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DIAGONAL CAUCHY SPACES

D.C. KENT AND G.D. RICHARDSON

A diagonal condition is defined which internally characterises those Cauchy spaces which have topological completions. The T_2 diagonal Cauchy spaces allow both a finest and a coarsest T_2 diagonal completion. The former is a completion functor, while the latter preserves uniformisability and has an extension property relative to θ-continuous maps.

INTRODUCTION

In 1954, Kowalsky [**2**] defined a diagonal axiom for convergence spaces subject to which every pretopological space is topological. In 1967, Cook and Fischer [**1**] gave a stronger version of the Kowalsky axiom relative to which a larger class of convergence spaces was shown to be topological. In [4], we showed that any convergence space satisfying the Cook-Fischer diagonal axiom is topological.

In this paper, we introduce a diagonal axiom for Cauchy spaces which reduces to the Cook-Fischer axiom when the Cauchy space is complete. A *diagonal Cauchy space* is one which satisfies this axiom. We show that a Cauchy space is diagonal if and only if every Cauchy equivalence class contains a smallest Cauchy filter, and this smallest Cauchy filter has a base of open sets. Equivalently, a Cauchy space is diagonal if and only if it allows a diagonal (that is, topological) completion. The category of diagonal Cauchy spaces and Cauchy continuous maps is shown to be a topological category.

It is shown that a T_2 diagonal Cauchy space has both a finest and a coarsest T_2 diagonal completion. The "fine diagonal completion" determines a completion functor on the category of T_2, diagonal Cauchy spaces. The "coarse diagonal completion" is finer than any T_3 completion, and thus if a T_3, diagonal completion exists, it necessarily coincides with the coarse diagonal completion. In particular, the coarse diagonal completion preserves uniformisability. A condition is given which is necessary and sufficient for the coarse diagonal completion of a T_3, diagonal Cauchy space to be T_3, and this is followed by an example which shows that the coarse diagonal completion of a totally bounded, T_3, diagonal Cauchy space can fail to be either T_3 or totally bounded.

Finally, we show that the coarse diagonal completion, although not functorial, has the following interesting extension property: Any Cauchy continuous map between T_2, diagonal Cauchy spaces has a θ-continuous extension to the respective coarse diagonal completions.

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1. Diagonal Cauchy spaces

Let X be a set, $\mathbf{F}(X)$ the set of all (proper) filters on X, and 2^X the set of all subsets of X. For $x \in X$, let $\dot{x}$ be the fixed ultrafilter generated by $\{x\}$. For $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$, we write $\mathcal{F} \leqslant \mathcal{G}$ if $\mathcal{F} \subseteq \mathcal{G}$. If $F \cap G \neq \emptyset$, for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$, let $\mathcal{F} \vee \mathcal{G}$ denote the filter generated by $\{F \cap G : F \in \mathcal{F}, G \in \mathcal{G}\}$. If, on the other hand, there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \cap G = \emptyset$, we say that "$\mathcal{F} \vee \mathcal{G}$ fails to exist".

Definition 1.1: A *convergence structure* q on a set X is a function $q : \mathbf{F}(X) \to 2^X$ such that:

(a) $x \in q(\dot{x})$, for all $x \in X$;
(b) $\mathcal{F} \leqslant \mathcal{G} \Rightarrow q(\mathcal{F}) \subseteq q(\mathcal{G})$;
(c) $x \in q(\mathcal{F}) \Rightarrow x \in q(\mathcal{F} \cap \dot{x})$.

The statement $x \in q(\mathcal{F})$ means "$\mathcal{F}$ q-converges to x," which is also written "$\mathcal{F} \xrightarrow{q} x$." If p, q are convergence structures on X, $p \leqslant q$ if and only if $\mathcal{F} \xrightarrow{q} x$ implies $\mathcal{F} \xrightarrow{p} x$. If $p \leqslant q$, we say that "p is coarser than q" or "q is finer than p." If q is a convergence structure on X, then (X, q) is a *convergence space.*

Given a convergence space (X, q) and $x \in X$, let $\mathcal{V}_q(x)$ denote the intersection of all filters which q-converge to x, and for a subset A of X, let $I_q(A) = \{x \in A : A \in \mathcal{V}_q(x)\}$ and $cl_q(A) = \{x \in X :$ there is $\mathcal{F} \xrightarrow{q} x$ such that $A \in \mathcal{F}\}$. $\mathcal{V}_q(x)$ is called the *q-neighbourhood filter at x*, $I_q(A)$ is the *q-interior of A*, and $cl_q(A)$ is the *q-closure of A*. A subset U of X is defined to be *q-open* if $I_q(U) = U$. A convergence structure q on X is called a *pretopology* if $\mathcal{V}_q(x) \xrightarrow{q} x$, for each $x \in X$; a pretopology in which every neighbourhood filter has a base of q-open sets is a *topology.* For any convergence structure q on X, the set of all q-open subsets of X determines a topology τ_q on X, called the *topological modification* of q, which is the finest topology coarser than q.

In 1967, Cook and Fischer [1] introduced a "diagonal condition" for convergence spaces which we shall call Condition F. This condition is defined by means of a "compression operator" for filters which is defined as follows: If J is any set, $\mathcal{F} \in \mathbf{F}(J)$, and $\sigma : J \to \mathbf{F}(X)$ is any function, let $\kappa\sigma\mathcal{F} = \bigcup_{F \in \mathcal{F}} (\bigcap \sigma(y) : y \in F)$. Condition F may now be stated for a convergence space (X, q) as follows:

F: Let J be any set, $\psi : J \to X$, and let $\sigma : J \to \mathbf{F}(X)$ have the property that $\dot{\psi}(y) \geqslant \sigma(y)$ and $\sigma(y) \xrightarrow{q} \psi(y)$, for all $y \in J$. If $\mathcal{F} \in \mathbf{F}(J)$ is such that $\psi(\mathcal{F}) \xrightarrow{q} x$, then $\kappa\sigma\mathcal{F} \xrightarrow{q} x$.

In [4], the authors proved the following theorem.

Theorem 1.2. *A convergence structure q on a set X is a topology if and only if (X, q) satisfies Condition F.*

We shall proceed to formulate an analogous diagonal condition for Cauchy spaces.

First we must review the basic theory of Cauchy spaces.

