi \in \mathfrak{g}^*$. If the algebras are clear from the context we drop the subscripts. We denote

$$\mathfrak{h} = \mathfrak{h}^\perp(\mathfrak{g}) = p_{\mathfrak{g},\mathfrak{h}}^{-1}(\{0\}) \subset \mathfrak{g}^*.$$

By a generic subset of $\mathfrak{g}^*$ we mean a subset, the complement of whose interior is Lebesgue null. More generally for any manifold W, we say a statement P_w, $w \in W$, is true generically if it holds for all points of W except for a set whose interior is co-null with respect to the canonical measure class.

3. Codimension One. In this section we present a detailed and complete description of the decomposition of the restriction, to a codimension 1 connected subgroup G_1, of a representation of a completely solvable Lie group G. We give the orbital paramenters of the decomposition as well as related information on various stabilitzers and orbit correspondence. We also give Mackey parameters for G_1, when it is normal, and for the canonical codimension 2 subgroup (see [**8**, Proposition 3.2] and below) when it is not. As with induced representations, we relate the Mackey and orbital parameters.

We start with G completely solvable, $N \subset G$ a codimension 1 closed connected and *normal* subgroup. Let $\varphi \in \mathfrak{g}^*$, $\pi = \pi_\varphi \in \hat{G}$ the corresponding Kirillov-Bernat irreducible unitary representation. The analysis of $\pi|_N$ is known in great detail (see [**4**] or [**6**]). We summarize in

THEOREM 3.1. *Let* $\varphi \in \mathfrak{g}^*$, $\theta = \varphi|_{\mathfrak{n}}$, $\gamma = \gamma_\theta \in \hat{N}$ *the corresponding Kirillov-Bernat representation. Select* $\alpha \in \mathfrak{n}^\perp$, $\alpha \neq 0$, *so that* $p^{-1}(\theta) = \{\varphi + t\alpha : t \in \Re\}$. *Also select* $X \in \mathfrak{g}$, $\alpha(X) = 1$. *Then there are two mutually exclusive possibilities:*

(i) $G \cdot \varphi \supset p^{-1}(\theta)$. *Then* $G_\theta = N_\theta$, $G_\varphi = N_\varphi$, $G_\gamma = N$, $N_\theta \cdot \varphi = \varphi + \mathfrak{n}^\perp$ *and*

$$\pi_\varphi|_N = \int_\Re^\oplus \gamma_{\theta_s}\, ds,$$

where $\theta_s = \exp sX \cdot \theta$. *The N-orbits* $N \cdot \theta_s$, $s \in \Re$, *are all distinct,* $p(G \cdot \varphi) = \bigcup_{s \in \Re} N \cdot \theta_s$, *and* $\dim G \cdot \varphi = \dim N \cdot \theta_s + 2$, $s \in \Re$.

(ii) The orbits $\{G \cdot (\varphi + t\alpha) : t \in \Re\}$ *are all distinct. Then* $N_\theta = N_\varphi$, $G_\theta = G_\varphi$, $G_\gamma = G$ *and*

$$\pi_\varphi|_N = \gamma_\theta.$$

Moreover, $p(G \cdot \varphi) = N \cdot \theta$ *and* $\dim G \cdot \varphi = \dim N \cdot \theta$.

Combining all the information in Theorem 3.1, we see that in either case, the orbital spectrum formula

$$\pi_\varphi|_N = \int_{p(G \cdot \varphi)/N}^\oplus n_\pi^\theta \gamma_\theta \; d\lambda_{G,N}^\pi(\theta)$$

is valid. Indeed, in case (i) we have

$$G \cdot \varphi \cap p^{-1}(N \cdot \theta_s) = G \cdot \varphi_s \cap p^{-1}(N \cdot \theta_s) = N \cdot \varphi_s, \qquad \varphi_s = \exp sX \cdot \varphi, \quad s \in \Re,$$

and (since $\mathfrak{g}_\varphi \subset \mathfrak{n}$) the push-forward of the invariant measure on $G \cdot \varphi = \bigcup_s N \cdot \varphi_s$ gives the Lebesgue class in s; whereas in case (ii) we have

$$G \cdot \varphi \cap p^{-1}(N \cdot \theta) = N \cdot \varphi.$$

Now we pass to the non-normal codimension 1 situation. We assume G is completely solvable with $G_1 \subset G$ a closed connected codimension 1 subgroup. We assume G_1 is *not* normal in G. We utilize the structure theory developed in [**8**, Proposition 3.2]. There exist canonical subalgebras $\mathfrak{g}_0$, $\mathfrak{g}_2$ of $\mathfrak{g}$ such that $\mathfrak{g}_2$ is a codimension 1 ideal in $\mathfrak{g}$, $\mathfrak{g}_0 = \mathfrak{g}_1 \cap \mathfrak{g}_2$ is a codimension 2 ideal in $\mathfrak{g}$, and $\mathfrak{g}/\mathfrak{g}_0$ is isomorphic to the $ax + b$-algebra. We select $X \in \mathfrak{g}_1 \setminus \mathfrak{g}_0$, $Y \in \mathfrak{g}_2 \setminus \mathfrak{g}_0$, satisfying

$$[X, Y] \equiv Y \bmod \mathfrak{g}_0.$$

This determines two linear functionals α, β on $\mathfrak{g}$ according to

$$\alpha \in \mathfrak{g}_1^{\perp} \quad \alpha(Y) = 1$$
$$\beta \in \mathfrak{g}_2^{\perp} \quad \beta(X) = 1.$$

We sometimes abuse notation by writing α for $\alpha|_{\mathfrak{g}_2}$ or β for $\beta|_{\mathfrak{g}_1}$.

Now let $\varphi \in \mathfrak{g}^*$, $\pi = \pi_\varphi \in \hat{G}$. We denote $\psi = \varphi|_{\mathfrak{g}_1}$, $\theta = \varphi|_{\mathfrak{g}_0}$, $\omega = \varphi|_{\mathfrak{g}_2}$. We denote the corresponding Kirillov-Bernat irreducible representations by the symbols

$$\nu = \nu_\psi \in \hat{G}_1 \quad \gamma = \gamma_\theta \in \hat{G}_0 \quad \sigma = \sigma_\omega \in \hat{G}_2.$$

Then in analogy with the codimension 1 induced representation situation [**8**, Section 3], there are five possibilities for the structure of the restricted representation $\pi|_{G_1}$. However the invariant that determines the structure is not $\mathfrak{g}_\gamma = \mathfrak{g}_0 + \mathfrak{g}_\theta$ (as in [**8**]), but rather $\mathfrak{g}_\varphi$. We shall discover that and examine the relationship between the invariants in our next

THEOREM 3.2. *One of the following five mutually exclusive possibilities obtains:*
(i) $\mathfrak{g}_\gamma = \mathfrak{g}_0$. *Then if we set* $\varphi_s = \exp sY \cdot \varphi$, $s \in \Re$, *we have* $p(G\cdot\varphi) = \bigcup_s G_1 \cdot \psi_s$, *where* $\psi_s = \varphi_s|_{\mathfrak{g}_1}$, *the* G_1*-orbits* $G_1 \cdot \psi_s$, $s \in \Re$ *are distinct, and*

$$\pi_\varphi|_{G_1} = \int_{\Re}^{\oplus} \nu_{\psi_s}\, ds.$$

(ii) $\mathfrak{g}_\gamma = \mathfrak{g}_2$. *Then* $p(G\cdot\varphi) = G_1 \cdot \psi$ *and* $\pi_\varphi|_{G_1} = \nu_\psi$ *is irreducible.*
(iii) $\mathfrak{g}_\gamma = \mathfrak{g}_1$. *Then the projection* $p(G\cdot\varphi)$ *is a union of three* G_1*-orbits*

$$p(G\cdot\varphi) = G_1 \cdot \psi^+ \cup G_1 \cdot \psi \cup G_1 \cdot \psi^-,$$

where $\psi^\pm = \psi_{\pm 1}$. *We have* $G_1 \cdot \psi_s = G_1 \cdot \psi_{\mathrm{sgn}(s)}$, $\dim G_1 \cdot \psi^+ = \dim G_1 \cdot \psi^- = \dim G_1 \cdot \psi + 2$ *and*

$$\pi_\varphi|_{G_1} = \nu_{\psi^+} \oplus \nu_{\psi^-}.$$

(iv) $\mathfrak{g}_\gamma = \mathfrak{g}$ *and* $\mathfrak{g}_\varphi = (\mathfrak{g}_0)_\theta$. *Then*

$$p(G\cdot\varphi) = \bigcup_{t\in\Re} G_1 \cdot (\psi + t\beta).$$

The orbits $G_1 \cdot (\psi + t\beta)$, $t \in \Re$ *are distinct and*

$$\pi_\varphi|_{G_1} = \int_{\Re}^{\oplus} \nu_{\psi + t\beta}\, dt.$$

(v) $\mathfrak{g}_\gamma = \mathfrak{g}$ and $\mathfrak{g}_\varphi = \mathfrak{g}_\theta$. Then $p(G\cdot\varphi) = G_1\cdot\psi$ and $\pi_\varphi|_{G_1} = \nu_\psi$ is irreducible. Moreover, in every one of the five cases we have the orbital spectrum formula

$$\pi_\varphi|_{G_1} = \int_{p(G\cdot\varphi)/G_1}^{\oplus} n_\varphi^\psi \nu_\psi \, d\lambda_{G,G_1}^\varphi(\psi)$$
$$n_\varphi^\psi = \#[G\cdot\varphi \cap p^{-1}(G_1\cdot\psi)]/G_1 \equiv 1.$$

NOTES. (1) The restricted representation $\pi_\varphi|_{G_1}$ is either irreducible, a sum of two inequivalent irreducibles, or a direct integral over a 1-parameter family of irreducibles. The first and third can happen in one of two ways. But, as with induced representations [**8**, Section 3, Note 1], these really represent different cases. In fact, in case (v) it is true that $\pi_\varphi|_{G_0}$ is actually irreducible, but in case (ii) it is not. As for the direct integrals, in case (i), $\pi_\varphi|_{G_0}$ is a direct integral over a 2-parameter family, whereas in case (iv) the restriction $\pi_\varphi|_{G_0}$ is an infinite multiple of an irreducible.

(2) We saw in [**8**] that the structure of the induced representaion $\mathrm{Ind}_{G_1}^G \nu_\psi$ is determined by the subalgebra $\mathfrak{g}_\gamma$, $\gamma = \gamma_\theta$, $\theta = \psi|_{\mathfrak{g}_0}$. The five mutually exclusive possibilities for $\mathfrak{g}_\gamma$ were: $\mathfrak{g}_0$, $\mathfrak{g}_2$, $\mathfrak{g}_1$, codimension 1 nonideal $\neq \mathfrak{g}_1$, or $\mathfrak{g}$. When restricting, the third and fourth cases coalesce. This is because any codimension 1 subalgebra other than $\mathfrak{g}_2$ is conjugate to $\mathfrak{g}_1$. Thus if for $\varphi \in \mathfrak{g}^*$, $\theta = \varphi|_{\mathfrak{g}_0}$, $\gamma = \gamma_\theta$, we have an element g satisfying $g\cdot\mathfrak{g}_\gamma = \mathfrak{g}_1$, then the functional $g\cdot\varphi$ satisfies $\mathfrak{g}_{\gamma_{g\cdot\varphi|\mathfrak{g}_0}} = \mathfrak{g}_1$. But $\pi_{g\cdot\varphi} \cong \pi_\varphi$. On the other hand, the fifth case, $\mathfrak{g}_\gamma = \mathfrak{g}$, actually splits into two distinct subcases according to whether $\mathfrak{g}_\varphi$ is $(\mathfrak{g}_0)_\theta$ or $\mathfrak{g}_\theta$. In fact, in the proof that follows, we will show that in the first three cases—i. e., $\mathfrak{g}_\gamma = \mathfrak{g}_0, \mathfrak{g}_2, \mathfrak{g}_1$, respectively—the stabilizer $\mathfrak{g}_\varphi$ satisfies the distinct conditions: $\mathfrak{g}_\varphi \subset (\mathfrak{g}_0)_\theta$ and $\dim(\mathfrak{g}_0)_\theta/\mathfrak{g}_\varphi = 2$; $\dim\mathfrak{g}_\varphi = \dim(\mathfrak{g}_0)_\theta$ and $(\mathfrak{g}_0)_\theta \neq \mathfrak{g}_\varphi \subset \mathfrak{g}_2$; $\dim\mathfrak{g}_\varphi = \dim(\mathfrak{g}_0)_\theta$ and $(\mathfrak{g}_0)_\theta \neq \mathfrak{g}_\varphi \not\subset \mathfrak{g}_2$. Thus, while $\mathfrak{g}_\gamma$ is the invariant that determines the structure of the induced representation, the subalgebra $\mathfrak{g}_\varphi$ is the invariant that determines the structure of the restricted representation $\pi_\varphi|_{G_1}$.

PROOF. The five possibilities for the stabilizers $\mathfrak{g}_\gamma$ and $\mathfrak{g}_\varphi$ enumerated in the statement of the therorem are manifestly mutually distinct. We handle each case separately. In each we verify the orbital facts asserted and derive the direct integral decomposition of $\pi_\varphi|_{G_1}$. To substantiate the orbital spectrum formula in each case, we must identify the spectrum, multiplicity and spectral measure. We treat the first two separately in each of the five cases. We consider the measures together at the end of the proof.

We use throughout that $G_\gamma = G_0G_\theta$, $\mathfrak{g}_\gamma = \mathfrak{g}_0 + \mathfrak{g}_\theta$.

