Orbit types for maps of the interval

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Abstract. Relationships between the permutations associated to periodic orbits of maps of the interval and the topology of the space of these maps are given. These extend the author's previous work on periodic orbits to orbit types.

Let I denote a compact interval on the real line, and for each positive integer n, let $F(n)$ denote the set of f in $C^0(I, I)$ such that f has a periodic point of (least) period n. A theorem of Šarkovskii [8] asserts that

$$F(3) \subset F(5) \subset \cdots \subset F(2 \cdot 3) \subset F(2 \cdot 5) \subset \cdots \subset F(4) \subset F(2) \subset F(1).$$

Furthermore, if $i \neq j$ and $F(i) \subset F(j)$, then $F(i) \subset \text{int}\,(F(j))$ [3], and if k is not a power two, then $F(k) \cap C^1(I, I)$ is a closed subset of $C^1(I, I)$ [5]. In general, if $i \neq j$ and $F(i) \subset F(j)$, then the closure of $F(i) \cap C^1(I, I)$ is a subset of the interior of $F(j) \cap C^1(I, I)$ [6]. Thus, the sets $F(k)$ form a *stratification* of $C^1(I, I)$ with strata $F(3)$, $F(5) - F(3)$, and so on. Any C^1-continuous family of maps f_s moving from one of the strata to another must pass through all the strata in between. New periods are created one at a time, in the Šarkovskii order.

However, inside one of the strata, for example $F(5) - F(3)$, many other bifurcations occur. In this paper we see that at least some of these occur in a predictable pattern, which can be described in terms of orbit types of periodic orbits and which refines the Šarkovskii ordering.

We will let C_n denote the set of cyclic permutations of $\{1, \ldots, n\}$. An *orbit type* is just an element of C_n for some positive integer n. If $X = \{x_1 < x_2 < \cdots < x_n\}$ is a periodic orbit of $f \in C^0(I, I)$, the *orbit type of* X is the permutation F defined by $F(i) = j$ if and only if $f(x_i) = x_j$.

If p is an orbit type we let $F(p)$ denote the set of maps in $C^0(I, I)$ which have a periodic orbit of type p. Unlike the Šarkovskii sets, the sets $F(p)$ are not linearly ordered by inclusion. However, in this paper, we show that any linearly ordered sublattice induces a stratification. More precisely, we prove the following.

THEOREM. *If p and q are orbit types, with $F(p) \subset F(q)$ and $p \neq q$, then $F(p) \subset \text{int}\,(F(q))$, and*

$$\overline{F(p) \cap C^1(I, I)} \subset \text{int}\,(F(q) \cap C^1(I, I)).$$

As noted in [1], it follows from the kneading theory, [7], that the class of unimodal

orbit types is linearly ordered. Thus, the theorem implies that this class yields a stratification of $C^1(I, I)$. This stratification refines the Šarkovskii stratification, in the sense that we may identify the Šarkovskii set $F(n)$ with the set $F(p)$, where p is a 'simple' ([2], [6], [9]) unimodal orbit type of period n. Of course, the theorem applies as well to orbit types which are not unimodal.

The main step in obtaining the desired theorem is the following. The remaining step is Theorem 5, below.

THEOREM 1. *If p and q are orbit types, with $F(p) \subset F(q)$ and $p \neq q$, then $F(p) \subset \text{int}(F(q))$.*

The proof of Theorem 1 relies on ideas from [1] and [4] concerning the relationship between loops in the Markov graph and periodic orbits. Suppose $f \in C^0(I, I)$ and $X = \{x_1, \ldots, x_n\}$ is a periodic orbit of f with $x_1 < x_2 < \cdots < x_n$. Let $I_1 = [x_1, x_2], \ldots,$ $I_{n-1} = [x_{n-1}, x_n]$. The *Markov graph of X* is the graph with vertices $I_1, \ldots, I_{n-1}$ and an arrow from I_i to I_j if and only if $f(I_i) \supset I_j$. If p is an orbit type, the Markov graph of p is the Markov graph of a map f_p with an orbit $X = \{x_1, \ldots, x_n\}$ of type p such that f_p is linear on each interval joining adjacent points of X. Given a loop in the Markov graph of p, the *type* of the loop may be obtained (as in [1]). Baldwin proves in [1] that given two orbit types p and q with $p \neq q$, $F(p) \subset F(q)$ if and only if the Markov graph of p has a non-repetitive loop of type q. We will use this result and also part of Baldwin's proof. We first note the following lemma.

LEMMA 2. *Let $f \in C^0(I, I)$ and let m be a positive integer. Let $K = [c, d]$ be a closed subinterval of I such that the intervals $K, f(K), \ldots, f^{m-1}(K)$ are pairwise disjoint. Suppose that at least one of the following holds:*
   (A) $K \subset \text{int}(f^m(K))$.
   (B) $f^m(c) > c$ and $f^m(d) < d$.
Let $q \in C_m$ be the permutation defined by labelling the disjoint intervals $K, \ldots, f^{m-1}(K)$ as $L_1, \ldots, L_m$ with $L_1 < L_2 < \cdots < L_m$ and setting $q(i) = j$ if $f(L_i) = L_j$ or $L_i = f^{m-1}(K)$ and $L_j = K$. Then $f \in \text{int}(F(q))$.

Proof. The hypothesis of the lemma holds for all g in a neighbourhood of f and implies the existence of a periodic orbit of type q.