DEFINITION 1.3: A *Cauchy structure* $\mathcal{C}$ on a set X is a collection of filters on X satisfying:

(C_1) $\dot{x} \in \mathcal{C}$, for all $x \in X$;
(C_2) $\mathcal{F} \in \mathcal{C}$ and $\mathcal{F} \leqslant \mathcal{G}$ implies $\mathcal{G} \in \mathcal{C}$;
(C_3) If $\mathcal{F}, \mathcal{G} \in \mathcal{C}$ and $\mathcal{F} \vee \mathcal{G}$ exists, then $\mathcal{F} \cap \mathcal{G} \in \mathcal{C}$.

If $\mathcal{C}_1, \mathcal{C}_2$ are Cauchy structures on X, we write $\mathcal{C}_1 \leqslant \mathcal{C}_2$ if and only if $\mathcal{C}_2 \subseteq \mathcal{C}_1$; in this case we say "$\mathcal{C}_2$ is finer than $\mathcal{C}_1$" or "$\mathcal{C}_1$ is coarser than $\mathcal{C}_2$." A pair $(X, \mathcal{C})$ consisting of a set X and a Cauchy structure $\mathcal{C}$ on X is called a *Cauchy space*.

For each Cauchy space $(X, \mathcal{C})$, there is an associated convergence structure q_c on X defined by $\mathcal{F} \xrightarrow{q_c} x$ if and only if $\mathcal{F} \cap \dot{x} \in \mathcal{C}$. A Cauchy space $(X, \mathcal{C})$ is $\mathbf{T}_2$ (or *Hausdorff*) if $\dot{x} \cap \dot{y} \in \mathcal{C}$ implies $x = y$. Equivalently, $(X, \mathcal{C})$ is T_2 if and only if (X, q_c) is T_2 in the sense that each q_c-convergent filter has a unique limit. A T_2 Cauchy space $(X, \mathcal{C})$ is $\mathbf{T}_3$ if $cl_{q_c} \mathcal{F} \in \mathcal{C}$ whenever $\mathcal{F} \in \mathcal{C}$, *complete* if every filter $\mathcal{F}$ in $\mathcal{C}$ is q_c-convergent, and *totally bounded* if every ultrafilter on X is in $\mathcal{C}$.

Let $(X, \mathcal{C})$ be a Cauchy space. An equivalence relation $\sim$ on $\mathcal{C}$ is defined as follows: For $\mathcal{F}, \mathcal{G} \in \mathcal{C}$, $\mathcal{F} \sim \mathcal{G}$ if and only if $\mathcal{F} \cap \mathcal{G} \in \mathcal{C}$. If $\mathcal{F} \in \mathcal{C}$, let $[\mathcal{F}]_\mathcal{C} = \{\mathcal{G} \in \mathcal{C} : \mathcal{F} \sim \mathcal{G}\}$ be the equivalence class determined by $\mathcal{F}$; this equivalence class is denoted simply by $[\mathcal{F}]$ if there is no ambiguity.

If $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ are two Cauchy spaces, $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is *Cauchy continuous* if $\mathcal{F} \in \mathcal{C}$ implies $f(\mathcal{F}) \in \mathcal{D}$. If $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is Cauchy continuous, then obviously $f : (X, q_c) \to (Y, q_\mathcal{D})$ is *continuous* in the sense that $\mathcal{F} \xrightarrow{q_c} x$ implies $f(\mathcal{F}) \xrightarrow{q_\mathcal{D}} f(x)$. If $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is a bijection such that f and f^{-1} are both Cauchy continuous, then f is a *Cauchy isomorphism*.

Let $(X, \mathcal{C})$ be a Cauchy space and $A \subseteq X$. $\mathcal{F} \in \mathbf{F}(X)$ has a *trace* on A if $F \cap A \neq \emptyset$, for all $F \in \mathcal{F}$; in this case $\mathcal{F}_A = \{F \cap A : F \in \mathcal{F}\}$ denotes the trace of $\mathcal{F}$ on A. $\mathcal{C}_A = \{\mathcal{F}_A : \mathcal{F} \in \mathcal{C}, \mathcal{F}$ has a trace on $A\}$ is a Cauchy structure on A, and $(A, \mathcal{C}_A)$ is a *Cauchy subspace* of $(X, \mathcal{C})$. If $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is such that $f : (X, \mathcal{C}) \to (f(X), \mathcal{D}_{f(X)})$ is a Cauchy isomorphism, then $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is called a *Cauchy embedding*.

We next define the diagonal Condition D for a Cauchy space $(X, \mathcal{C})$. The following notation will be useful: $\mathcal{N}_\mathcal{C} = \{\mathcal{G} \in \mathcal{C} : \mathcal{G}$ is non-q_c-convergent $\}$.

D: Let J be any set, $\psi : J \to X \cup \{[\mathcal{G}] : \mathcal{G} \in \mathcal{N}_\mathcal{C}\}$, and let $\sigma : J \to \mathcal{C}$ be such that $\dot{\psi(y)} \geqslant \sigma(y)$ and $\sigma(y) \xrightarrow{q_c} \psi(y)$ if $\psi(y) \in X$ and $\sigma(y) \in [\mathcal{G}]$ if $\psi(y) = [\mathcal{G}]$, where $\mathcal{G} \in \mathcal{N}_\mathcal{C}$. If $\mathcal{F}$ is a filter on J such that $\psi(\mathcal{F})$ is finer than the filter on $X \cup \{[\mathcal{G}] : \mathcal{G} \in \mathcal{N}_\mathcal{C}\}$ generated by $\mathcal{H} \cap [\dot{\mathcal{H}}]$, for some $\mathcal{H} \in \mathcal{C}$, then $\kappa\sigma\mathcal{F} \in \mathcal{C}$.

Observe that Condition D is equivalent to Condition F if $(X,\mathcal{C})$ is a complete Cauchy space (in which case $\mathcal{C}$ can be identified with q_c).

A Cauchy space $(X,\mathcal{C})$ which satisfies Condition D will be called a *diagonal Cauchy space*. We shall now establish some properties of diagonal Cauchy spaces.