(i) $\mathfrak{g}_\gamma = \mathfrak{g}_0$. This implies that $\mathfrak{g}_\varphi \subset (\mathfrak{g}_0)_\theta$ and $\dim(\mathfrak{g}_0)_\theta/\mathfrak{g}_\varphi = 2$. Indeed, if $\mathfrak{g}_\gamma = \mathfrak{g}_0$, then $\mathfrak{g}_\varphi \subset \mathfrak{g}_\theta = (\mathfrak{g}_0)_\theta$. Hence $(\mathfrak{g}_0)_\varphi \subset \mathfrak{g}_\varphi \subset (\mathfrak{g}_0)_\theta \cap \mathfrak{g}_\varphi = (\mathfrak{g}_0)_\varphi$. But $(G_0)_\theta\cdot\varphi = \varphi + \mathfrak{g}_0^\perp$ by Lemma 2.4. Thus

$$\begin{aligned}\dim(\mathfrak{g}_0)_\theta / \mathfrak{g}_\varphi &= \dim(\mathfrak{g}_0)_\theta / (\mathfrak{g}_0)_\varphi \\ &= 2.\end{aligned}$$

Now when $\mathfrak{g}_\gamma = \mathfrak{g}_0$, we can carry over the facts from [**8**, Theorem 3.3, case (i)] to obtain

$$\begin{aligned}\pi_\varphi &= \mathrm{Ind}_{G_0}^{G}\, \gamma_\theta \\ &= \mathrm{Ind}_{G_2}^{G}\, \sigma_\omega.\end{aligned}$$

Therefore,

$$\begin{aligned}\pi_\varphi|_{G_1} &= (\mathrm{Ind}_{G_2}^{G} \sigma_\omega)|_{G_1} \\ &= \mathrm{Ind}_{G_0}^{G_1}(\sigma_\omega|_{G_1}) \quad \text{(Mackey Supgroup Theorem)} \\ &= \mathrm{Ind}_{G_0}^{G_1} \int_{\mathfrak{R}}^{\oplus} \gamma_{\theta_s}\, ds \quad \text{(Theorem 3.1)} \\ &= \int_{\mathfrak{R}}^{\oplus} \mathrm{Ind}_{G_0}^{G_1} \gamma_{\theta_s}\, ds \quad \text{(Commutation of induction} \\ &\qquad\qquad\qquad\qquad\qquad \text{and direct integral)} \\ &= \int_{\mathfrak{R}}^{\oplus} \nu_{\psi_s}\, ds \quad \text{below}.\end{aligned}$$

To justify the last equivalence we have to show that $(\mathfrak{g}_1)_{\theta_s} = (\mathfrak{g}_0)_{\theta_s}$ for every $s \in \mathfrak{R}$. Indeed

$$\begin{aligned}(\mathfrak{g}_1)_{\theta_s} &= (\mathfrak{g}_1)_{\exp sY \cdot \theta} \\ &= \{Z \in \mathfrak{g}_1 : \exp sY \cdot \theta\, [Z, \mathfrak{g}_0] = 0\} \\ &= \{Z \in \mathfrak{g}_1 : \theta\, [\exp -sY(Z_1), \mathfrak{g}_0] = 0\} \\ &= \exp sY\{\exp -sY \cdot Z \in \mathfrak{g}_1 : \theta\, [\exp -sY(Z_1), \mathfrak{g}_0] = 0\} \\ &\subset sY \cdot \mathfrak{g}_\theta \\ &= \exp sY \cdot (g_0)_\theta \\ &= (g_0)_{\theta_s}.\end{aligned}$$

The functionals ψ_s, $s \in \mathfrak{R}$, are in distinct G_1-orbits, for if $g_1 \cdot \psi_s = \psi_{s'}$, $g_1 \in G_1$, then $g_1 \cdot \theta_s = \theta_{s'} \Rightarrow \exp -s'Yg_1\ \exp sY \in G_\theta = (G_0)_\theta$. If $g_1 = \exp tXg_0$, $g_0 \in G_0$, then

$$\exp -s'Yg_1\ \exp sY \equiv \exp tX \exp(-s'e^t + s)Y \bmod G_0.$$

Hence $t = 0$ and $s = s'$.

The equality $p(G \cdot \varphi) = \bigcup_s G_1 \cdot \psi_s$ is easy to verify. The observation $\psi_s = \varphi_s|_{\mathfrak{g}_1} = \exp sY \cdot \varphi|\mathfrak{g}_1$ proves the $\supset$ inclusion. The reverse inclusion is true since $g_1 \exp sY \cdot \varphi|_{\mathfrak{g}_1} = g_1 \cdot \varphi_s|_{\mathfrak{g}_1} = g_1 \cdot \psi_s$. Finally we verify that

$$G \cdot \varphi \cap p^{-1}(G_1 \cdot \psi_s) = G_1 \cdot \varphi_s.$$

The only point not obvious is that if $g \cdot \varphi$ satisfies $g \cdot \varphi|_{\mathfrak{g}_1} = g_1 \cdot \psi_s$, then $g \cdot \varphi \in G_1 \cdot \varphi_s$ But the hypothesis gives $g \cdot \theta = g_1 \cdot \theta_s = g_1 \exp sY \cdot \theta$, which implies $g^{-1} g_1 \exp sY \in G_\theta = (G_0)_\theta$. Write $g = g_1' \exp s'Y$. Then

$$g^{-1} g_1 \exp sY = \exp -s'Y {g_1'}^{-1} g_1 \exp sY \in \exp -s'Y G_1 \exp sY \cap G_0.$$

Exactly as in the previous paragraph, this implies $s = s'$. Hence $g = g_1' \exp sY \Rightarrow g \cdot \varphi \in G_1 \cdot \varphi_s$.

(ii) $\mathfrak{g}_\gamma = \mathfrak{g}_2$. In this case $\mathfrak{g}_2 = \mathfrak{g}_0 + \mathfrak{g}_\theta$ forces $\mathfrak{g}_\varphi \subset \mathfrak{g}_\theta \subset \mathfrak{g}_2$. Moreover we can carry over the facts from [**8**, Theorem 3.3, case (iii)]. In particular, from [**8**, Theorem 4.1 ($\mathcal{U}_2 a$)] we have

$$\dim \mathfrak{g} \cdot \varphi = \dim \mathfrak{g}_0 \cdot \theta + 2.$$

Therefore

$$\dim \mathfrak{g} / \mathfrak{g}_\varphi = \dim \mathfrak{g}_0 / (\mathfrak{g}_0)_\theta + 2,$$

from which it follows that $\dim \mathfrak{g}_\varphi = \dim (\mathfrak{g}_0)_\theta$. These algebras are not equal, because $\mathfrak{g}_\varphi = (\mathfrak{g}_0)_\theta \Rightarrow (\mathfrak{g}_0)_\varphi \subset \mathfrak{g}_\varphi = (\mathfrak{g}_0)_\theta \cap \mathfrak{g}_\varphi = (\mathfrak{g}_0)_\varphi$. But the equation $(G_0)_\theta \cdot \varphi = \varphi + \mathfrak{g}_\gamma^\perp$ (Lemma 2.4) implies $\dim (\mathfrak{g}_0)_\theta / (\mathfrak{g}_0)_\varphi = 1$. Thus $(\mathfrak{g}_0)_\varphi = \mathfrak{g}_\varphi = (\mathfrak{g}_0)_\theta$ is not a possibility.

Now we turn our attention to $\pi_\varphi|_{G_1}$ in this case. Using the information from [**8**, loc. cit.] again, we have:

$$\begin{aligned} \mathfrak{g}_\omega &= (\mathfrak{g}_2)_\omega \quad \text{from which} \quad \pi_\varphi = \operatorname{Ind}_{G_2}^G \sigma_\omega; \\ (\mathfrak{g}_2)_\theta &\neq (\mathfrak{g}_0)_\theta \quad \text{from which} \quad \sigma_\omega|_{G_0} = \gamma_\theta; \\ (\mathfrak{g}_1)_\theta &= (\mathfrak{g}_0)_\theta \quad \text{from which} \quad \operatorname{Ind}_{G_0}^{G_1} \gamma_\theta = \nu_\psi \quad \text{is irreducible.} \end{aligned}$$

Combining these with the Subgroup Theorem, we obtain

$$\begin{aligned} \pi_\varphi|_{G_1} &= (\operatorname{Ind}_{G_2}^G \sigma_\omega)|_{G_1} \\ &= \operatorname{Ind}_{G_0}^{G_1} (\sigma_\omega|_{G_0}) \\ &= \operatorname{Ind}_{G_0}^{G_1} \gamma_\theta \\ &= \nu_\psi. \end{aligned}$$

Next we show $p(G \cdot \varphi) = G_1 \cdot \psi$. The inclusion $\supset$ is obvious. The reverse inclusion requires some tricky reasoning. We have $(\mathfrak{g}_2)_\theta \underset{\neq}{\supset} (\mathfrak{g}_0)_\theta$. Hence $\forall s \in \Re, \exists W \in \mathfrak{g}_0$ such that $sY + W_0 \in (\mathfrak{g}_2)_\theta$. It follows, since G_0 is normal, that $\forall s \in \Re, \exists g_0 \in G_0$ such that $\exp sY \cdot \theta = g_0 \cdot \theta$. We also have—since $(\mathfrak{g}_1)_\theta = (\mathfrak{g}_0)_\theta$—that

(3.1) $\qquad (G_0)_\theta \cdot \psi = \psi + \mathfrak{g}_0^\perp(\mathfrak{g}_1).$

Now we know that

$$(\exp sY \cdot \varphi - g_0 \cdot \varphi)|_{\mathfrak{g}_0} = 0.$$

If we apply equation (3.1) to $g_0 \cdot \psi$ (to which it applies equally well), then we obtain an element $g_0' \in (G_0)_{g_0 \cdot \theta}$ such that

$$(\exp sY \cdot \varphi - g_0 \cdot \varphi)|_{\mathfrak{g}_1} = g_0' g_0 \cdot \psi - g_0 \cdot \psi.$$

That is

$$\exp sY \cdot \varphi|_{\mathfrak{g}_1} = g_0' g_0 \cdot \psi.$$

Then for any $g = g_1 \exp sY \in G$, we have for $U \in \mathfrak{g}_1$ that

$$\begin{aligned} g \cdot \varphi(U) &= \exp sY \cdot \varphi(g_1^{-1} \cdot U) \\ &= g_0' g_0 \cdot \psi(g_1^{-1} \cdot U). \end{aligned}$$

That is, $p(G \cdot \varphi) \subset G_1 \cdot \psi$.

Finally we prove that $G \cdot \varphi \cap p^{-1}(G_1 \cdot \psi) = G_1 \cdot \varphi$. Of course the inclusion $\supset$ is obvious. The reverse is obtained as follows. Let $\varphi' = g \cdot \varphi \in p^{-1}(G_1 \cdot \psi)$. Then $g_1 \cdot \varphi'|_{\mathfrak{g}_1} = \psi$ for some $g_1 \in G$. Then $g_1 \cdot \varphi' = \varphi + s\alpha$ for some $s \in \Re$. In other words, $g_1 g \cdot \varphi = \varphi + s\alpha$. But [**8**, Proof of Theorem 3.3(iii)] in this instance, the functionals φ, $\varphi + s\alpha$ lie in distinct orbits unless $s = 0$. Therefore $g_1 g \cdot \varphi = \varphi$, and so $\varphi' = g \cdot \varphi = g_1^{-1} \cdot \varphi \in G_1 \cdot \varphi$.

(iii) $\mathfrak{g}_\gamma$ is a codimension 1 subalgebra $\neq \mathfrak{g}_2$. As explained in Note 2, there is no loss of generality in assuming $\mathfrak{g}_\gamma = \mathfrak{g}_1$. Then we can carry over the results of [**8**, Theorem 3.3 (iv)]. In particular, $\mathfrak{g}_1 = \mathfrak{g}_0 + \mathfrak{g}_\theta \Rightarrow \mathfrak{g}_\varphi \subset \mathfrak{g}_1$. We also have in this case that

$$\dim \mathfrak{g} \cdot \varphi = \dim \mathfrak{g}_0 \cdot \theta + 2,$$

thus again $\dim \mathfrak{g}_\varphi = \dim (\mathfrak{g}_0)_\theta$. The same reasoning as in the previous case—using that $(G_0)_\theta \cdot \varphi = \varphi + \mathfrak{g}_\gamma^\perp$ again implies $\dim (\mathfrak{g}_0)_\theta / (\mathfrak{g}_0)_\varphi = 1$—gives $\mathfrak{g}_\varphi \neq (\mathfrak{g}_0)_\theta$. Any conjugate of φ will therefore satisfy:

$$\dim \mathfrak{g}_\varphi = \dim (\mathfrak{g}_0)_\theta; \ \mathfrak{g}_\varphi \neq (\mathfrak{g}_0)_\theta; \ \mathfrak{g}_\varphi \not\subset \mathfrak{g}_2.$$

Now we consider the restriction. Once again π_φ is induced, so we can use the Subgroup Theorem. In fact, from [**8**, Theorem 3.3 (iv)] we see that

$$\pi_\varphi = \mathrm{Ind}_{G_1}^{G}\ \nu_\psi .$$

Hence

$$\pi_\varphi|_{G_1} = (\mathrm{Ind}_{G_1}^{G}\ \nu_\psi)|_{G_1},$$

which, in order to decompose, requires us to compute the G_1-double cosets in G. In fact,

$$G = G_1 \cup G_1 \exp YG_1 \cup G_1 \exp -YG_1,$$

a disjoint union of three double cosets, the latter two of which are of full dimension. Indeed, for any $s \in \mathfrak{R}$, we have

$$\begin{aligned} \exp sYG_1 &= \{\exp sY \exp tXG_0 : t \in \mathfrak{R}\} \\ &= \{\exp tX \exp e^{-t}sYG_0 : t \in \mathfrak{R}\} \\ \Rightarrow G_1 \exp sYG_1 &= \{G_1 \exp uYG_1 : \mathrm{sgn}(u) = \mathrm{sgn}(s)\}. \end{aligned}$$