Proof of Theorem 1. Suppose $p \in C_n$ and $q \in C_m$. Let $f \in F(p)$, and let $X = \{x_1, \ldots, x_n\}$ be a periodic orbit of type p. By the theorem of Baldwin, there is a non-repetitive loop

$$I_{a_1} \to I_{a_2} \to \cdots \to I_{a_m} \to I_{a_1}$$

in the graph of X which has type q. As in the proof of Theorem 3.3 of [1] we can form closed intervals $J_{2m}, J_{2m-1}, \ldots, J_1, J_0$ such that the following hold:
   (i) $J_{2m} = I_{a_m}, J_{2m-1} \subset I_{a_{m-1}}, \ldots, J_{m+1} \subset I_1$;
        $J_m \subset J_{2m}, J_{m-1} \subset J_{2m-1}, \ldots, J_1 \subset J_{m+1}; J_0 \subset J_m$.
   (ii) $f(\text{int}(J_i)) = \text{int}(J_{i+1})$ for $i = 0, \ldots, 2m - 1$.
   (iii) $J_0, J_1, \ldots, J_{m-1}$ are pairwise disjoint.
   (iv) If z is a periodic point of f in J_0 of period m, then the orbit of z has type q.
We have two cases.

Case 1. $J_0 \subset \text{int} (I_{a_m})$.

Since each endpoint of J_0 is mapped by f^{2m} to an endpoint of I_{a_m}, neither endpoint of J_0 is a fixed point of f^m. Also, the endpoints of J_0 are mapped by f^m onto the endpoints of J_m, and $J_0 \subset J_m$. Thus, if $K = J_0$, the hypothesis of Lemma 2 is satisfied. Hence, $f \in \text{int} (F(q))$.

Case 2. J_0 is not a subset of int (I_{a_m}).

In this case, each of the intervals $J_0, J_1, \ldots, J_{m-1}$ contains a point of X. Thus, if $m = n$, then $p = q$, a contradiction. Hence, $m \neq n$.

   We will show that m is not a multiple of n. Suppose m is a multiple of n. Then $m \geq 2n$. It follows that three of the intervals $J_0, J_1, \ldots, J_{m-1}$ are in the same I_j. Hence for some $k = 0, \ldots, m - 1$, J_k contains no elements of X. It follows that J_0 contains no elements of X, a contradiction to Case 2. Thus, m is not a multiple of n.

   Let $J_0 = [c, d]$. Either c or d is an endpoint of I_{a_m}; we may suppose without loss of generality that c is. Then $c \in X$, and hence, also $f^m(c) \in X$. Since $f^m(c) \neq c$, it follows that f^m reverses the order of the endpoints of J_0. Thus, if $K = J_0$, the hypothesis of Lemma 2 is satisfied, with (B) holding. Hence, $f \in \text{int} (F(q))$, and the proof of Theorem 1 is complete.

Definition. Let $p \in C_n$ and $q \in C_k$. We say $q = p/2$ if the following hold:
   (i) $k = n/2$.
   (ii) If $\beta_1 = \{1, 2\}$, $\beta_2 = \{3, 4\}, \ldots, \beta_k = \{n - 1, n\}$, then $p(\beta_i) = \beta_j$ if and only if $q(i) = j$.

The case $q = p/2$ corresponds to an orbit of type p formed from an orbit of type q by period doubling.

LEMMA 3. *Suppose p and q are orbit types and $q = p/2$. Then*
$$\overline{F(p) \cap C^1(I, I)} \subset \text{int} (F(q) \cap C^1(I, I)).$$

Proof. Suppose f is the limit of a sequence of maps in $F(p) \cap C^1(I, I)$. By Theorem 1 of [5], f has a periodic orbit of type p or q. By Theorem 1, we need only consider the case where f has an orbit X of type q. In this case, the derivative of f^k must be less than or equal to zero at each point of X. In particular, the derivative of f^k is not equal to one at each point of X. Hence, the orbit is preserved under perturbations, and thus, $f \in \text{int} (F(q) \cap C^1(I, I))$.

The following lemma was observed by Baldwin, [1].

LEMMA 4. *Suppose p, q, and r are orbit types with $q = p/2$, $p \neq r$, and $F(p) \subset F(r)$. Then $F(q) \subset F(r)$.*

From Lemma 3 and its proof, Lemma 4, and Theorem 1, we immediately obtain the following.

THEOREM 5. *If p and q are orbit types with $F(p) \subset F(q)$ and $p \neq q$, then*
$$\overline{F(p) \cap C^1(I, I)} \subset \text{int} (F(q) \cap C^1(I, I)).$$

We conclude with an intermediate value type result concerning orbit types and families of maps.

COROLLARY 6. *Suppose f_t is a continuous one-parameter family of C^1 maps of the interval I for $0 \le t \le 1$ (i.e. the map $t \to f_t$ from $[0, 1]$ to $C^1(I, I)$ is continuous). Suppose p, r, and q are orbit types with $F(p) \subset F(r) \subset F(q)$ and $r \ne q$. If $f_1 \in F(p)$ and $f_0 \notin F(r)$ then there is a parameter t_1 with $f_{t_1} \in F(q) - F(r)$.*

Proof. Let t_0 denote the infimum of $\{t \in [0, 1]: f_t \in F(r)\}$. If $f_{t_0} \in F(r)$, then for t_1 slightly less than t_0, $f_{t_1} \in F(q) - F(r)$. If $f_{t_0} \notin F(r)$, then for $t_1 = t_0$, $f_{t_1} \in F(q) - F(r)$.

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