THEOREM 1.4. *If $(X,\mathcal{C})$ is a diagonal Cauchy space, then q_c is a topology.*

PROOF: By Theorem 1.2, it is sufficient to show that (X,q_c) satisfies Condition F. If J,ψ, and σ are as specified in Condition F and $\mathcal{F}$ is a filter on J such that $\psi(\mathcal{F}) \xrightarrow{q_c} x$ (which is equivalent to $\psi(\mathcal{F}) \cap \dot{x} \in \mathcal{C}$), then we are dealing with a special case of Condition D where $\psi(J) \subseteq X$, and we conclude by the latter condition that $\kappa\sigma\mathcal{F} \in \mathcal{C}$. But $\psi(\mathcal{F}) \geqslant \kappa\sigma\mathcal{F}$, and $(\psi(\mathcal{F}) \cap \dot{x}) \vee \kappa\sigma\mathcal{F}$ exists, so by Definition 1.3 (C_3), $(\kappa\sigma\mathcal{F}) \cap \dot{x} \in \mathcal{C}$, or in other words, $\kappa\sigma\mathcal{F} \xrightarrow{q_c} x$. Thus Condition F is satisfied. ∎

PROPOSITION 1.5. *If $(X,\mathcal{C})$ is a diagonal Cauchy space and $\mathcal{G} \in \mathcal{C}$, then there is $\mathcal{H} \in \mathcal{C}$ such that $\mathcal{H} \leqslant \mathcal{G}$ and $\mathcal{H}$ has a filter base of q_c-open sets.*

PROOF: Let $J = X$ and let $\mathcal{G} \in \mathcal{C}$. Let $\sigma(x) = \mathcal{V}_{q_c}(x)$, the q_c-neighbourhood filter at x, for all $x \in X$. Let $\mathcal{H} = \kappa\sigma\mathcal{G}$. If $G \in \mathcal{G}$, then $\bigcap\limits_{x\in G} \sigma(x) = \bigcap\limits_{x\in G} \mathcal{V}_{q_c}(x)$ is a filter with a base of q_c-open sets by Theorem 1.4, and $\mathcal{H} = \kappa\sigma\mathcal{G} = \bigcup\limits_{G\in\mathcal{G}} \Big(\bigcap\limits_{x\in G} \sigma(x)\Big)$ is likewise a filter with a base of q_c-open sets. Note that $\mathcal{H} \in \mathcal{C}$ by Condition D, and $\mathcal{H} \leqslant \mathcal{G}$ is obvious from the construction of $\kappa\sigma\mathcal{G}$. ∎

THEOREM 1.6. *If $(X,\mathcal{C})$ is a diagonal Cauchy space, then for each $\mathcal{G} \in \mathcal{C}$, $[\mathcal{G}]$ contains a smallest filter $\mathcal{G}_{\min}$, and $\mathcal{G}_{\min}$ has a filter base of q_c-open sets.*

PROOF: If $\mathcal{G} \xrightarrow{q_c} x$, then $\mathcal{G}_{\min} = \mathcal{V}_{q_c}(x)$, and $\mathcal{V}_{q_c}(x)$ has a filter base of q_c-open sets by Theorem 1.4. Assume $\mathcal{G} \in \mathcal{N}_C$ and let $\{\mathcal{G}_\alpha : \alpha \in A\}$ be a set of filters in $\mathcal{G}$ such that $\bigcap[\mathcal{G}] = \bigcap\{\mathcal{G}_\alpha : \alpha \in A\}$. Let $J = X \cup A$. Let $\psi : J \to X \cup \{[\mathcal{H}] : \mathcal{H} \in \mathcal{N}_C\}$ be defined by $\psi(x) = x$, for all $x \in X$, and $\psi(\alpha) = [\mathcal{G}]$, for all $\alpha \in A$. Let $\sigma : J \to \mathcal{C}$ be defined by $\sigma(x) = \mathcal{V}_{q_c}(x)$, for all $x \in X$ and $\sigma(\alpha) = \mathcal{G}_\alpha$, for all $\alpha \in A$. Let $\mathcal{F}$ be the filter on J with filter base $\{G \cup A : G \in \mathcal{G}\}$. Then $\psi(\mathcal{F})$ is the filter generated by $\{G \cup \{[\mathcal{G}]\} : G \in \mathcal{G}\}$, and so by Condition D, $\kappa\sigma\mathcal{F} \in \mathcal{C}$. Note that for each $G \cup A \in \mathcal{F}$, $\bigcap\limits_{y\in G\cup A} \sigma(y) \leqslant \bigcap\{\mathcal{G}_\alpha : \alpha \in A\} = \bigcap[\mathcal{G}]$, and since $\kappa\sigma\mathcal{F} \leqslant \bigcap[\mathcal{G}]$ and $\kappa\sigma\mathcal{F} \in [\mathcal{G}]$, $\kappa\sigma\mathcal{F} = \mathcal{G}_{\min}$ is the smallest filter in $[\mathcal{G}]$. The fact that $\mathcal{G}_{\min}$ has a base of q_c-open sets follows by Proposition 1.5. ∎

Let DCHY be the category whose objects are diagonal Cauchy spaces and whose morphisms are Cauchy continuous maps.

THEOREM 1.7. *DCHY is a topological category.*

PROOF: It suffices to show that Condition D is preserved under the formation of initial structures. Let $\{(X_\alpha, \mathcal{C}_\alpha) : \alpha \in A\}$ be a collection of Cauchy spaces, each satisfying Condition D. Let X be a set and $f_\alpha : X \to X_\alpha$ a family of maps indexed by A. Let $\mathcal{C}$ be the initial Cauchy structure on X induced by the collections $\{f_\alpha : \alpha \in A\}$ and $\{(X_\alpha, \mathcal{C}_\alpha) : \alpha \in A\}$. As is well known, $\mathcal{F} \in \mathcal{C}$ if and only if $f_\alpha(\mathcal{F}) \in \mathcal{C}_\alpha$, for all $\alpha \in A$.

Let J be any set, $\psi : J \to X \cup \{[\mathcal{G}]_\mathcal{C} : \mathcal{G} \in \mathcal{N}_\mathcal{C}\}$ and $\sigma : J \to \mathcal{C}$ such that $\dot{\psi}(y) \geqslant \sigma(y)$ and $\sigma(y) \overset{q_\mathcal{C}}{\to} \psi(y)$ if $\psi(y) \in X$ and $\sigma(y) \in [\mathcal{G}]$ if $\psi(y) = [\mathcal{G}]$. Let $\mathcal{F} \in \mathbf{F}(J)$ be such that $\psi(\mathcal{F})$ is finer than the filter on $X \cup \{[\mathcal{G}]_\mathcal{C} : \mathcal{G} \in \mathcal{N}_\mathcal{C}\}$ generated by $\mathcal{H} \cap [\dot{\mathcal{H}}]$, for some $\mathcal{H} \in \mathcal{C}$. We must show $\kappa\sigma\mathcal{F} \in \mathcal{C}$.