Therefore, using the Subgroup Theorem, we obtain

$$\begin{aligned} \pi_\varphi|_{G_1} &= \mathrm{Ind}_{G_1 \cap \exp Y \cdot G_1}^{G_1}\ \nu_\psi^{\exp Y} \oplus \mathrm{Ind}_{G_1 \cap \exp -Y \cdot G_1}^{G_1}\ \nu_\psi^{\exp -Y} \\ &= \mathrm{Ind}_{G_0}^{G_1}\ (\nu_\psi^{\exp Y}|_{G_0}) \oplus \mathrm{Ind}_{G_0}^{G_1}\ (\nu_\psi^{\exp -Y}|_{G_0}). \end{aligned}$$

But in this case (see [**8**, loc. cit.]) we have $(\mathfrak{g}_1)_\theta \neq (\mathfrak{g}_0)_\theta$ so that $\nu_\psi|_{G_0} = \gamma_\theta$. Therefore

$$\nu_\psi^{\exp \pm Y}|_{G_0} = \gamma_{\theta^\pm},$$

where

$$\theta^\pm = \exp \pm Y \cdot \theta .$$

Combining, we have

$$\pi_\varphi|_{G_1} = \mathrm{Ind}_{G_0}^{G} \gamma_{\theta^+}\ \oplus\ \mathrm{Ind}_{G_0}^{G} \gamma_{\theta^-}.$$

Next we shall show the latter two representations are irreducible by demonstrating that

$$(3.2) \qquad (\mathfrak{g}_1)_{\theta^\pm} = (\mathfrak{g}_0)_{\theta^\pm}.$$

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We consider θ^+, the other being similar. If (3.2) were false, then $\exists(a_1X + W_1) \in \mathfrak{g}_1, a_1 \neq 0, W_1 \in \mathfrak{g}_0$, such that

$$\exp Y \cdot \theta\,[a_1X + W_1, \mathfrak{g}_0] = 0.$$

Therefore,

$$\begin{aligned}\theta\,[\exp -Y \cdot (a_1X + W_1), \mathfrak{g}_0] &= 0\\ &= \theta\,[a_1X + a_1Y + W_2, \mathfrak{g}_0], \quad \text{some } W_2 \in \mathfrak{g}_0\\ &\Rightarrow \mathfrak{g}_\theta \neq (\mathfrak{g}_1)_\theta, \quad \text{a contradiction.}\end{aligned}$$

The equation (3.2) also says that if we set

$$\varphi^\pm = \exp \pm Y \cdot \varphi, \ \psi^\pm = \psi^\pm|_{\mathfrak{g}_1},$$

then

$$\nu_{\psi^\pm} = \operatorname{Ind}_{G_0}^{G_1} \gamma_{\theta^\pm} \ \text{ and } \pi_\varphi|_{G_1} = \nu_{\psi^+} \oplus \nu_{\psi^-}.$$

Next we show the two representations $\nu_{\psi^\pm}$ are pairwise inequalivalent. For that we only need demonstrate that $\psi^\pm$ lie in distinct G_1-orbits. We prove more generally that if $\psi^s = \varphi^s|_{\mathfrak{g}_1}, \varphi^s = \exp sY \cdot \varphi$, then ψ^s and ψ^t are in the same G_1-orbit $\iff$ $\operatorname{sgn}(s) = \operatorname{sgn}(t)$. In fact, if $g_1 \cdot \psi^s = \psi^t$, then $g_1 \cdot \theta^s = \theta^t$. Writing $g_1 = \exp uXg_0$, we obtain

$$\exp uXg_0 \exp sY \cdot \theta = \exp tY \cdot \theta.$$

Therefore

$$\exp -tY \exp uX \exp sY \in G_0G_\theta = G_\gamma = G_1.$$

But

$$\begin{aligned}\exp -tY \exp uX \exp sY = \exp uX \exp(s - te^{-u})Y &\in G_1\\ &\iff \operatorname{sgn}(s) = \operatorname{sgn}(t).\end{aligned}$$

Conversely, let us prove that for $s > 0, \psi^s$ and ψ^+ are in the same G_1-orbit (we leave ψ^- to the reader). Since $(\mathfrak{g}_1)_\theta \underset{\neq}{\supset} (\mathfrak{g}_0)_\theta$, we know that for every $t \in \Re, \exists g_0 \in G_0$ such that $g_0 \exp tX \cdot \theta = \theta$. Then if $e^{-t} = s$, we have

$$\begin{aligned}\theta &= g_0 \exp tX \exp(-e^{-t} + s)Y \cdot \theta\\ &= g_0 \exp tX \exp(-e^{-t}Y) \cdot \theta^s\\ &= \exp -Yg_0' \exp tX \cdot \theta^s\\ \Longrightarrow \theta^+ &= g_1 \cdot \theta^s \ \text{ if } g_1 = g_0' \exp tX \in G_1.\end{aligned}$$

But $(\mathfrak{g}_1)_{\theta^+} = (\mathfrak{g}_0)_{\theta^+} \Longrightarrow (G_0)_{\theta^+} \cdot \psi^+ = \psi^+ + \mathfrak{g}_0^{\perp}(\mathfrak{g}_1)$. Hence we can choose an element $g_1' \in G_1$ which satisfies $\psi^+ = g_1' g_1 \cdot \psi^s$. This proves in particular that ν_{ψ^+} and ν_{ψ^-} are inequivalent.

Next we observe that

$$p(G \cdot \varphi) = \bigcup_s G_1 \cdot \psi^s = G_1 \cdot \psi^+ \cup G_1 \cdot \psi^- \cup G_1 \cdot \psi.$$

We already know (by [**8**, Theorem 3.3 (iv)]) that

$$\dim \mathfrak{g}_1 \cdot \psi = \dim \mathfrak{g}_0 \cdot \theta = \dim \mathfrak{g} \cdot \varphi - 2.$$

We also have (from [**8**, Theorem 4.1 ($\mathcal{U}_{nn}a$)]

$$\dim \mathfrak{g}_1 \cdot \psi^{\pm} = \dim \mathfrak{g}_0 \cdot \theta^{\pm} + 2 = \dim \mathfrak{g}_0 \cdot \theta + 2 = \dim \mathfrak{g} \cdot \varphi.$$

Thus $G_1 \cdot \psi^{\pm}$ are generic and $G_1 \cdot \psi$ is of lower dimension. It remains to prove

$$G \cdot \varphi \cap p^{-1}(G_1 \cdot \psi^{\pm}) = G_1 \cdot \varphi^{\pm}.$$

Consider the plus sign. Since $G \cdot \varphi = G \cdot \varphi^+$, it is enough to show

$$G \cdot \varphi^+ \cap p^{-1}(G_1 \cdot \psi^+) = G_1 \cdot \varphi^+.$$

But precisely this equality is proven in [**8**, Theorem 3.3 (ii)]—which argument is pertinent here, since for $\theta^+ = \varphi^+|_{\mathfrak{g}_0}$, the algebra $\mathfrak{g}_0 + \mathfrak{g}_{\theta^+}$ falls into that case.

(iv) $\mathfrak{g}_\gamma = \mathfrak{g}$ and $\mathfrak{g}_\varphi = (\mathfrak{g}_0)_\theta$. This time we find ourselves in the situation of [**8**, Theorem 3.3 (v)]. We adopt the notation from there—in particular, we have that

$$\mathrm{Ind}_{G_1}^G \nu_\psi = \pi^+ \oplus \pi^-,$$

where there is a fixed real number s_0 (equal to $-\frac{\theta(U_1)}{a_1}$ in the notation of [**8**]), such that the only G-orbits lying over ψ are

$$G \cdot (\varphi + s_0\alpha) \quad G \cdot (\varphi + s_1\alpha),\ s_1 < s_0 \quad G \cdot (\varphi + s_2\alpha),\ s_2 > s_0$$

and $\pi^+ = \pi_{\varphi + s_2\alpha}$, $\pi^- = \pi_{\varphi + s_1\alpha}$. Moreover, for $j = 1, 2$, we have

$$\text{(3.3)} \qquad \begin{aligned} \dim G \cdot (\varphi + s_j\alpha) &= \dim G_0 \cdot \theta + 2 \\ &= \dim G \cdot (\varphi + s_0\alpha) + 2. \end{aligned}$$

Now I claim that

$$\mathfrak{g}_\varphi = (\mathfrak{g}_0)_\theta \iff s_0 \neq 0.$$

Indeed, if $s_0 \neq 0$, then by (3.3) we have $\dim \mathfrak{g}_\varphi = \dim (\mathfrak{g}_0)_\theta$. But since $(G_0)_\theta \cdot \varphi = \varphi + \mathfrak{g}_\gamma^\perp = \varphi$ (Lemma 2.4), it follows that $(\mathfrak{g}_0)_\theta = (\mathfrak{g}_0)_\varphi$. Therefore $(\mathfrak{g}_0)_\theta = (\mathfrak{g}_0)_\varphi \subset \mathfrak{g}_\varphi$, and so they are equal. The inclusion $(\mathfrak{g}_0)_\theta \subset \mathfrak{g}_\varphi$ is true regardless of the value of s_0 (when $\mathfrak{g}_\gamma = \mathfrak{g}$). But if $s_0 = 0$, then by (3.3) again, we have $\dim \mathfrak{g}_\varphi = \dim (\mathfrak{g}_0)_\theta + 2$. Since $\mathfrak{g}_\varphi \subset \mathfrak{g}_\theta$ and $\dim \mathfrak{g}_\theta / (\mathfrak{g}_0)_\theta = 2$, the case $s_0 = 0$ is equivalent to $\mathfrak{g}_\varphi = \mathfrak{g}_\theta$—which will be our last case (v).

Let us proceed with $s_0 \neq 0$. We take $s_0 < 0$, the opposite sign being virtually identical. Then $\pi_\varphi \cong \pi^+ = \mathrm{Ind}_{G_2}^G \sigma_\omega$ (see [**8**, Theorem 3.3 (v)]). Therefore, continuing to use [**8**], we have

$$\begin{aligned}\pi_\varphi|_{G_1} &= (\mathrm{Ind}_{G_2}^G \sigma_\omega)|_{G_1} \\ &= \mathrm{Ind}_{G_0}^{G_1}(\sigma_\omega|_{G_0}) \\ &= \mathrm{Ind}_{G_0}^{G_1} \gamma_\theta \\ &= \int_{\mathfrak{R}}^{\oplus} \nu_{\psi + t\beta}\, dt.\end{aligned}$$

The representations $\nu_{\psi+t\beta}$ are inequivalent for distinct t by Lemma 2.4 (applied to the case $G_0 \lhd G_1$). Next we show that

$$p(G \cdot \varphi) = \bigcup_t G_1 \cdot (\psi + t\beta).$$

Indeed this follows immediately from the facts:

$$G = G_1 (G_2)_\theta \quad (G_2)_\theta = (G_2)_\omega \quad (G_2)_\omega \cdot \varphi = \varphi + \mathfrak{g}_2^\perp.$$

Finally we assert that if we set $\varphi^t = \varphi + t\beta$, $\psi^t = \varphi^t|_{\mathfrak{g}_1}$, then

$$G \cdot \varphi \cap p^{-1}(G_1 \cdot \psi^t) = G_1 \cdot \varphi^t.$$

One inclusion is clear since $\varphi^t \in (G_2)_\theta \cdot \varphi$. Conversely, suppose $\varphi' = g \cdot \varphi \in G \cdot \varphi$ and $g_1 \cdot \varphi'|_{\mathfrak{g}_1} = \psi + t\beta$. Then $g_1^{-1} g \in G_\theta = (G_1)_\theta (G_2)_\theta \subset G_1 (G_2)_\theta$. Therefore

$$\varphi' = g \cdot \varphi \in G_1 \cdot \varphi^{t_1} \qquad \text{for some } t_1.$$

Hence $G_1 \cdot \psi^t = G_1 \cdot \psi^{t_1}$. But this can happen only if $t = t_1$, and so the assertion is proven.

(v) $\mathfrak{g}_\gamma = \mathfrak{g}$ and $\mathfrak{g}_\varphi = \mathfrak{g}_\theta$. As we saw above, this means $s_0 = 0$ and

$$\dim \mathfrak{g} \cdot \varphi = \dim \mathfrak{g}_2 \cdot \omega = \dim \mathfrak{g}_0 \cdot \theta.$$

Therefore $\pi_\varphi|_{G_2} = \sigma_\omega$ is irreducible, and furthermore $\pi_\varphi|_{G_0} = \sigma_\omega|_{G_0} = \gamma_\theta$ is irreducible. It follows *a fortiori* that $\pi_\varphi|_{G_1}$ must be irreducible. Next it must be shown that it is ν_ψ. But of course the only irreducible representations of G_1 which restrict to γ_θ on G_0 are $\nu_{\psi+t\beta}$, $t \in \mathfrak{R}$. Hence

$$\pi_\varphi|_{G_1} = \nu_{\psi+t_0\beta}, \qquad \text{for some } t_0 \in \mathfrak{R}.$$

I claim $t_0 = 0$. To see this, let $\mathfrak{b}$ be a real polarization for φ which satisfies the Pukanszky conditon. Then $\mathfrak{b} \not\subset \mathfrak{g}_1$, since π_φ is not induced from G_1. Hence $\mathfrak{b}_1 = \mathfrak{b} \cap \mathfrak{g}_1$ is codimension 1 in $\mathfrak{b}$. Now we know

$$\dim \mathfrak{g} \cdot \varphi = \dim \mathfrak{g}_0 \cdot \theta = \dim \mathfrak{g}_1 \cdot (\psi + t\beta), \text{ any } t \in \mathfrak{R}.$$

Also,

$$\psi[\mathfrak{b}_1, \mathfrak{b}_1] \subset \varphi[\mathfrak{b}, \mathfrak{b}] = 0.$$

So $\mathfrak{b}_1$ is a real polarization for ψ. In fact $\mathfrak{b}_1$ must satisfy Pukanszky. For we have $G = BG_1$ and so

$$\pi_\varphi|_{G_1} = \operatorname{Ind}_B^G \chi_\varphi|_{G_1} = \operatorname{Ind}_{B\cap G_1}^{G_1} \chi_\varphi|_{B\cap G_1} = \operatorname{Ind}_{B_1}^{G_1} \chi_\varphi,$$

which is irreducible. It follows therefore that $\pi_\varphi|_{G_1} = \nu_\psi$.