Let $\alpha \in A$ be fixed. Define $\psi_\alpha : J \to X_\alpha \cup \{[\mathcal{G}]_{\mathcal{C}_\alpha} : \mathcal{G} \in \mathcal{N}_{\mathcal{C}_\alpha}\}$ as follows:

1. $\psi_\alpha(y) = f_\alpha(\psi(y))$ if $\psi(y) \in X$.
2. $\psi_\alpha(y) = \lim f_\alpha(\mathcal{G})$ if $\psi(y) = [\mathcal{G}]_\mathcal{C}$ and $f_\alpha(\mathcal{G})$ is $q_{\mathcal{C}_\alpha}$-convergent.
3. $\psi_\alpha(y) = [f_\alpha(\mathcal{G})]_{\mathcal{C}_\alpha}$ if $\psi(y) = [\mathcal{G}]_\mathcal{C}$ and $f_\alpha(\mathcal{G})$ is not $q_{\mathcal{C}_\alpha}$-convergent.

Let $\sigma_\alpha : J \to \mathcal{C}_\alpha$ be defined by: $\sigma_\alpha(y) = \mathcal{V}_{q_{\mathcal{C}_\alpha}}(\psi_\alpha(y))$ if $\psi_\alpha(y)$ is defined by 1 or 2; $\sigma_\alpha(y) = f_\alpha(\mathcal{G}_{\min})$ if $\psi_\alpha(y)$ is defined as in 3.

Note that for all y in J, $f_\alpha(\sigma(y)) \geqslant \sigma_\alpha(y)$. Let $\mathcal{F} \in \mathbf{F}(J)$ be as above. Then for each $\alpha \in A$, $\psi_\alpha(\mathcal{F})$ is finer than the filter on $X_\alpha \cup \{[\mathcal{G}]_{\mathcal{C}_\alpha} : \mathcal{G} \in \mathcal{N}_{\mathcal{C}_\alpha}\}$ generated by $f_\alpha(\mathcal{H}) \cap f_\alpha(\dot{\mathcal{H}})$. By the assumption that $(X_\alpha, \mathcal{C}_\alpha)$ satisfies D, we have that $\kappa\sigma_\alpha(\mathcal{F}) \in \mathcal{C}_\alpha$. But since $f_\alpha(\sigma(y)) \geqslant \sigma_\alpha(y)$ holds for all $y \in J$, it follows that $f_\alpha(\kappa\sigma\mathcal{F}) \geqslant \kappa\sigma_\alpha\mathcal{F}$, which implies $f_\alpha(\kappa\sigma\mathcal{F}) \in \mathcal{C}_\alpha$. Since $\alpha \in A$ is arbitrary, $\kappa\sigma\mathcal{F} \in \mathcal{C}$. ∎

2. Diagonal Cauchy completions

Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be Cauchy spaces and $\phi : X \to Y$. Then $((Y, \mathcal{D}), \phi)$ is a completion of $(X, \mathcal{C})$ if:

(1) $(Y, \mathcal{D})$ is complete;
(2) $\phi : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is a dense embedding.

Here, "dense" means that $cl_{q_\mathcal{D}} \phi(X) = Y$, where $cl_{q_\mathcal{D}}$ denotes the $q_\mathcal{D}$-closure operator.

A completion $((Y, \mathcal{D}), \phi)$ of $(X, \mathcal{C})$ is said to be *diagonal* (respectively, $\mathbf{T}_2$) if $(Y, \mathcal{D})$ is a diagonal (respectively, T_2) Cauchy space. Since the properties of being diagonal and T_2 are both hereditary (in the former case, by Theorem 1.7), any Cauchy space having a diagonal (or T_2) completion must itself be diagonal (or T_2). Note that a complete Cauchy space $(Y, \mathcal{D})$ is diagonal if and only if any one of the following equivalent conditions holds: (1) $(Y, \mathcal{D})$ satisfies Condition D; (2) $(Y, q_\mathcal{D})$ satisfies Condition F; (3) $(Y, q_\mathcal{D})$ is a topological space.

Theorem 2.1. *The following statements about a Cauchy space $(X, \mathcal{C})$ are equivalent.*

(1) *$(X, \mathcal{C})$ is a diagonal Cauchy space.*

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(2) *Each equivalence class* $[\mathcal{F}]$ *in* $\mathcal{C}$ *contains a smallest filter* $\mathcal{F}_{\min}$, *and* $\mathcal{F}_{\min}$ *has a base of* q_c*-open sets.*

(3) $(X,\mathcal{C})$ *has a diagonal Cauchy completion.*

PROOF: (1)⇒(2). This is Theorem 1.6.

(3)⇒(1). This follows by a remark in the paragraph preceding the theorem.

(2)⇒(3). Let $(X,\mathcal{C})$ be a Cauchy space as described in (2). If $\mathcal{F} \xrightarrow{q_c} X$, then $\mathcal{F}_{\min} = \mathcal{V}_{q_c}(x)$, and it follows by (2) that q_c is a topology.

Let $X^\sim = X \cup \{[\mathcal{F}] : \mathcal{F} \in \mathcal{N}_\mathcal{C}\}$, and let $\psi : X \to X^\sim$ be the identity injection. For each $A \subseteq X$, let $A^\sim = A \cup \{[\mathcal{F}] : \mathcal{F} \in \mathcal{N}_\mathcal{C}$ and $A \in \mathcal{F}_{\min}\}$. Noting that $(A \cap B)^\sim = A^\sim \cap B^\sim$, let $\mathcal{G} \in \mathbf{F}(X)$, and define $\mathcal{G}^\sim \in \mathbf{F}(X^\sim)$ to be the filter with base $\{G^\sim : G \in \mathcal{G}\}$. Define $\mathcal{C}^\sim = \{\mathcal{A} \in \mathbf{F}(X^\sim) : \exists \mathcal{F} \in \mathcal{C}$ such that $\mathcal{A} \geqslant \mathcal{F}^\sim_{\min}\}$; it is a simple matter to verify that $\mathcal{C}^\sim$ is a Cauchy structure on $X^\sim$. For $\mathcal{F} \in \mathcal{C}$, $\psi(\mathcal{F}) \geqslant \mathcal{F}^\sim$ and $\psi^{-1}(\mathcal{F}^\sim) = \mathcal{F}$; from these observations, it follows that $\psi : (X,\mathcal{C}) \to (X^\sim,\mathcal{C}^\sim)$ is a Cauchy embedding. Since $[\dot{\mathcal{F}}] \geqslant \mathcal{F}^\sim_{\min}$ for all $\mathcal{F} \in \mathcal{N}_\mathcal{C}$, $\mathcal{F}^\sim_{\min} \cap [\dot{\mathcal{F}}] \in \mathcal{C}^\sim$, and hence $\mathcal{F}^\sim_{\min} \xrightarrow{q^\sim} [\mathcal{F}]$ for all $\mathcal{F} \in \mathcal{N}_\mathcal{C}$, where $q^\sim$ denotes the convergence structure associated with $\mathcal{C}^\sim$. Since $\mathcal{V}_{q_c}(x)^\sim \xrightarrow{q^\sim} X$, for all $x \in X$, it follows that $(X^\sim,\mathcal{C}^\sim)$ is complete. Finally, $\mathcal{F}^\sim \xrightarrow{q^\sim} [\mathcal{F}]$ for all $\mathcal{F} \in \mathcal{N}_\mathcal{C}$ and $\psi(\mathcal{F}) \geqslant \mathcal{F}^\sim$ imply that $cl_{q^\sim}\psi(X) = X^\sim$, so the embedding ψ is dense.

It remains to show that $(X^\sim,\mathcal{C}^\sim)$ is a diagonal Cauchy space. First observe that $(X^\sim, q^\sim)$ is pretopological (meaning that the neighbourhood filter at each point in $X^\sim$ converges to that point). To show that each $q^\sim$-neighbourhood filter has a filter base of $q^\sim$-open sets, it suffices to show that $U^\sim$ is $q^\sim$-open whenever U is q_c-open. If $x \in U^\sim$, then $U \in \mathcal{V}_{q_c}(x)$, and hence $U^\sim \in \mathcal{V}q_c(x)^\sim = \mathcal{V}_{q^\sim}(x)$. If $[\mathcal{F}] \in U^\sim$, where $\mathcal{F} \in \mathcal{N}_\mathcal{C}$, then $U \in \mathcal{F}_{\min}$ implies $U^\sim \in \mathcal{F}^\sim_{\min} = \mathcal{V}_{q^\sim}([\mathcal{F}])$. Thus $U^\sim$ is a $q^\sim$-open set, since it is a $q^\sim$-neighbourhood of each of its points, and the proof is complete. ∎

Observe that the implication (2)⇒(1) in Theorem 2.1 is the converse of Theorem 1.6.

THEOREM 2.2. *If* $(X,\mathcal{C})$ *is a* T_2, *diagonal Cauchy space, the diagonal completion* $((X^\sim,\mathcal{C}^\sim),\psi)$ *constructed in the preceding proof is likewise* T_2.

PROOF: If $[\dot{\mathcal{F}}] \cap [\dot{\mathcal{G}}] \in \mathcal{C}^\sim$, where $\mathcal{F},\mathcal{G} \in \mathcal{N}_\mathcal{C}$, then $\mathcal{F}_{\min} \cap \mathcal{G}_{\min} \in \mathcal{C}$, which implies $\mathcal{F}_{\min} = \mathcal{G}_{\min}$, and hence $[\mathcal{F}] = [\mathcal{G}]$. If $\dot{x} \cap \dot{y} \in \mathcal{C}^\sim$, then the intersection of the restrictions of $\dot{x}$ and $\dot{y}$ to X is in $\mathcal{C}$, and since $(X,\mathcal{C})$ is T_2, $x = y$. If $[\dot{\mathcal{F}}] \cap \dot{x} \in \mathcal{C}^\sim$, where $\mathcal{F} \in \mathcal{N}_\mathcal{C}$ and $x \in X$, it follows easily that $\mathcal{F}_{\min} \xrightarrow{q_c} x$, a contradiction, and so this case is impossible. We conclude that $(X^\sim,\mathcal{C}^\sim)$ is T_2. ∎

Let $(X,\mathcal{C})$ be any T_2 Cauchy space, let $X^* = \{[\mathcal{F}] : \mathcal{F} \in \mathcal{C}\}$, and let $j : X \to X^*$ be defined by $j(x) = [\dot{x}]$, for all $x \in X$. A T_2 completion $((Y,\mathcal{D}),\phi)$ of $(X,\mathcal{C})$ is said

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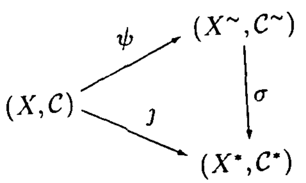
to be in *standard form* if $Y = X^*$, $\phi = \jmath$ and $\jmath(\mathcal{F}) \xrightarrow{q_{\mathcal{D}}} [\mathcal{F}]$. Reed, [**3**], showed that every T_2 completion of a Cauchy space $(X,\mathcal{C})$ is equivalent to one in standard form.

In case $(X,\mathcal{C})$ is a T_2, diagonal Cauchy space, it will be convenient to give a description of the preceding completion in standard form. Let $\mathcal{C}^\circ = \{\mathcal{F} \in \mathcal{C} : \mathcal{F}$ has a filter base of $q_{\mathcal{C}}$-open sets $\}$. Let X^* and $\jmath$ be defined as in the preceding paragraph. For $A \subseteq X$ and $\mathcal{G} \in \mathcal{C}^\circ$, let $A^* = \{[\mathcal{F}] \in X^* : A \in \mathcal{F}_{\min}\}$ and let $\mathcal{G}^*$ be the filter on X^* generated by $\{A^* : A \in \mathcal{G}\}$. Let $\mathcal{C}^* = \{\mathcal{A} \in \mathbf{F}(X^*) :$ there is $\mathcal{F} \in \mathcal{C}^\circ$ such that $\mathcal{F}^* \leqslant \mathcal{A}\}$. One easily verifies that $((X^*,\mathcal{C}^*),\jmath)$ is a T_2, diagonal completion of $(X,\mathcal{C})$ in standard form, and if $\sigma : X^\sim \to X^*$ is defined by $\sigma(x) = [\dot{x}]$ for $x \in X$, and $\sigma([\mathcal{F}]) = [\mathcal{F}]$ for $\mathcal{F} \in \mathcal{N}_{\mathcal{C}}$, then $\sigma : (X^\sim,\mathcal{C}^\sim) \to (X^*,\mathcal{C}^*)$ is a Cauchy isomorphism which makes the following diagram commute:



This establishes the equivalence of the completions $((X^*,\mathcal{C}^*),\jmath)$ and $((X^\sim,\mathcal{C}^\sim),\psi)$ of a T_2, diagonal Cauchy space $(X,\mathcal{C})$.