It remains to demonstrate that $p(G\cdot\varphi) = G_1 \cdot \psi$ and $G \cdot \varphi \cap p^{-1}(G_1 \cdot \psi) = G_1 \cdot \varphi$. The inclusion $p(G \cdot \varphi) \supset G_1 \cdot \psi$ is obvious. The reverse comes from the facts:

$$G = G_1(G_2)_\theta \quad (G_2)_\theta \cdot \varphi = (G_2)_\omega \cdot \varphi = \varphi.$$

It is easy to check that the equality $G \cdot \varphi \cap p^{-1}(G_1 \cdot \psi) = G_1 \cdot \varphi$ is a consequence of the same facts.

To complete the proof of Theorem 3.2 we must prove the equality of the spectral measure classes obtained in cases (i)–(v)—that is, the point mass, 2-point measure or Lebesgue measure on the line—with the obital measure (class) λ^φ_{G,G_1}. Let us examine the latter more carefully in the codimension 1 situation. In case $p(G \cdot \varphi)/G_1$ is (generically) discrete—i. e., cases (ii), (iii) or (iv)—it is clear that λ^φ_{G,G_1}, being the push-forward of canonical measure on $G \cdot \varphi$, gives a discrete measure concentrated on the generic orbit classes. What about the continuous measures in (i) or (iv)? In case (i), it is obvious from the description

$$G \cdot \varphi = \bigcup_s G_1 \exp sY \cdot \varphi,$$

and from $\mathfrak{g}_\varphi \subset \mathfrak{g}_1$ that the canonical measure pushes forward to the Lebesgue measure class in the parameter s on $p(G \cdot \varphi)/G_1$. The same is true in case (iv), since again $\mathfrak{g}_\varphi \subset \mathfrak{g}_1$ but this time

$$G \cdot \varphi = \bigcup_t G_1 \cdot (\varphi + t\beta).$$

Thus the push-forward yields a measure in the Lebesgue class of the parameter t. This completes our argument.

Each of the five cases described in Theorem 3.2 actually occurs. Examples may be found in section 6.

4. Arbitrary Codimension. In this section we prove our main result, Theorem 4.1, giving the orbital spectrum formula for an arbitrary restricted representation. G is completely solvable and $H \subset G$ is closed and connected. We fix $\varphi \in \mathfrak{g}^*$, $\pi = \pi_\varphi$ the associated Kirillov-Bernat irreducible representation. The orbital spectrum formula is

THEOREM 4.1. *We have the direct integral decomposition*

$$\pi_\varphi|_H = \int^{\oplus}_{p_{\mathfrak{g},\mathfrak{h}}(G\cdot\varphi)/H} n^{\omega}_{\varphi}\rho_\omega \, d\lambda^{\varphi}_{G,H}(\omega), \tag{4.1}$$

where $p_{\mathfrak{g},\mathfrak{h}}: \mathfrak{g}^* \longrightarrow \mathfrak{h}^*$ *is the canonical projection,* $\lambda^{\varphi}_{G,H}$ *is the push-forward of the canonical measure on* $G \cdot \varphi$ *and*

$$n^{\omega}_{\varphi} = \#H\text{-orbits on } G\cdot\varphi \cap p^{-1}_{\mathfrak{g},\mathfrak{h}}(H\cdot\omega).$$

PROOF. The proof of formula (4.1) is by induction on $\dim G/H$. It follows from Theorems 3.1 and 3.2 that it is true if $\dim G/H = 1$. Now let $\dim G/H$ be larger than 1 and assume by induction that formula (4.1) is true for lower codimension. Since G is completely solvable we can find a closed connected subgroup G_1 such that

$$H \subset G_1 \subset G \text{ and } \dim G/G_1 = 1.$$

Because we will use Theorem 3.2 extensively, we preserve the notation $\psi \in \mathfrak{g}_1^*$, $\nu_\psi \in \hat{G}_1$. It is for that reason that we alter the notation of Definition 2.1 and write $\omega \in \mathfrak{h}^*$, $\rho = \rho_\omega$ for the orbital data on $\mathfrak{h}^*$ and $\hat{H}$.

Now the induction assumption applies to the pair (G_1, H). Hence for any $\psi \in \mathfrak{g}_1^*$, we have the orbital spectrum formula

$$\nu_\psi|_H = \int_{p^{\oplus}_{\mathfrak{g}_1,\mathfrak{h}}(G_1\cdot\psi)/H} n^{\omega}_{\psi}\rho_\omega \, d\lambda^{\psi}_{G_1,H}(\omega).$$

Now we restrict in stages and use the fact that the orbital spectrum formula is true in lower codimension

$$\begin{aligned}
\pi_\varphi|_H &= (\pi_\varphi|_{G_1})|_H \\
&= \left[\int^{\oplus}_{p_{\mathfrak{g},\mathfrak{g}_1}(G\cdot\varphi)/G_1} n^{\psi}_{\varphi}\nu_\psi \, d\lambda^{\varphi}_{G,G_1}(\psi)\right]\Bigg|_H \\
&= \int^{\oplus}_{p_{\mathfrak{g},\mathfrak{g}_1}(G\cdot\varphi)/G_1} n^{\psi}_{\varphi}\nu_\psi|_H \, d\lambda^{\varphi}_{G,G_1}(\psi) \\
&= \int^{\oplus}_{p_{\mathfrak{g},\mathfrak{g}_1}(G\cdot\varphi)/G_1} n^{\psi}_{\varphi} \int^{\oplus}_{p_{\mathfrak{g}_1,\mathfrak{h}}(G_1\cdot\psi)/H} n^{\omega}_{\psi}\rho_\omega d\lambda^{\psi}_{G_1,H}(\omega)\, d\lambda^{\varphi}_{G,G_1}(\psi).
\end{aligned}$$

Thus we must prove the equivalence of the two direct integrals

$$(4.2)\qquad \int_{p_{\mathfrak{g},\mathfrak{h}}(G\cdot\psi)/H}^{\oplus} n_{\varphi}^{\omega}\rho_{\omega}\, d\lambda_{G,H}^{\varphi}(\omega) = \int_{p_{\mathfrak{g},\mathfrak{g}_1}(G\cdot\varphi)/G_1}^{\oplus} \int_{p_{\mathfrak{g}_1,\mathfrak{h}}(G_1\cdot\psi)/H}^{\oplus} n_{\varphi}^{\psi} n_{\psi}^{\omega}\rho_{\omega}\, d\lambda_{G_1,H}^{\psi}(\omega)\, d\lambda_{G,G_1}^{\varphi}(\psi).$$

Now the subgroup G_1 may or may not be normal. In the former case there are two possibilities for the structure of the restriction $\pi_{\varphi}|_{G_1}$ (Theorem 3.1); in the latter case five (Theorem 3.2). In every case the multiplicity function $n_{\varphi}^{\psi} \equiv 1$. In any event the remainder of the argument is the demonstration of the equivalence of the two direct integrals in (4.2) in these seven cases. We must prove equality of spectrucm, multiplicity and spectral measure (see the material after [**8**, equation (4.2)] for a more elaborate discussion of what that means). As with the case of induced representations, we shall see that the spectrum and measure are handled without too much labor. It is the verification of equal multiplicity in (4.2) that is difficult, and which occupies the major portion of the argument.

In fact, we can dispose of the spectrum question immediately. The equality of spectrum in (4.2) follows instantly from

$$p_{\mathfrak{g},\mathfrak{h}}(G\cdot\varphi) = p_{\mathfrak{g}_1,\mathfrak{h}} \circ p_{\mathfrak{g},\mathfrak{g}_1}(G\cdot\varphi).$$

The proof of equal multiplicity and spectral measure requires that we handle the seven cases separately. We proceed to that now.

We first assume that G_1 is normal. In keeping with Theorem 3.1 then, we denote it by N—so $H \subset N \lhd G$, $\dim G/N = 1$. We set $\theta = \varphi|_{\mathfrak{n}}$, $\gamma = \gamma_{\theta}$. We have two subcases to consider.

(1) $\pi_{\varphi}|_N = \gamma_{\theta}$. This is case (ii) of Theorem 3.1. In this case formula (4.2) becomes

$$(4.3)\qquad \int_{p_{\mathfrak{g},\mathfrak{h}}(G\cdot\varphi)/H}^{\oplus} n_{\varphi}^{\omega}\rho_{\omega}\, d\lambda_{G,H}^{\varphi}(\omega) = \int_{p_{\mathfrak{n},\mathfrak{h}}(N\cdot\theta)/H}^{\oplus} n_{\theta}^{\omega}\rho_{\omega}\, d\lambda_{N,H}^{\theta}(\omega).$$

To show this we consider the projection map

$$G\cdot\varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H\cdot\omega) \longrightarrow N\cdot\theta \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(H\cdot\omega)$$

determined by $p_{\mathfrak{g},\mathfrak{n}}$. It is clearly a surjective H-equivariant map. I claim it is actually bijective. In fact $p_{\mathfrak{g},\mathfrak{n}}\colon G\cdot\varphi \longrightarrow N\cdot\theta$ is already injective, since if

$$g_1\cdot\varphi|_{\mathfrak{n}} = g_2\cdot\varphi|_{\mathfrak{n}}, \quad g_1, g_2 \in G,$$

then $g_2^{-1}g_1 \in G_{\theta} = G_{\varphi}$. Therefore

$$\begin{aligned} n_\varphi^\omega &= \#[G \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega)]/H \\ &= \#[N \cdot \theta \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(H \cdot \omega)]/H \\ &= n_\theta^\omega. \end{aligned}$$

It is also clear that the projection $G \cdot \varphi \longrightarrow N \cdot \theta$ carries the class of the G-invariant measure on $G \cdot \varphi$ to that of the N-invariant measure on $N \cdot \theta$. Hence the spectral measures also agree in (4.3), whence its proof is completed.

(2) $\pi_\varphi|_N = \int^\oplus \gamma_{\theta_s}\, ds$, $\theta_s = \exp sX \cdot \theta$, $\mathfrak{g} = \mathfrak{n} + \Re X$. This is case (i) of Theorem 3.1. Now formula (4.2) becomes

$$(4.4) \quad \int^\oplus_{p_{\mathfrak{g},\mathfrak{h}}(G\cdot\varphi)/H} n_\varphi^\omega \rho_\omega \, d\lambda_{G,H}^\varphi(\omega) = \int_\Re^\oplus \int^\oplus_{p_{\mathfrak{n},\mathfrak{h}}(N\cdot\theta_s)/H} n_{\theta_s}^\omega \rho_\omega \, d\lambda_{N,H}^{\theta_s}(\omega) ds.$$

The proof of equal multiplicity here is considerably more subtle. We base it on the argument in [**6**, Section 4]. And for that we must employ the analog of [**8**, Lemma 4.2].

LEMMA 4.2. *Let $H \subset N \lhd G$ be simply connected exponential solvable Lie groups, N normal. Fix $\omega \in \mathfrak{h}^*$. Then generically on $p_{\mathfrak{n},\mathfrak{h}}^{-1}(H \cdot \omega)$ we have*

$$G \cdot \theta \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(H \cdot \omega) \text{ has the same dimension as } \mathfrak{g} \cdot \theta \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(\mathfrak{h} \cdot \omega).$$

In addition for fixed $\theta \in \mathfrak{n}^*$, the same statement is true generically on $p_{\mathfrak{n},\mathfrak{h}}(G\cdot\theta)$.

PROOF. The first statement is proven in [**6**, Proposition 1.7] for the case that G is nilpotent. In that case generic means Zariski-open. It is generalized to exponential solvable groups—where generic means holding on a set whose interior is co-null—in [**8**, Section 2]; but only under the assumption that $H \cdot \omega$ is a singleton. That restriction was made in [**8**] because that was all we needed there. We observe now that the proof of [**6**, Proposition 1.7], adapted as in [**8**, Lemma 4.2], works fine for bona fide orbits $H \cdot \omega$ as well as singletons—thereby giving us the stated result. The second result also follows easily from the reasoning in [**6**] and [**8**].

Now the proof in case (2) requires the usual splitting according to generic orbit-intersection dimensions. This is reminiscent of arguments in [**2**], [**6**], [**8**]. To wit, generically on $p_{\mathfrak{g},\mathfrak{h}}(G \cdot \varphi)$ we are in one of the two following mutually exclusive situations:

$$(4.5a) \quad \dim[G \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega)] > \dim H \cdot \varphi$$

$$(4.5b) \quad \dim[G \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega)] = \dim H \cdot \varphi$$

According to Lemma 4.2 these are equivalent to

$$(4.6a)\qquad \dim[\mathfrak{g}\cdot\varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(\mathfrak{h}\cdot\omega)] > \dim\mathfrak{h}\cdot\varphi$$

$$(4.6b)\qquad \dim[\mathfrak{g}\cdot\varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(\mathfrak{h}\cdot\omega)] = \dim\mathfrak{h}\cdot\varphi$$

And finally by the next result, Lemma 4.3, these are further equivalent to

$$(4.7a)\qquad \dim\mathfrak{g}\cdot\varphi > 2\dim\mathfrak{h}\cdot\varphi - \dim\mathfrak{h}\cdot\omega$$

$$(4.7b)\qquad \dim\mathfrak{g}\cdot\varphi = 2\dim\mathfrak{h}\cdot\varphi - \dim\mathfrak{h}\cdot\omega$$

LEMMA 4.3. *We have*

$$\dim[\mathfrak{g}\cdot\varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(\mathfrak{h}\cdot\omega)] = \dim\mathfrak{g}\cdot\varphi - \dim\mathfrak{h}\cdot\varphi + \dim\mathfrak{h}\cdot\omega.$$

PROOF. This is obtained as follows.

$$\begin{aligned}
\dim[\mathfrak{g}\cdot\varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(\mathfrak{h}\cdot\omega)] &= \dim[\mathfrak{g}_\varphi^\perp \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}\big(\mathfrak{h}_\omega^\perp(\mathfrak{h})\big)] \\
&= \dim[\mathfrak{g}_\varphi^\perp \cap \mathfrak{h}_\omega^\perp(\mathfrak{g})] \\
&= \dim(\mathfrak{g}_\varphi + \mathfrak{h}_\omega)^\perp \\
&= \dim\mathfrak{g}/(\mathfrak{g}_\varphi + \mathfrak{h}_\omega) \\
&= \dim\mathfrak{g} - \dim\mathfrak{g}_\varphi - \dim\mathfrak{h}_\omega + \dim\mathfrak{g}_\varphi \cap \mathfrak{h}_\omega \\
&= (\dim\mathfrak{g} - \dim\mathfrak{g}_\varphi) - (\dim\mathfrak{h}_\omega - \dim\mathfrak{h}_\varphi) \\
&= \dim\mathfrak{g}/\mathfrak{g}_\varphi - (\dim\mathfrak{h} - \dim\mathfrak{h}_\varphi) \\
&\qquad\qquad + (\dim\mathfrak{h} - \dim\mathfrak{h}_\omega) \\
&= \dim\mathfrak{g}\cdot\varphi - \dim\mathfrak{h}\cdot\varphi + \dim\mathfrak{h}\cdot\omega.
\end{aligned}$$

REMARK 4.4 Case (b) characterizes finite multiplicity when the groups are nilpotent (see [**2**], [**6**]). For exponential solvable groups, infinite multiplicity can occur in case (b). For completely solvable groups, no example is known of infinite multiplicity in cases (b)—see [**8**, Remark before Lemma 4.4].

Now we treat the two cases (2a), (2b) separately.

(2a) $\pi_\varphi|_N = \int^\oplus \gamma_{\theta_s} ds$ and $\dim\mathfrak{g}\cdot\varphi > 2\dim\mathfrak{h}\cdot\varphi - \dim\mathfrak{h}\cdot\omega$ generically on $p_{\mathfrak{g},\mathfrak{h}}(G\cdot\varphi)$. Now we need to examine the multiplicity in formula (4.4) instead of (4.3). In this case it follows from Definition 2.1 that the multiplicity on the left side of (4.4) is uniformly $+\infty$. We show that the multiplicity on the right side of (4.4) is also uniformly infinite. Of course, we use the facts in Theorem 3.1 (part (i)). We select any generic $\omega \in p_{\mathfrak{g},\mathfrak{h}}(G\cdot\varphi)$ for which $n_\varphi^\omega = +\infty$. Then, mimicking the proof of Lemma 4.3, we have

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$$\begin{aligned}\dim \mathfrak{g}\cdot\theta \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(\mathfrak{h}\cdot\omega) &= \dim\left[\mathfrak{n}_{\varphi}^{\perp} \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}\left(\mathfrak{h}_{\omega}^{\perp}(\mathfrak{h})\right)\right]\\ &= \dim[\mathfrak{n}_{\varphi}^{\perp} \cap \mathfrak{h}_{\omega}^{\perp}(\mathfrak{n})]\\ &= \dim(\mathfrak{n}_{\varphi} + \mathfrak{h}_{\omega})^{\perp}\\ &= (\dim \mathfrak{n} - \dim \mathfrak{n}_{\varphi}) - (\dim \mathfrak{h}_{\omega} - \dim \mathfrak{h}_{\varphi})\\ &= \dim \mathfrak{g}\cdot\theta - \dim \mathfrak{h}\cdot\varphi + \dim \mathfrak{h}\cdot\omega.\end{aligned}$$

(Note we used $\mathfrak{g}\cdot\theta = \mathfrak{n}_{\varphi}^{\perp}$ which follows from Theorem 3.1 (i) and the equations $\mathfrak{g}\cdot\theta(\mathfrak{n}_{\varphi}) = \theta[\mathfrak{g},\mathfrak{n}_{\varphi}] = 0, \dim\mathfrak{g}\cdot\theta = \dim\mathfrak{g}/\mathfrak{g}_{\theta} = \dim\mathfrak{n}/\mathfrak{n}_{\theta}+1 = \dim\mathfrak{n}/\mathfrak{n}_{\varphi}$.) We also have

$$\dim \mathfrak{g}\cdot\theta = \dim\mathfrak{g}\cdot\varphi - 1.$$

Hence, since both sides of the inequality

$$\dim \mathfrak{g}\cdot\theta > 2\dim\mathfrak{h}\cdot\varphi - \dim\mathfrak{h}\cdot\omega$$

are even, we also obtain

$$\dim \mathfrak{g}\cdot\theta > 2\dim\mathfrak{h}\cdot\varphi - \dim\mathfrak{h}\cdot\omega.$$

Then, combining these facts with Lemma 4.2 we deduce

$$\begin{aligned}\dim G\cdot\theta \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(H\cdot\omega) &= \dim\mathfrak{g}\cdot\theta \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(\mathfrak{h}\cdot\omega)\\ &= \dim\mathfrak{g}\cdot\theta - \dim\mathfrak{h}\cdot\varphi + \dim\mathfrak{h}\cdot\omega\\ &> \dim\mathfrak{h}\cdot\varphi\\ &\geq \dim H\cdot\theta.\end{aligned}$$

Moreover, using φ_s instead of φ, the same strict inequality applies to θ_s.

Now we cannot at this point simply deduce that $n_{\theta_s}^{\omega} = +\infty$, because $\dim N\cdot\theta_s \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(H\cdot\omega)$ may be one less that $\dim G\cdot\theta_s \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(H\cdot\omega)$. Thus we reason as follows. We have

$$G\cdot\theta = \bigcup_s N\exp sX\cdot\theta = \bigcup_s N\cdot\theta_s.$$

Therefore

$$G\cdot\theta \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(H\cdot\omega) = \bigcup_s N\cdot\theta_s \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(H\cdot\omega).$$

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Now suppose there is an $s \in \Re$ such that

$$\dim G \cdot \theta_s \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(H \cdot \omega) = \dim N \cdot \theta_s \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(H \cdot \omega).$$

Then clearly $n_{\theta_s}^{\omega} = +\infty$, and so $\rho = \rho_\omega$ occurs with infinite multiplicity. On the other hand if this condition fails, then we must have $N \cdot \theta_s \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(H \cdot \omega) \neq \emptyset$ for $s \in S$, a set of positive Lebesgue measure. Then for any $s \in S$, we have $H \cdot \omega \subset p_{\mathfrak{n},\mathfrak{h}}(N \cdot \theta_s)$, which says that $\rho = \rho_\omega$ occurs with infinite multiplicity in

$$\int_S^{\oplus} \int_{P(N \cdot \theta_s)/H}^{\oplus} \gamma_{\theta_s}|_H.$$

Either way, we get infinite multiplicity on both sides of equation (4.4).

(2b) $\pi_\varphi|_N = \int^{\oplus} \gamma_{\theta_s} ds$ and $\dim \mathfrak{g} \cdot \varphi = 2 \dim \mathfrak{h} \cdot \varphi - \dim \mathfrak{h} \cdot \omega$ generically on $p_{\mathfrak{g},\mathfrak{h}}(G \cdot \varphi)$. Once again we must prove equality of multiplicity for the two sides of (4.4). Exactly as in case (1) we consider the projection

$$G \cdot \varphi \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(H \cdot \omega) \longrightarrow G \cdot \theta \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(H \cdot \omega).$$

It is not a bijection this time, but it does set up a bijection of H-orbits. The argument for that is identical to the one in case (ia) of [**6**, Section 2]. In short, if $g \cdot \theta = h \cdot \theta$, then $h^{-1} g \in G_\theta = N_\theta$. But $N_\theta \cdot \varphi = \varphi + \Re\alpha = H_\theta \cdot \varphi$, because $H_\theta \cdot \varphi$ can only fail to be $\varphi + \Re\alpha$ if $H_\theta = H_\varphi$—which is not so (see below). We have the decomposition

$$(4.8) \qquad G \cdot \theta \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(H \cdot \omega) = \bigcup_s N \cdot \theta_s \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(H \cdot \omega).$$

Now $\dim G \cdot \theta \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(H \cdot \omega) = \dim H \cdot \theta$. This is because each side has dimension one less than the corresponding variety when θ is replaced by φ. (The first is shown in part (a), the second below.) Therefore, the left side of (4.8) is a countable union of disjoint H-orbits. Thus at most countably many of the intersections in the right side of (4.8) are non-empty. We suppose they are indexed by $s_1, s_2, \ldots$, so that

$$G \cdot \theta \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(H \cdot \omega) = \bigcup_{j=1}^{\infty} N \cdot \theta_{s_j} \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(H \cdot \omega).$$

We next show that the dimensions are all the same. Assume first that $s_1 = 0$. Then we have

$$\begin{aligned} 2 \dim \mathfrak{h} \cdot \theta &\leq \dim \mathfrak{n} \cdot \theta + \dim \mathfrak{h} \cdot \omega \\ &= \dim \mathfrak{g} \cdot \varphi - 2 + \dim \mathfrak{h} \cdot \omega \\ &= 2 \dim \mathfrak{h} \cdot \varphi - 2. \end{aligned}$$

Thus

$$\dim \mathfrak{h} \cdot \theta \leq \dim \mathfrak{h} \cdot \varphi - 1.$$

The reverse inequality is obvious. Hence

$$\begin{aligned}\dim \mathfrak{n} \cdot \theta &= \dim \mathfrak{g} \cdot \varphi - 2 \\ &= 2 \dim \mathfrak{h} \cdot \varphi - 2 - \dim \mathfrak{h} \cdot \omega \\ &= 2 \dim \mathfrak{h} \cdot \theta - \dim \mathfrak{h} \cdot \omega.\end{aligned}$$

By Lemma 4.2, this says

$$\dim N \cdot \theta \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(H \cdot \omega) = \dim H \cdot \theta.$$

Now the point is that, after replacing φ by φ_{s_j}, the same argument applies and establishes the claim. Thus finally

$$\begin{aligned}(4.9) \qquad n_\varphi^\omega &= \#[G \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega)]/H \\ &= \sum_{j=1}^{\infty} \#[N \cdot \theta_{s_j} \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(H \cdot \omega)]/H,\end{aligned}$$

which is precisely the multiplicity in the direct integral of the right side of equation (4.4). (This last computation is nicely illustrated by Example 4 in section 6.)

This completes the proof of equal multiplicity in case(2). To complete the proof of formula (4.4) therefore, we only need to demonstrate that the spectral measures are equivalent. Indeed, the argument for that is word-for-word identical to that of the nilpotent situation—[**6**, Section 4]—and so we do not repeat it.

Now we drop the assumption that G_1 is normal. Then the restriction from G to G_1 is controlled by Theorem 3.2 instead of Theorem 3.1. We have $H \subset G_1 \subset G$, $\psi = \varphi|_{\mathfrak{g}_1}, \nu = \nu_\psi$ and five cases to consider. In some of these we need to split the argument into two subcases according to the generic dimension of orbit intersections, sometimes we don't. Actually the situation turns out to be analogous to the normal situation in that we need to split exactly when the codimension 1 restriction is *not* irreducible. That occurs in cases (i), (iii) and (iv) of Theorem 3.2. Hence we will consider those last.

(1) (Case (ii) of Theorem 3.2) $\mathfrak{g}_\gamma = \mathfrak{g}_2$. In this case $\pi_\varphi|_{G_1} = \nu_\psi$ is irreducible and formula (4.2) becomes

$$(4.10) \qquad \int_{p_{\mathfrak{g},\mathfrak{h}}(G\cdot\varphi)/H}^{\oplus} n_\varphi^\omega \rho_\omega \, d\lambda_{G,H}^{\varphi}(\omega) = \int_{p_{\mathfrak{g}_1,\mathfrak{h}}(G_1\cdot\psi)/H}^{\oplus} n_\psi^\omega \rho_\omega \, d\lambda_{G_1,H}^{\psi}(\omega).$$

Since, as explained earlier, we are only dealing with multiplicities at this stage, we must show now that $n_\varphi^\omega = n_\psi^\omega$. Consider the projection

$$G \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) \longrightarrow G_1 \cdot \psi \cap p_{\mathfrak{g}_1,\mathfrak{h}}^{-1}(H \cdot \omega).$$

This is well-defined (even though G_1 is not normal) since $p_{\mathfrak{g},\mathfrak{g}_1}(G \cdot \varphi) = G_1 \cdot \psi$ in this case. The map is obviously an H-equivariant surjection. We only need to show it is injective. So suppose $\varphi', \varphi'' \in G \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega)$ have the same restriction on $\mathfrak{g}_1$. Since $\mathfrak{g}_2$ is an ideal, it is no loss of generality to replace φ by φ'. Then we must take $\varphi, g \cdot \varphi \in G \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega)$ satisfying

$$\varphi|_{\mathfrak{g}_1} = g \cdot \varphi|_{\mathfrak{g}_1},$$

and deduce that $g \in G_\varphi$. But equality on $\mathfrak{g}_1$ says that $g \cdot \varphi = \varphi + s\alpha$ for some $s \in \Re$. However, we know that when $\mathfrak{g}_\gamma = \mathfrak{g}_2$ (see [**8**, Theorem 3.3 (iii)]), the functionals $\varphi + s\alpha$ lie in distinct G-orbits as s varies. Hence $g \cdot \varphi = \varphi$.

The same argument works in the other instance of an irreducible codimension 1 restriction, namely

(2) (Case (v) of Theorem 3.2) $\mathfrak{g}_\gamma = \mathfrak{g}$ and $\mathfrak{g}_\varphi = \mathfrak{g}_\theta$. In this case we again have $\pi_\varphi|_{G_1} = \nu_\psi$ is irreducible. Equal multiplicity in (4.10) is proven by the identical argument to the previous case, since $\mathfrak{g}_\gamma = \mathfrak{g}$ is an ideal and the functionals φ and $\varphi + s\alpha$, $s \neq 0$, lie in distinct orbits. (In this case the orbit of $\varphi + s\alpha$ is determined by sgn(s).)