THEOREM 2.3. *Let $(X,\mathcal{C})$ be a T_2, diagonal Cauchy space. Let $((X^*,\mathcal{C}^*),\jmath)$ be the T_2, diagonal completion of $(X,\mathcal{C})$ defined in the preceding paragraph, and let $((X^*,\mathcal{D}),\jmath)$ be any T_2, diagonal completion of $(X,\mathcal{C})$ in standard form. Then $\mathcal{D} \subseteq \mathcal{C}^*$.*

PROOF: By analogy to the proof of Theorem 2.1, we see that sets of the form U^*, where U is $q_{\mathcal{C}}$-open, form a basis for the topology q^* of $(X^*,\mathcal{C}^*)$. Thus it suffices to show that if U is $q_{\mathcal{C}}$-open, then U^* is $q_{\mathcal{D}}$-open.

Let U be $q_{\mathcal{C}}$-open and let $[\mathcal{F}] \in U^*$. Then $U \in \mathcal{F}_{\min}$ and $\jmath(\mathcal{F}_{\min}) \geqslant \mathcal{V}_{q_{\mathcal{D}}}([\mathcal{F}])$, the $q_{\mathcal{D}}$-neighbourhood filter at $[\mathcal{F}]$. Thus $\mathcal{F}_{\min} \geqslant \jmath^{-1}\left(\mathcal{V}_{q_{\mathcal{D}}}([\mathcal{F}])\right)$, and since the latter filter is in $\mathcal{C}$, it follows from the minimality of $\mathcal{F}_{\min}$ that $\mathcal{F}_{\min} = \jmath^{-1}\left(\mathcal{V}_{q_{\mathcal{D}}}([\mathcal{F}])\right)$. Since $U \in \mathcal{F}_{\min}$, there is a $q_{\mathcal{D}}$-open set $W \in \mathcal{V}_{q_{\mathcal{D}}}([\mathcal{F}])$ such that $\jmath^{-1}(W) \subseteq U$. To see that $W \subseteq U^*$, let $[\mathcal{G}] \in W$; then $\jmath(\mathcal{G}_{\min}) \xrightarrow{q_{\mathcal{D}}} [\mathcal{G}]$, and since W is $q_{\mathcal{D}}$-open, $\jmath^{-1}(W) \in \mathcal{G}_{\min}$, which implies $U \in \mathcal{G}_{\min}$, and hence $[\mathcal{G}] \in U^*$. Thus U^* is $q_{\mathcal{D}}$-open. ∎

The preceding theorem shows that $((X^*,\mathcal{C}^*),\jmath)$ is (up to equivalence) the coarsest T_2, diagonal completion of a T_2, diagonal Cauchy space $(X,\mathcal{C})$; for this reason, $((X^*,\mathcal{C}^*),\jmath)$ will be called the *coarse diagonal completion* of $(X,\mathcal{C})$. The next example shows that a T_2, diagonal Cauchy space may have T_2, diagonal completions which are not equivalent to the coarse diagonal completion.

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EXAMPLE 2.4. Let X be the set Q of rational numbers, and let $\mathcal{C}$ be the usual Cauchy structure for Q. In this case, we can make the usual identification between the set Q^* and the set R of real numbers, in which case $\jmath : Q \to R$ becomes the identity injection. The Cauchy structure $\mathcal{C}^*$ on R of the coarse diagonal completion is the usual (complete) Cauchy structure for R which gives rise to the usual topology. Next, consider the finest topology τ on R which contains all sets open in the usual topology along with the set Q. If $y \in R \setminus Q$, $\mathcal{V}_\tau(y)$ has a filter base of open intervals of the form $(y-\varepsilon, y+\varepsilon)$, where $\varepsilon > 0$, but if $y \in Q$, $\mathcal{V}_\tau(y)$ has a filter base of sets of the form $(y-\varepsilon, y+\varepsilon) \cap Q$. If $\mathcal{C}' = \{\mathcal{F} \in \mathbf{F}(R) :$ there is $y \in R$ such that $\mathcal{F} \geqslant \mathcal{V}_\tau(y)\}$, then $\left(\left(R, \mathcal{C}'\right), \jmath\right)$ is a T_2, diagonal completion of $(X, \mathcal{C})$ which is obviously not equivalent to $((R, \mathcal{C}^*), \jmath)$.

Recall that a Cauchy space $(X, \mathcal{C})$ is T_3 if it is T_2 and has the property that $cl_{q_\mathcal{C}}(\mathcal{F}) \in \mathcal{C}$ wherever $\mathcal{F} \in \mathcal{C}$. The next result shows that the coarse diagonal completion is at least as fine as any T_3 completion of a T_2, diagonal Cauchy space.

PROPOSITION 2.5. *If $((X^*, \mathcal{D}), \jmath)$ is any T_3 completion (in standard form) of a T_2, diagonal Cauchy space $(X, \mathcal{C})$, then $\mathcal{C}^* \subseteq \mathcal{D}$.*

Proof: If $\mathcal{A} \in \mathcal{C}^*$, then $\mathcal{A} \geqslant \mathcal{F}^*_{\min}$, for some $\mathcal{F} \in \mathcal{C}$. But $\mathcal{F}^*_{\min} \geqslant cl_{\mathcal{D}}\jmath(\mathcal{F}_{\min})$, and it follows by the T_3 property that $\mathcal{A} \in \mathcal{D}$. □

The next theorem establishes that the coarse diagonal completion preserves uniformisability.