(3) (Case (iii) of Theorem 3.2) $\mathfrak{g}_\gamma$ is non-ideal codimension 1 subalgebra. Conjugating φ if necessary, we may assume $\mathfrak{g}_\gamma = \mathfrak{g}_1$. Then, in the notation of Section 3, we have $\pi_\varphi|_{G_1} = \nu_{\psi^+} \oplus \nu_{\psi^-}$. Formula (4.2) becomes

$$\begin{aligned}(4.11)\qquad &\int_{p_{\mathfrak{g},\mathfrak{h}}(G\cdot\varphi)/H}^{\oplus} n_\varphi^\omega \rho_\omega \, d\lambda_{G,H}^{\varphi}(\omega) \\ &= \int_{p_{\mathfrak{g}_1,\mathfrak{h}}(G_1\cdot\psi^+)/H}^{\oplus} n_{\psi^+}^\omega \rho_\omega \, d\lambda_{G_1,H}^{\psi^+}(\omega) \oplus \int_{p_{\mathfrak{g}_1,\mathfrak{h}}(G_1\cdot\psi^-)/H}^{\oplus} n_{\psi^-}^\omega \rho_\omega \, d\lambda_{G_1,H}^{\psi^-}(\omega).\end{aligned}$$

In fact we know from Theorem 3.2 (iii) that

$$p_{\mathfrak{g},\mathfrak{g}_1}(G \cdot \varphi) = G_1 \cdot \psi^+ \cup G_1 \cdot \psi^- \cup G_1 \cdot \psi$$

where the G_1-orbit $G_1 \cdot \psi$ has dimension 2 less than $G \cdot \varphi$, and the other two G_1-orbits have the same dimension as that of $G \cdot \varphi$. Of course we have $G \cdot \varphi = \bigcup_s G_1 \cdot \varphi^s$. Furthermore, we assert that $G_1 \cdot \varphi^s = G_1 \cdot \varphi^{s'}$ if and only if $\mathrm{sgn}(s) = \mathrm{sgn}(s')$. This follows easily from the facts established in Section 3, namely

$$G_1 \cdot \varphi^s = G \cdot \varphi^s \cap p_{\mathfrak{g},\mathfrak{g}_1}^{-1}(G_1 \cdot \psi^s) \text{ and } G_1 \cdot \psi^s = G_1 \cdot \psi^{s'} \quad \text{if and only if } \mathrm{sgn}(s) = \mathrm{sgn}(s').$$

Thus we have a disjoint union

$$G \cdot \varphi = G_1 \cdot \varphi^+ \cup G_1 \cdot \varphi^- \cup G_1 \cdot \varphi$$

Moreover, the first two are of full dimension (since $\dim \mathfrak{g}_1 \cdot \varphi^+ \geq \mathfrak{g}_1 \cdot \psi^+ = \dim \mathfrak{g} \cdot \varphi$), and the last is of lower dimension (since $\mathfrak{g}_\varphi = (\mathfrak{g}_1)_\varphi$). Thus we also have

$$G \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) = [G_1 \cdot \varphi^+ \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega)] \cup$$
$$[G_1 \cdot \varphi^- \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega)] \cup [G_1 \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega)].$$

Furthermore, reasoning as in [**8**, proof of Lemma 4.3 (iv)], we see that either

$$\text{(4.12)} \quad \dim[G_1 \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega)] < \dim[G_1 \cdot \varphi^\pm \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega)], \text{ or}$$
$$G_1 \cdot \varphi^\pm \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) = \emptyset.$$

But we only need to pay attention to generic $\omega \in p_{\mathfrak{g},\mathfrak{h}}(G \cdot \varphi)$. Thus we can ignore the subvariety $p_{\mathfrak{g},\mathfrak{h}}(G_1 \cdot \varphi)$. That is, within the choices in (4.12), it must be true that the first one holds generically—in particular for generic ω, at least one of

$$\text{(4.13)} \quad G_1 \cdot \varphi^\pm \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) \neq \emptyset.$$

Now we are ready to bifurcate according to orbit intersection dimensions.

(3a) $\dim \mathfrak{g} \cdot \varphi > 2 \dim \mathfrak{h} \cdot \varphi - \dim \mathfrak{h} \cdot \omega$ generically on $p_{\mathfrak{g},\mathfrak{h}}(G \cdot \varphi)$. Then of course $n_\varphi^\omega = +\infty$ on the left side of (4.11). It is infinite on the right side as well, which we show as follows. We may suppose $G_1 \cdot \varphi^+ \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) \neq \emptyset$, and by the dimension condition

$$\dim G_1 \cdot \varphi^+ \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) > \dim H \cdot \varphi.$$

We show

$$\dim G_1 \cdot \psi^+ \cap p_{\mathfrak{g}_1,\mathfrak{h}}^{-1}(H \cdot \omega) > \dim H \cdot \psi.$$

The argument is modelled after [**8**, Theorem 4.1 ($\mathcal{U}_{n1}a$)]. Set $2n = \dim \mathfrak{g} \cdot \varphi^+ = \dim \mathfrak{g}_1 \cdot \varphi^+$, $m = \dim \mathfrak{h} \cdot \varphi^+$, $2r = \dim \mathfrak{h} \cdot \omega$, so that $2n > 2m - 2r$. Then

$$\dim G_1 \cdot \varphi^+ \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) = \dim \mathfrak{g}_1 \cdot \varphi^+ - \dim \mathfrak{h} \cdot \varphi + \dim \mathfrak{h} \cdot \omega$$
$$= 2n - m + 2r.$$

But the surjective projection

$$G_1 \cdot \varphi^+ \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) \longrightarrow G_1 \cdot \psi^+ \cap p_{\mathfrak{g}_1,\mathfrak{h}}^{-1}(H \cdot \omega)$$

has fiber of dimension at most 1. Therefore

$$\dim G_1 \cdot \psi^+ \cap p_{\mathfrak{g}_1,\mathfrak{h}}^{-1}(H \cdot \omega) \geq 2n - m + 2r - 1.$$

On the other hand

$$\dim H \cdot \psi^+ \leq \dim H \cdot \varphi^+ = m.$$

Since $2n - m + 2r - 1 > m$, we are done.

(3b) $\dim \mathfrak{g} \cdot \varphi = 2 \dim \mathfrak{h} \cdot \varphi - \dim \mathfrak{h} \cdot \omega$ generically. In this case we will show that

$$G_1 \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) = \emptyset.$$

In fact, we can reason as in Lemmas 4.3 and 4.4 to show that, generically on $p_{\mathfrak{g},\mathfrak{h}}(G \cdot \varphi)$, we have

$$\begin{aligned} \dim G_1 \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) &= \dim \mathfrak{g}_1 \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(\mathfrak{h} \cdot \omega) \\ &= \dim\big((\mathfrak{g}_1)_{\psi}^{\perp}(\mathfrak{g}) \cap \mathfrak{h}_{\omega}^{\perp}\big) \\ &= \dim \mathfrak{g} - \dim (\mathfrak{g}_1)_{\psi} - \dim \mathfrak{h} \cdot \psi + \dim \mathfrak{h} \cdot \omega \\ &= \dim \mathfrak{g} \cdot \varphi - 1 - \dim \mathfrak{h} \cdot \psi + \dim \mathfrak{h} \cdot \omega. \end{aligned}$$

But in case (b) we also have

$$\begin{aligned} 2 \dim \mathfrak{h} \cdot \psi &\leq \dim \mathfrak{g}_1 \cdot \psi + \dim \mathfrak{h} \cdot \omega \\ &= \dim \mathfrak{g} \cdot \varphi - 2 + \dim \mathfrak{h} \cdot \omega \\ &= 2 \dim \mathfrak{h} \cdot \varphi - 2. \end{aligned}$$

Since the codimension cannot be any less than 1, we have

$$\dim \mathfrak{h} \cdot \psi = \dim \mathfrak{h} \cdot \varphi - 1.$$

Combining, we obtain

$$\dim G_1 \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) = \dim H \cdot \varphi.$$

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This is incompatible with (4.12) and (4.13) unless $G_1 \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) = \emptyset$. Thus to prove equality of multiplicity in (4.11), we only need to show

$$\#[G_1 \cdot \varphi^{\pm} \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega)]/H = \#[G_1 \cdot \psi^{\pm} \cap p_{\mathfrak{g}_1,\mathfrak{h}}^{-1}(H \cdot \omega)]/H.$$

That is, we must show that the projection

$$G_1 \cdot \varphi^{\pm} \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) \longrightarrow G_1 \cdot \psi^{\pm} \cap p_{\mathfrak{g}_1,\mathfrak{h}}^{-1}(H \cdot \omega)$$

yields a bijection of H-orbits. It is clearly surjective and H-equivariant. In fact, it is injective. For if

$$g_1 \cdot \varphi^{+}|_{\mathfrak{g}_1} = \varphi^{+}|_{\mathfrak{g}_1}, \; g_1 \in G_1,$$

then $g_1 \in G_{1\psi^+} \subset G_{\varphi^+}$ (or alternatively $\varphi^+, \varphi^+ + s\alpha$ lie in distinct orbits in this case).

(4) (Case (i) of Theorem 3.2) $\mathfrak{g}_\gamma = \mathfrak{g}_0$. In this case $\pi_\varphi|_{G1} = \int^{\oplus} \nu_{\psi_s} ds$, $\psi_s = \varphi_s|_{\mathfrak{g}_1}$, $\varphi_s = \exp sY \cdot \varphi$. Thus formula (4.2) becomes

$$(4.14) \quad \int_{p_{\mathfrak{g},\mathfrak{h}}(G\cdot\varphi)/H}^{\oplus} n_{\varphi}^{\omega} \rho_{\omega} \; d\lambda_{G,H}^{\varphi}(\omega) = \int_{\Re}^{\oplus} \int_{p(G_1,\psi_s)/H}^{\oplus} n_{\psi_s}^{\omega} \rho_{\omega} \; d\lambda_{G_1,H}^{\psi_s}(\omega) ds.$$

The reasoning begins like the corresponding part in the normal case. We have

$$\begin{aligned} G \cdot \varphi &= \bigcup_s G_1 \exp sY \cdot \varphi \\ &= \bigcup_s G_1 \cdot \varphi_s. \end{aligned}$$

Therefore,

$$G \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) = \bigcup_s G_1 \cdot \varphi_s \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega).$$

Now we split the argument.

(4a) Assume $\dim \mathfrak{g} \cdot \varphi > 2 \dim \mathfrak{h} \cdot \varphi - \dim \mathfrak{h} \cdot \omega$ generically on $p_{\mathfrak{g},\mathfrak{h}}(G \cdot \varphi)$. Then we have $n_{\varphi}^{\omega} = +\infty$ on the left side of equation (4.14). We demonstrate infinite multiplicity for the right side. Suppose there is an $s \in \Re$ such that

$$\dim G \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) = \dim G_1 \cdot \varphi_s \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega).$$

We reason as in case (3a). Consider the projection

$$G_1 \cdot \varphi_s \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) \longrightarrow G_1 \cdot \psi_s \cap p_{\mathfrak{g}_1,\mathfrak{h}}^{-1}(H \cdot \omega),$$

the fiber of which has dimension at most 1. Set $2n = \dim \mathfrak{g} \cdot \varphi_s$, $m = \dim \mathfrak{h} \cdot \varphi_s$, $2r = \dim \mathfrak{h} \cdot \omega$. By hypothesis $2n > 2m - 2r$. Then

$$\begin{aligned}\dim G_1 \cdot \varphi_s \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) &= \dim G \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega)\\ &= 2n - m + 2r.\end{aligned}$$

Therefore

$$\begin{aligned}\dim G_1 \cdot \psi_s \cap p_{\mathfrak{g}_1,\mathfrak{h}}^{-1}(H \cdot \omega) &\geq 2n - m + 2r - 1\\ &> m\\ &= \dim H \cdot \varphi_s\\ &\geq \dim H \cdot \psi_s.\end{aligned}$$

It follows that $n_{\psi_s}^{\omega} = +\infty$ and so $\rho = \rho_\omega$ occurs with infinite multiplicity. Thus we can assume that no such s exists. Then we must have

$$G_1 \cdot \varphi_s \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) \neq \emptyset$$

for $s \in S$, a set of positive Lebesgue measure. Consequently

$$G_1 \cdot \psi_s \cap p_{\mathfrak{g}_1,\mathfrak{h}}^{-1}(H \cdot \omega) \neq \emptyset$$

for s in the same set S. In particular, $H \cdot \omega \subset p_{\mathfrak{g}_1,\mathfrak{h}}(G_1 \cdot \psi_s)$, $s \in S$, which guarantees that ρ_ω occurs with infinite multiplicity in

$$\int_{s \in S}^{\oplus} \int_{p_{\mathfrak{g}_1,\mathfrak{h}}(G_1 \cdot \psi_s)/H}^{\oplus} \nu_{\psi_s}|_H.$$

Either way, we get infinite multiplicity on both sides of equation (4.14).