THEOREM 2.6. *Let $(X, \mathcal{U})$ be a T_2 uniform space, and let $\mathcal{C} = \{\mathcal{F} \in \mathbf{F}(X) : \mathcal{U} \leqslant \mathcal{F} \times \mathcal{F}\}$ be the associated Cauchy structure. Let $((X^*, \mathcal{U}^\wedge), \jmath)$ be the uniform completion of $(X, \mathcal{U})$ in standard form, and let $\mathcal{C}^\wedge = \{\mathcal{A} \in \mathbf{F}(X^*) : \mathcal{U}^\wedge \leqslant \mathcal{A} \times \mathcal{A}\}$. Then $\mathcal{C}^* = \mathcal{C}^\wedge$.*

PROOF: Observe that $(X, \mathcal{C})$ is a T_2, diagonal Cauchy space, and $((X^*, \mathcal{C}^\wedge), \jmath)$ is the "uniformisable Cauchy completion" of $(X, \mathcal{C})$. The latter completion is clearly a T_2, diagonal completion, so $\mathcal{C}^\wedge \subseteq \mathcal{C}^*$ follows by Theorem 2.3. But any uniformisable completion is also T_3, so it follows by Proposition 2.5 that $\mathcal{C}^* \subseteq \mathcal{C}^\wedge$. □

Indeed, the following more general result follows as in the preceding proof from Theorem 2.3 and Proposition 2.5.

COROLLARY 2.7. *Any T_3, diagonal completion of a T_3, diagonal Cauchy space is the coarse diagonal completion.*

The next theorem gives a condition which is necessary and sufficient for the coarse diagonal completion of a T_3, diagonal Cauchy space to be T_3.

Let $(X, \mathcal{C})$ be a T_2, diagonal Cauchy space, and for $A \subseteq X$, let $A^{**} = \{[\mathcal{F}] \in X^* : \mathcal{F}_{\min}$ has a trace on $A\}$. Note that $A^* \subseteq A^{**}$ and $\jmath^{-1}(A^{**}) = cl_{q_\mathcal{C}} A$ for all $A \subseteq X$; furthermore, if $\mathcal{F} \in \mathbf{F}(X)$, then the filter $\mathcal{F}^{**}$ generated by $\{F^{**} : F \in \mathcal{F}\}$ is a proper filter on X^*.

THEOREM 2.8. *Let $(X,\mathcal{C})$ be a T_3, diagonal Cauchy space. Then the coarse diagonal completion of $(X,\mathcal{C})$ is T_3 if and only if $\mathcal{F}^*_{\min} = \mathcal{F}^{**}_{\min}$, for each $\mathcal{F} \in \mathcal{C}$.*

PROOF: Let q^* be the convergence structure associated with $(X^*,\mathcal{C}^*)$. One easily verifies that for any $\mathcal{F} \in \mathcal{C}$, $\mathcal{F}^{**}_{\min} = cl_{q^*}(\mathcal{F}_{\min})$. If $(X^*,\mathcal{C}^*)$ is T_3, then $\mathcal{F}^*_{\min} = \mathcal{F}^{**}_{\min}$ follows immediately. On the other hand, if $\mathcal{F}^*_{\min} = \mathcal{F}^{**}_{\min}$ for every $\mathcal{F} \in \mathcal{C}$, then $\mathcal{F}^*_{\min}$ has a filter base of q^*-closed sets for every $\mathcal{F} \in \mathcal{C}$, and therefore $(X^*,\mathcal{C}^*)$ is T_3. ∎

The following example shows that the coarse diagonal completion need not preserve either the T_3 property or total boundedness.

EXAMPLE 2.9. Let X be an infinite set, partitioned into infinite subsets $\{X_i : i \in N\}$, where N is the set of natural numbers. For each $i \in N$, let $\mathcal{G}_i$ be the filter on X with filter base $\{X_i \setminus A : A \text{ a finite subset of } X_i\}$. Let $\mathcal{C} = \{\mathcal{H} \in \mathbf{F}(X) : \exists\, i \in N$ such that $\mathcal{H} \geqslant \mathcal{G}_i\} \cup \{\mathcal{F} : \mathcal{F}$ a free ultrafilter on $X\} \cup \{\dot{x} : x \in X\}$. Note that $\mathcal{C}$ is a Cauchy structure on X and q_c is discrete; hence $(X,\mathcal{C})$ is a totally bounded, T_3, diagonal Cauchy space.

Let $\mathcal{G}$ be any free ultrafilter on X such that each $G \in \mathcal{G}$ has an infinite intersection with infinitely many of the X_i's. Then $\mathcal{F} = \mathcal{G}_{\min}$ and $\mathcal{G}^*_{\min} = \jmath(\mathcal{G}) \cap [\dot{\mathcal{G}}]$, but $\mathcal{G}^{**}_{\min}$ has the property that for each $G \in \mathcal{G}$, G^{**} contains infinitely many of the $[\mathcal{G}_i]$'s. Thus $\mathcal{G}^{**}_{\min} \neq \mathcal{G}^*_{\min}$, so by Theorem 2.8, $(X^*,\mathcal{C}^*)$ is not T_3. Furthermore, $(X^*,\mathcal{C}^*)$ is not totally bounded; otherwise, $((X^*,q^*),\jmath)$ would be a T_2, topological compactification of (X,q_c), which would imply that $(X^*,\mathcal{C}^*)$ is T_3.

It comes as no surprise that the coarse diagonal completion for T_2, diagonal Cauchy spaces does not behave "functorially". For if $(Q,\mathcal{C})$ and $\left(R,\mathcal{C}'\right)$ are the T_2, diagonal Cauchy spaces defined in Example 2.4 and $\jmath : (Q,\mathcal{C}) \to \left(R,\mathcal{C}'\right)$ is the identity injection, then $\jmath$ is Cauchy continuous, but $\jmath$ obviously has no Cauchy continuous extension $\jmath^* : (Q^*,\mathcal{C}^*) \to \left(R^*,\left(\mathcal{C}'\right)^*\right) = \left(R,\mathcal{C}'\right)$.

There is, however, a completion functor on the category T_2DCHY of T_2 diagonal Cauchy spaces and Cauchy continuous maps. This completion is, for obvious reason, called the *fine diagonal completion* and is denoted (in standard form) by $((X^*,\mathcal{C}^*),\jmath)$. The construction of this completion is briefly described below.