(4b) $\dim \mathfrak{g} \cdot \varphi = 2 \dim \mathfrak{h} \cdot \varphi - \dim \mathfrak{h} \cdot \omega$, generically. Then consider the natural projection

$$G_1 \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) \longrightarrow G_1 \cdot \psi \cap p_{\mathfrak{g}_1,\mathfrak{h}}^{-1}(H \cdot \omega).$$

It is clearly an H-equivariant surjection. It is not injective now, but it is a bijection of H-orbits. To see this , it is enough to show that if for $g_1 \in G_1$ we have

$$g_1 \cdot \varphi|_{\mathfrak{g}_1} = \varphi|_{\mathfrak{g}_1},$$

then φ and $g_1 \cdot \varphi$ are in the same H-orbit. For that, it suffices to prove

$$H_\psi \cdot \varphi = \varphi \cdot \mathfrak{g}_1^{\perp}.$$

It is obvious that $H_\psi \cdot \varphi \subset \varphi + \mathfrak{g}_1^\perp$. But in the situation (4b) we have

$$\begin{aligned} 2\dim \mathfrak{h} \cdot \psi &\leq \dim \mathfrak{g}_1 \cdot \psi + \dim \mathfrak{h} \cdot \omega \\ &= \dim \mathfrak{g} \cdot \varphi - 2 + \dim \mathfrak{h} \cdot \omega \\ &= 2\dim \mathfrak{h} \cdot \varphi - 2. \end{aligned}$$

It follows that

$$\dim \mathfrak{h} \cdot \psi = \dim \mathfrak{h} \cdot \varphi - 1,$$

from which $H_\psi \cdot \varphi$ is an open connected subset of $\varphi + \mathfrak{g}_1^\perp$. But the same argument applies to any other $\varphi' \in \varphi + \mathfrak{g}_1^\perp$ (since $\varphi' \in G \cdot \varphi$ in this case, see [**8**, Proof of Theorem 3.3 (i)]), and so the result follows.

Now the same reasoning is valid for any φ_s. Hence

$$(4.15) \qquad G \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) = \bigcup_s G_1 \cdot \varphi_s \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) \longrightarrow \bigcup_s G_1 \cdot \psi_s \cap p_{\mathfrak{g}_1,\mathfrak{h}}^{-1}(H \cdot \omega)$$

sets up a bijection of H-orbits. The left side of (4.15) is a countable union of H-orbits. Thus at most countably many of the intersections on the right side of (4.15) are non-empty, say for $s_1, s_2, \ldots$,

$$G \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega) = \bigcup_{j=1}^{\infty} G_1 \cdot \varphi_{s_j} \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega).$$

Next we show that every intersection $G_1 \cdot \psi_{s_j} \cap p_{\mathfrak{g}_1,\mathfrak{h}}^{-1}(H \cdot \omega)$ is of full dimension. Just as in the normal case, the following argument is independent of s_j. So assume $s_1 = 0$ and show

$$(4.16) \qquad \dim G_1 \cdot \psi \cap p_{\mathfrak{g}_1,\mathfrak{h}}^{-1}(H \cdot \omega) = \dim H \cdot \psi.$$

After that we compute

$$\begin{aligned} n_\varphi^\omega &= \#[G \cdot \varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(H \cdot \omega)]/H \\ &= \sum_{j=1}^{\infty} \#[G_1 \cdot \psi_s \cap p_{\mathfrak{g}_1,\mathfrak{h}}^{-1}(H \cdot \omega)]/H, \end{aligned}$$

the equality of multiplicity in formula (4.14). It remains to prove (4.16). This is done by

$$\begin{aligned}\dim G_1\cdot\psi\cap p_{\mathfrak{g}_1,\mathfrak{h}}^{-1}(H\cdot\omega) &= \dim\mathfrak{g}_1\cdot\psi-\dim\mathfrak{h}\cdot\psi+\dim\mathfrak{h}\cdot\omega\\ &= \dim\mathfrak{g}\varphi-2-\dim\mathfrak{h}\cdot\psi+\dim\mathfrak{h}\cdot\omega\\ &= 2\dim\mathfrak{h}\cdot\psi-\dim\mathfrak{h}\cdot\omega-2-\dim\mathfrak{h}\cdot\psi\\ &\qquad +\dim\mathfrak{h}\cdot\omega\\ &= 2\dim(\mathfrak{h}\cdot\varphi-1)-\dim\mathfrak{h}\cdot\psi\\ &= \dim\mathfrak{h}\cdot\psi\end{aligned}$$

We arrive at the last case.

(5) (Case (iv) of Theorem 3.2) $\mathfrak{g}_\gamma=\mathfrak{g}$ and $\mathfrak{g}_\varphi=(\mathfrak{g}_0)_\theta$. Then $\pi_\varphi|_{G_1}=\int^\oplus\nu_{\psi+t\beta}\,dt$, and formula (4.2) becomes

$$\int^\oplus_{p_{\mathfrak{g},\mathfrak{h}}(G\cdot\varphi)/H} n_\varphi^\omega\rho_\omega\,d\lambda^\varphi_{G,H}(\omega)=\int^\oplus_{\mathfrak{R}}\int^\oplus_{p_{\mathfrak{g},\mathfrak{h}}(G_1\cdot\psi+t\beta)/H} n^\omega_{\psi+t\beta}\rho_\omega d\lambda^\varphi_{G,H}(\omega)\,dt. \tag{4.17}$$

This case is handled in a manner analogous to the previous case, this time using the partition

$$G\cdot\varphi=\bigcup_t G_1\cdot(\varphi+t\beta).$$

To prove that, we note the inclusion $\supset$ is clear from $(G_2)_\omega\cdot\varphi=\varphi+\mathfrak{g}_2^\perp$. The reverse inclusion comes from $G=G_1(G_2)_\omega$. The partition is disjoint because if we write $\varphi^t=\varphi+t\beta$, $\psi^t=\psi+t\beta=\varphi^t|_{\mathfrak{g}_1}$, we have $G_1\cdot\varphi^t=G_1\cdot\varphi^{t'}\Longrightarrow G_1\cdot\psi^t=G_1\cdot\psi^{t'}$—which impossible by the normal $\mathfrak{g}_0\triangleleft\mathfrak{g}_1$ theory (since $(\mathfrak{g}_1)_\theta\neq(\mathfrak{g}_0)_\theta$). We split the argument now.

(5a) $\dim\mathfrak{g}\cdot\varphi>2\dim\mathfrak{h}\cdot\varphi-\dim\mathfrak{h}\cdot\omega$, generically. We have $n^\omega_\varphi=+\infty$ on the left in (4.17). To show infinite multiplicity on the right, we reason virtually word-for-word as in case (4a), replacing the parameter s everywhere by t.

(5b) $\dim\mathfrak{g}\cdot\varphi=2\dim\mathfrak{h}\cdot\varphi-\dim\mathfrak{h}\cdot\omega$, ω generic on $p^{-1}_{\mathfrak{g},\mathfrak{h}}(G\cdot\varphi)$. Now consider the projection

$$G_1\cdot\varphi\cap p^{-1}_{\mathfrak{g},\mathfrak{h}}(H\cdot\omega)\longrightarrow G_1\cdot\psi\cap p^{-1}_{\mathfrak{g}_1,\mathfrak{h}}(H\cdot\omega).$$

It is clearly an H-equivariant surjection—we prove it is a bijection of H-orbits. As in (4b), to see this it is enough to prove that for $g_1\in G_1$, if we have $g_1\cdot\varphi|_{\mathfrak{g}_1}=\varphi|_{\mathfrak{g}_1}$, then φ and $g_1\cdot\varphi$ are in the same H-orbit. Now the equality insures that $g_1\cdot\varphi=\varphi_s$ for some $s\in\mathfrak{R}$. But in this case, the G-orbit structure forces s and 0 to lie on the same side of s_0 (see **[8]**, Theorem 4.1 ($\mathcal{U}$b)]). Reasoning exactly as in that case **[8]**

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as well as in (4b), we see that $H_\psi \cdot \varphi_s$ is an open and connected subset of $\varphi + \mathfrak{g}_1^\perp$, for any s on the same side of s_0 as 0. Hence $g_1 \cdot \varphi = h \cdot \varphi$ for some $h \in H_\psi$. The rest of the argument is finished precisely as in (4b).

To complete the proof of formula (4.2), and so of Theorem 4.1, we must demonstrate the equivalence of the measures that appear on each side of the equation. We have already accomplished that when the intermediary subgroup G_1 is normal, the argument is identical to that of [**6**, Section 4]. Now for the non-normal situation. In subcases (1) and (2), wherein $p_{\mathfrak{g},\mathfrak{g}_1}(G \cdot \varphi) = G_1 \cdot \psi$, the equivalence of measures in (4.10) is obvious. It is also evident in case (3) since

$$p_{\mathfrak{g},\mathfrak{g}_1}(G \cdot \varphi) = G_1 \cdot \psi^+ \cup G_1 \cdot \psi^- \cup G_1 \cdot \psi,$$

and the third orbit is of smaller dimension than the first two. The equivalence of measure in (4.11) is then clear, since the canonical measure class on $G \cdot \varphi$ must project to those on $G_1 \cdot \psi^\pm$. The remaining two cases—where $p_{\mathfrak{g},\mathfrak{g}_1}(G \cdot \varphi)$ is a 1-parameter family of G_1 -orbits—are handled very similarly. I include the argument for case (4) and leave the other to the reader. We have $p_{\mathfrak{g},\mathfrak{g}_1}(G \cdot \varphi) = \bigcup_s G_1 \cdot \psi_s$, and a natural fiber space

$$\begin{array}{ccc} G_1 \cdot \psi_s & \longrightarrow & p_{\mathfrak{g},\mathfrak{g}_1}(G \cdot \varphi) \\ & & \downarrow \\ & & \Re(\approx \exp \Re X). \end{array}$$

Moreover, the push-forward of the canonical measure on $G \cdot \varphi$ is the natural fiber measure—that is, Lebesgue measure on the base the canonical G_1-invariant measure on the fiber. The picture is H-equivariant, and so factors to a fiber space

$$\begin{array}{ccc} p_{\mathfrak{g}_1,\mathfrak{h}}(G_1 \cdot \psi_s)/H & \longrightarrow & p_{\mathfrak{g},\mathfrak{h}}(G \cdot \varphi)/H \\ & & \downarrow \\ & & \Re; \end{array}$$

from which it is evident that the measures on the two sides of (4.14) are equivalent. This concludes the proof of Theorem 4.1.

5. Distinguishing like cases. For $G_1 \subset G$ of codimension 1 and non-normal, we have enumerated five possibilities for the structure of an induced representation $\mathrm{Ind}_{G_1}^G \nu_\psi$ or of a restricted representation $\pi_\varphi|_{G_1}$. The distinguishing invariant for the former is the subalgebra $\mathfrak{g}_0 + \mathfrak{g}_\theta$, where $\mathfrak{g}_0$ is the canonical codimension 2 ideal determined by $\mathfrak{g}_1$ and $\theta = \psi|_{\mathfrak{g}_0}$; for the latter it is $\mathfrak{g}_\varphi$. However, in both categories only three possible configurations are manifested by the representation—namely irreducible, a sum of two inequivalent irreducibles, or a 1-parameter direct integral of irreducibles. In either category, the sum of two irreducibles occurs in only one way, so no confusion can arise. But the other two possibilities (irreducible or a direct integral) occur in pairs. The invariants are different within each pair, but it is not transparent how that is reflected in the structure of the representation itself. The answer is to be found in the reciprocal process. To illustrate, the representation $\pi = \mathrm{Ind}_{G_1}^G \nu_\psi$ will be irreducible if $\mathfrak{g}_0 + \mathfrak{g}_\theta$ equals either $\mathfrak{g}_0$ or $\mathfrak{g}_1$. That these irreducible induced representations really reflect different cases can be understood by restricting each back to G_1. Thus if

$$\pi = \pi_\varphi \text{ and } \mathfrak{g}_0 + \mathfrak{g}_\theta = \mathfrak{g}_0, \text{ then } \pi_\varphi|_{G_1} = \int^\oplus \nu_{\psi_s} ds;$$

while if

$$\pi = \pi_\varphi \text{ and } \mathfrak{g}_0 + \mathfrak{g}_\theta = \mathfrak{g}_1, \text{ then } \pi_\varphi|_{G_1} = \nu_{\psi^+} \oplus \nu_{\psi^-}.$$

Similarly, if π is a 1-parameter direct integral of irreducibles, then each of the components in the direct integral restricts back to G_1 to give the same representation—but that restriction differs according to the case. Thus if

$$\pi = \int^\oplus \pi_{\varphi+s\alpha} ds \text{ and } \mathfrak{g}_0 + \mathfrak{g}_\theta = \mathfrak{g}_2, \text{ then } \pi_{\varphi+s\alpha}|_{G_1} = \nu_\psi \ \forall s;$$

while if

$$\pi = \int^\oplus \pi_{\varphi+s\beta} dt \text{ and } \mathfrak{g}_0 + \mathfrak{g}_\theta \text{ is non-ideal, } \neq \mathfrak{g}_1, \\ \text{then } \pi_{\varphi+t\beta}|_{G_1} = \nu_\psi \oplus \nu_{\psi'}, \ \forall t,$$

where ψ' is determined as follows. There is a unique s_0 such that if $\varphi_{s_0} = \exp s_0 Y \cdot \varphi$, $\theta_{s_0} = \varphi_{s_0}|_{\mathfrak{g}_0}$, then $\mathfrak{g}_{\theta_{s_0}} + \mathfrak{g}_0 = \mathfrak{g}_1$. Then $\psi' = \varphi_{2s_0}|_{\mathfrak{g}_1}$.