Given an arbitrary T_2 Cauchy space $(X,\mathcal{C})$, let $((X^*,\mathcal{W}),\jmath)$ denote the *Wyler completion* (see [5]) of $(X,\mathcal{C})$, where $\mathcal{W} = \{\mathcal{A} \in \mathbf{F}(X^*)$: there is $\mathcal{F} \in \mathcal{C}$ such that $\mathcal{A} \geqslant \jmath(\mathcal{F}) \cap [\dot{\mathcal{F}}]\}$. The Wyler completion defines a completion functor on the category T_2CHY of all T_2 Cauchy spaces, but this completion does not preserve the diagonal property. Let $\mathcal{C}^\star$ be the complete Cauchy structure on X^* associated with the topological modification $\tau_{q_{\mathcal{W}}}$ of $q_{\mathcal{W}}$.

Now let $(X,\mathcal{C})$ be a T_2, diagonal Cauchy space. It is clear that $\mathcal{C}^* \subseteq \mathcal{C}^\star \subseteq \mathcal{W}$. Since $\jmath : (X,\mathcal{C}) \to (X^*,\mathcal{C}^*)$ and $\jmath : (X,\mathcal{C}) \to (X^*,\mathcal{W})$ are both Cauchy embeddings, $\jmath : (X,\mathcal{C}) \to (X^*,\mathcal{C}^\star)$ is also a Cauchy embedding, and hence $((X^*,\mathcal{C}^\star),\jmath)$ is a T_2, diagonal completion of $(X,\mathcal{C})$. Since $\mathcal{W}$ is the finest T_2 completion structure for $(X,\mathcal{C})$ in standard form, $\mathcal{C}^\star$ is the finest diagonal completion structure for a T_2, diagonal Cauchy space $(X,\mathcal{C})$ in standard form. Furthermore, since the Wyler completion and the topological modification are both functorial, we obtain the following extension theorem for the fine diagonal completion.

THEOREM 2.10. *Let $(X,\mathcal{C})$ and $(Y,\mathcal{D})$ be T_2, diagonal Cauchy spaces, and let $f : (X,\mathcal{C}) \to (Y,\mathcal{D})$ be Cauchy continuous. Then there is a unique Cauchy continuous function $f^\star : (X^*,\mathcal{C}^\star) \to (Y^*,\mathcal{D}^\star)$ such that the following diagram commutes:*

$$\begin{array}{ccc} (X,\mathcal{C}) & \xrightarrow{\ f\ } & (Y,\mathcal{D}) \\ \big\downarrow{\scriptstyle \jmath_X} & & \big\downarrow{\scriptstyle \jmath_Y} \\ (X^*,\mathcal{C}^\star) & \xrightarrow{\ f^\star\ } & (Y^*,\mathcal{D}^\star) \end{array}$$

As noted in the remarks following Example 2.9, Theorem 2.10 is not valid for the coarse diagonal completion. However, the latter completion does exhibit an interesting extension property relative to a weaker type of continuity.

Let (X,q) and (Y,p) be topological spaces. $f : (X,q) \to (Y,p)$ is said to be *θ-continuous* if, for every $x \in X$ and every neighbourhood V of $f(x)$, there is a neighbourhood U of x such that $f(cl_q U) \subseteq cl_p(V)$. If $(X,\mathcal{C})$ and $(Y,\mathcal{D})$ are complete Cauchy spaces such that $q_{\mathcal{C}}$ and $q_{\mathcal{D}}$ are topologies, we define $f : (X,\mathcal{C}) \to (Y,\mathcal{D})$ to be *θ-continuous* if $f : \left(X, q_{\mathcal{C}}\right) \to \left(Y, q_{\mathcal{D}}\right)$ is θ-continuous.

THEOREM 2.11. *Let $(X,\mathcal{C})$ and $(Y,\mathcal{D})$ be T_2, diagonal Cauchy spaces and let $f : (X,\mathcal{C}) \to (Y,\mathcal{D})$ be Cauchy continuous. Then there is a θ-continuous function $f^* : (X^*,\mathcal{C}^*) \to (Y^*,\mathcal{D}^*)$ such that the following diagram commutes:*

$$\begin{array}{ccc} (X,\mathcal{C}) & \xrightarrow{\ f\ } & (Y,\mathcal{D}) \\ \big\downarrow{\scriptstyle \jmath_X} & & \big\downarrow{\scriptstyle \jmath_Y} \\ (X^*,\mathcal{C}^*) & \xrightarrow{\ f^*\ } & (Y^*,\mathcal{D}^*) \end{array}$$

PROOF: Let q^* be the topology on X^* induced by $\mathcal{C}^*$ and p^* the topology on Y^* induced by $\mathcal{D}^*$. If $[\mathcal{F}]_{\mathcal{C}} \in X^*$, let $f^*([\mathcal{F}]_{\mathcal{C}}) = [f(\mathcal{F})]_{\mathcal{D}}$. In particular, if $\mathcal{F} = \dot{x}$, then $f^*([\dot{x}]_{\mathcal{C}}) = [\dot{f(x)}]_{\mathcal{D}}$, and it follows that the above diagram commutes.

Next, we show that for any $A \subseteq X$, $f^*(cl_{q^*}(A^*)) \subseteq cl_{p^*}\jmath_Y(f(A))$. Given $A \subseteq X$, let $[\mathcal{F}] \in cl_{q^*}(A^*)$. Since $F^* \cap A^* \neq \emptyset$ for all $F \in \mathcal{F}_{\min}$, $F \cap A \neq \emptyset$ for all $F \in \mathcal{F}_{\min}$, and therefore $f(\mathcal{F}_{\min})$ has a trace on $f(A)$. Consequently, $(f(\mathcal{F}))^*_{\min}$ has a trace on $\jmath_Y f(A)$, which implies that $f^*([\mathcal{F}]_C) = [f(\mathcal{F})]_\mathcal{D} \in cl_{p^*}\jmath_Y(f(A))$.

Finally, let $[\mathcal{F}]_C \in X^*$ and $V^* \in (f(\mathcal{F}))^*_{\min}$, where $V \in (f(\mathcal{F}))_{\min}$. Choose $U \in \mathcal{F}_{\min}$ such that $f(U) \subseteq V$. Then U^* is a q^*-neighbourhood of $[\mathcal{F}]_C$ such that $f^*(cl_{q^*}(U^*)) \subseteq cl_{p^*}\jmath_Y(f(U)) \subseteq cl_{p^*}(V^*)$. ∎

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Department of Pure and Applied Mathematics
Washington State University
Pullman WA 99164-3113
United States of America

Department of Mathematics
University of Central Florida
Orlando FL 32816
United States of America