We can reverse the roles of induction and restriction. The restricted representation $\pi_\varphi|G_1$ can be a sum of two inequivalent irreducibles in only one way. But it can be irreducible or a 1-parameter direct integral in two ways. These can be distinguished by inducing back up to G. Thus if

$$\pi_\varphi|_{G_1} = \nu_\psi \text{ and } \dim \mathfrak{g}_\varphi = \dim(\mathfrak{g}_0)_\theta, \ (\mathfrak{g}_0)_\theta \neq \mathfrak{g}_\varphi \subset \mathfrak{g}_2, \\ \text{then } \mathrm{Ind}_{G_1}^G \nu_\psi = \int^\oplus \pi_{\varphi+s\alpha} ds;$$

while if

$$\pi_\varphi|_{G_1} = \nu_\psi \text{ and } \mathfrak{g}_\varphi = \mathfrak{g}_\theta, \text{ then } \mathrm{Ind}_{G_1}^G \nu_\psi = \pi_{\varphi^+} \oplus \pi_{\varphi^-},$$

where $\varphi^{\pm}$ are determined as follows. There exists a unique s_0 such that $\dim G \cdot (\varphi + s_0\alpha) = \dim G_1 \cdot \psi$. Then $\varphi^- = \varphi + s_1\alpha$, any $s_1 < s_0$, $\varphi^+ = \varphi + s_2\alpha$, any $s_2 > s_0$. Finally, if $\pi_\varphi|_{G_1}$ is a 1-parameter direct integral of irreducibles, each of the components induces to the same (class of) representation(s) on G—but that class depends on the invariant $\mathfrak{g}_\varphi$. To wit, if

$$\pi_\varphi|_{G_1} = \int^{\oplus} \nu_{\psi_s} ds \text{ and } \dim(\mathfrak{g}_0)_\theta / \mathfrak{g}_\varphi = 2, \quad \text{then } \operatorname{Ind}_{G_1}^G \nu_{\psi_s} = \pi_\varphi, \ \forall s;$$

while if

$$\pi_\varphi|_{G_1} = \int^{\oplus} \nu_{\psi+t\beta}\, dt \text{ and } \mathfrak{g}_\varphi = (\mathfrak{g}_0)_\theta,$$
$$\text{then } \operatorname{Ind}_{G_1}^G \nu_{\psi+t\beta} = \pi_{\varphi^+} \oplus \pi_{\varphi^-}, \ \forall t$$

where $\varphi^{\pm}$ are determined as above.

6. Examples. We give several examples to illustrate Theorems 3.1, 3.2 and 4.1. These are analogous to the examples provided in [**8**]. We also give an example to illustrate the multiplicity computation (formula 4.9) in Theorem 4.1. The symbol Ω will denote co-adjoint orbits.

(1) $ax + b$ algebra $\mathfrak{g} = \operatorname{sp}\{A, X\}, [A, X] = X, \varphi = \varphi_{\alpha,\xi} = \alpha A^* + \xi X^* \in \mathfrak{g}^*$, $\Omega_\alpha = \alpha A^*, \alpha \in \Re$; $\Omega^{\pm} = \{\alpha A^* + \xi X^* : \alpha \in \Re, \xi \gtrless 0\}$

(a) $\mathfrak{n} = \operatorname{sp}\{X\}$ $\theta_\xi = \varphi|_{\mathfrak{n}} = \xi X^*$

$$\pi_\varphi|_N = \begin{cases} 1, & \text{if } \xi = 0 \\ \int^{\oplus} \gamma_{\theta_s}\, ds = \int_{\operatorname{sgn}(\xi') = \operatorname{sgn}(\xi)} \gamma_{\theta_{\xi'}}\, d\xi', & \text{if } \xi \neq 0 \end{cases}$$

(b) $\mathfrak{g}_1 = \operatorname{sp}\{A\}$, $\mathfrak{g}_0 = \{0\}$, $\psi = \psi_\alpha = \alpha A^*$

$$\pi_\varphi|_{G_1} = \begin{cases} \chi_{\psi_\alpha}, & \text{if } \xi = 0 \\ \int_{\Re}^{\oplus} \chi_{\psi_{\alpha'}} d\alpha', & \text{if } \xi \neq 0. \end{cases}$$

(2) $\mathfrak{g} = \operatorname{sp}\{A, X, Y, Z\}$, $[X, Y] = Z$, $[A, X] = X$, $[A, Y] = Y$, $[A, Z] = 2Z$

$$\varphi = \varphi_{\alpha,\xi,\eta,\zeta} = \alpha A^* + \xi X^* + \eta Y^* + \zeta Z^* \in \mathfrak{g}^*$$

$$\Omega_{\pm} = \{\alpha A^* + \xi X^* + \eta Y^* + \varepsilon r Z^* : \alpha, \xi, \eta \in \Re, r > 0\}, \ \varepsilon = \pm 1$$

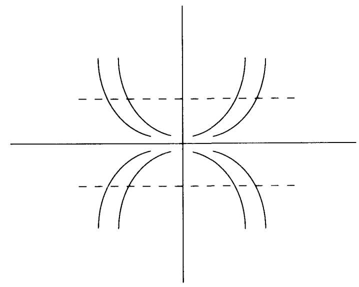
$$\mathfrak{g}_1 = \operatorname{sp}\{A, X, Z\}, \ \mathfrak{g}_0 = \operatorname{sp}\{X, Z\}$$

$$G_1 \text{ -- orbits : } \Omega_\alpha^1 = \alpha A^*, \ \alpha \in \Re$$

$\Omega^1_{\xi,\zeta} = \{\alpha A^* + r\xi X^* + r^2\zeta Z^* \colon \alpha \in \Re, r > 0\}$, $(\xi,\zeta) \in C$ a Borel cross-section for the action of $\Re$ on $\Re^2 \setminus \{0\}$ by $a \cdot (\xi,\zeta) = (e^a\eta, d^{2a}\zeta)$.

$$\varphi = \varphi_{(0,0,0,1)}$$

$$\pi_\varphi|_{G_1} = \int^{\oplus} \nu_{\psi_s}\, ds = \int_{-\infty}^{\infty\oplus} \nu_{\psi_{(0,s,1)}}\, ds,$$



since $p(G \cdot \varphi) = \{\psi_{\alpha,\xi,\zeta} : \alpha, \xi \in \Re, \zeta > 0\}$; and if we write $\psi = \varphi|_{\mathfrak{g}_1} = \psi_{(0,0,1)}$, then $\psi_s = (\exp sY \cdot \varphi_{0,0,0,1})|_{\mathfrak{g}_1} = \psi_{0,-s,1}$. Similarly $\pi_{(0,0,0,-1)}|_{G_1} = \int^{\oplus} \nu_{\psi_{(0,s,-1)}}\, ds$.

(3) $\mathfrak{g} = \mathrm{sp}\{A, X, Y, Z\}$, $[X,Y] = Z$, $[A,X] = X$, $[A,Y] = -Y$

$$\varphi = \varphi_{\alpha,\xi,\eta,\zeta} = \alpha A^* + \xi X^* + \eta Y^* + \zeta Z^* \in \mathfrak{g}^*$$

$$\Omega_\alpha = \alpha A^*, \ \alpha \in \Re$$

$\Omega_{\xi,\eta} = \{\alpha A^* + r\xi X^* + r^{-1}\eta Y^* \colon \alpha \in \Re, r > 0\}$, $(\xi,\zeta) \in C$ a Borel cross-section for the action $\Re$ on $\Re^2 \setminus \{0\}$ by $a(\xi,\eta) = (e^a\xi, e^{-a}\eta)$

$$\Omega(\alpha,\zeta) = G \cdot \varphi_{(\alpha,0,0,\zeta)'}\ \alpha \in \Re, \zeta \neq 0$$

(a) $\mathfrak{g}_1 = \mathrm{sp}\{A, X, Z\}$, $\mathfrak{g}_0 = \mathrm{sp}\{X, Z\}$

G_1-orbits: $\Omega^1_{\alpha,\zeta} = \alpha A^* + \zeta Z^*, \alpha, \zeta \in \Re$

$$\Omega^1_{\pm,\zeta} = \{\alpha A^* + r\varepsilon X^* + \zeta Z^* \colon \alpha \in \Re, r > 0\},\ \varepsilon = \pm 1, \zeta \in \Re$$

$$\varphi = \varphi_{(\alpha,0,0,0)} \qquad \pi_\varphi|_{G_1} = \nu_{\psi_\alpha},\ \psi_\alpha = \psi_{(\alpha,0,0)}$$

$$\varphi = \varphi_{(\alpha,0,\eta,0)} \qquad \pi_\varphi|_{G_1} = \int^{\oplus} \nu_{\psi_{\alpha'}}\, d\alpha'$$

$$\varphi = \varphi_{(\alpha,\xi_{(\xi\neq 0)},\eta,0)} \qquad \pi_\varphi|_{G_1} = \nu_{\psi_{(\alpha,\xi,0)}}$$

$$\varphi = \varphi_{(\alpha,0,0,\zeta_{(\zeta\neq 0)})} \qquad \pi_\varphi|_{G_1} = \nu_{\psi_{(\alpha,1,\zeta)}} \oplus \nu_{\psi_{(\alpha,-1,\zeta)}}.$$

The preceding four decompositions represent cases (v), (iv), (ii), (iii) of Theorem 3,2, respectively. Case (i) is exemplified by example (2). The following are also illustrative.

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(b) $\mathfrak{h} = \mathrm{sp}\{A, Z\}$

$$\varphi = \varphi_{(\alpha,0,0,\zeta)}\ \pi_\varphi|_H = 2\int^{\oplus} \rho_{(\alpha',\zeta)}\, d\alpha'.$$

(c) $\mathfrak{h} = \mathrm{sp}\{A, X\}$

$$\varphi = \varphi_{(\alpha,0,0,\zeta)}\ \pi_\varphi|_H = \rho_{(0,1)} \oplus \rho_{(0,-1)}.$$

(4) The following example actually involves nipotent groups, but it illustrates very nicely the equality of multiplicity computation in Theorem 4.1.

$$\mathfrak{g} = \mathrm{sp}\{A, X, Y, Z\},\ [A, X] = Y,\ [A, Y] = Z$$
$$\varphi = \varphi_{\alpha,\xi,\eta,\zeta} = \alpha A^* + \xi X^* + \eta Y^* + \zeta Z^* \in \mathfrak{g}^*$$

$$(\exp zZ\ \exp yY\ \exp xX\ \exp aA)^{-1} \cdot \varphi_{\alpha,\xi,\eta,\zeta}$$
$$= \varphi_{\alpha - x\eta - y\zeta,\, \xi + a\eta + (\frac{1}{2})a^2\zeta,\, \eta + a\zeta,\, \zeta}.$$

Consider $\varphi = \varphi_{0,\xi,0,\zeta}$, $\xi \in \Re$, $\zeta > 0$, and $\mathfrak{h} = \Re X$. $\mathfrak{h}$ is contained in a codimension 1 ideal in $\mathfrak{g}$—namely $\mathfrak{n} = \mathrm{sp}\{X, Y, Z\}$, which is abelian. Now

$$p_{\mathfrak{g},\mathfrak{h}}(G\cdot\varphi) = p_{\mathfrak{g},\mathfrak{h}}\{\varphi_{-y\zeta,\xi+(\frac{1}{2})a^2\zeta,a\zeta,\zeta} : a, y \in \Re\}$$
$$= \{\xi_1 X^* : \xi_1 \geqq \xi\}$$

Moreover, for $\xi_1 \geq \xi$, we have

$$G\cdot\varphi \cap p_{\mathfrak{g},\mathfrak{h}}^{-1}(\{\xi_1\}) = \{\varphi_{-y\zeta,\xi+(\frac{1}{2})a^2\zeta,a\zeta,\zeta} : y \in \Re, \xi + (\frac{1}{2})a^2\zeta = \xi_1\};$$

which has one component or H-orbit if $\xi_1 = \xi$, namely $\{\varphi_{(\alpha,\xi_1,0,\zeta)} : \alpha \in \Re\}$, but two components or H-orbits if $\xi_1 > \xi$, namely $\{\varphi_{\alpha,\xi_1,\pm\sqrt{2(\xi_1-\xi)\zeta},\zeta} : \alpha \in \Re\}$. Thus

$$\pi_\varphi|_H = 2\int_{\xi_1 \geqq \xi}^{\oplus} \rho_{\xi_1} d\xi_1.$$

On the other hand, we can compute the restriction in stages. We have

$$p_{\mathfrak{g},\mathfrak{n}}(G\cdot\varphi) = \{\theta_{(\xi_1,\eta,\zeta)} : \xi_1 \geqq \xi,\ \eta \in \Re\},$$

which is not a point, so we are in case (i) of Theorem 3.1. In particular,

$$\pi_\varphi|_N = \int_{-\infty}^{\infty\oplus} \gamma_{\theta_s} ds,$$

where $\theta_s = \exp sA \cdot \theta$, of which a computation reveals that if $\theta = \theta_{(\xi,0,\zeta)}$, then $\theta_s = \theta_{(\xi+(\frac{1}{2})s^2\zeta,-s\zeta,\zeta)}$, $s \in \Re$. Now

$$p_{\mathfrak{n},\mathfrak{h}}(N\cdot\theta_s) = p_{\mathfrak{n},\mathfrak{h}}(\theta_s) = (\xi + (\frac{1}{2})s^2\zeta)X^*,$$
$$\bigcup_s p_{\mathfrak{n},\mathfrak{h}}(N\cdot\theta_s) = p_{\mathfrak{g},\mathfrak{h}}(G\cdot\varphi) = \{\xi_1 X^* : \xi_1 \geqq \xi\}.$$

Moreover, for $s \in \Re$ and $\xi_1 \geqq \xi$, we have

$$N\cdot\theta_s \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(\{\xi_1\}) = \{\theta_{(\xi+(\frac{1}{2})a^2\zeta,a\zeta,\zeta)},\ a \in \Re\} \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(\{\xi_1\})$$
$$= \{\theta_{\xi_1,\pm\sqrt{2(\xi_1-\xi)\zeta},\zeta}\}.$$

Thus

$$G\cdot\theta \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(\{\xi_1\}) = \bigcup\{N\cdot\theta_s \cap p_{\mathfrak{n},\mathfrak{h}}^{-1}(\{\xi_1\}) : s = \pm\sqrt{2(\xi_1-\xi)/\zeta}\}.$$

Also, for every $s \in \Re$, there exists precisely one ξ, such that $\xi_1 = \xi + \frac{1}{2}s^2\zeta$; and conversely, given $\xi_1 > \xi$, there exist precisely two s for each of which $n_{\theta_s}^{\xi_1} = 1$. This illustrates the computation (formula (4.9)) of equal multiplicity in formula (4.4) when both H and N are abelian.

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