Canad. Math. Bull. Vol. **55** (4), 2012 pp. 815–820
http://dx.doi.org/10.4153/CMB-2011-064-6

Restricted Radon Transforms and Projections of Planar Sets

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Abstract. We establish a mixed norm estimate for the Radon transform in $\mathbb{R}^2$ when the set of directions has fractional dimension. This estimate is used to prove a result about an exceptional set of directions connected with projections of planar sets. That leads to a conjecture analogous to a well-known conjecture of Furstenberg.

1 Introduction

For each $\omega \in S^1$, fix $\omega^\perp$ with $\omega^\perp \perp \omega$. Define a Radon transform R for functions f on $\mathbb{R}^2$ by

$$Rf(t,\omega) = \int_{-1}^{1} f(t\,\omega + s\,\omega^\perp)\,ds.$$

Suppose $0 < \alpha < 1$ and fix a nonnegative Borel measure λ on S^1 which is α-dimensional in the sense that $\lambda(B(\omega,\delta)) \lesssim \delta^\alpha$ for $\omega \in S^1$. We are interested in mixed norm estimates for R of the following form:

$$\Big[\int_{S^1}\Big(\int_{-1}^{1} |Rf(t,\omega)|^s\,dt\Big)^{q/s} d\lambda(\omega)\Big]^{1/q} \lesssim \|f\|_p. \tag{1.1}$$

Here are some conditions that are necessary for (1.1): testing on $f = \chi_{B(0,\delta)}$ shows that

$$\frac{2}{p} \le 1 + \frac{1}{s}; \tag{1.2}$$

if there is $\omega_0 \in S^1$ such that $\lambda(B(\omega_0,\delta)) \gtrsim \delta^\alpha$ for small positive δ, then testing on 1 by δ rectangles centered at the origin in the direction $\omega_0^\perp$ gives

$$\frac{1}{p} \le \frac{1}{s} + \frac{\alpha}{q}; \tag{1.3}$$

if the Lebesgue measure in S^1 of the δ-neighborhood in S^1 of the support of λ is $\lesssim \delta^{1-\alpha}$, then testing on unions of 1 by δ rectangles in the directions of the support of λ gives

$$\frac{1-\alpha}{p} \le \frac{1}{s}. \tag{1.4}$$

Our first result is that these necessary conditions are almost sufficient.

Received by the editors November 18, 2009.
Published electronically April 6, 2011.
This work was supported in part by NSF grant DMS-0552041.
AMS subject classification: **44A12**, 28A78.

Theorem 1.1 *Suppose $p, q, r \in [1, \infty]$ satisfy the conditions* (1.2), (1.3), *and* (1.4) *with strict inequality. Then the estimate* (1.1) *holds.*

Now suppose that μ is a nonnegative Borel measure on $\mathbb{R}^2$. If $\omega \in S^1$, define the projection μ_ω of μ in the direction of ω by

$$\int_{\mathbb{R}} f(y)\, d\mu_\omega(y) \doteq \int_{\mathbb{R}^2} f(x \cdot \omega)\, d\mu(x),$$

where $x \cdot \omega$ denotes the inner product in $\mathbb{R}^2$. Fix $\alpha \in (0, 1)$ and suppose that λ is an α-dimensional measure on S^1. Then for $\epsilon > 0$ there is $C = C(\epsilon)$ such that

$$\int_{S^1} \frac{d\lambda(\omega)}{|\omega \cdot \omega_0|^{\alpha-\epsilon}} \le C(\epsilon)$$

for all $\omega_0 \in S^1$. The computation

$$\begin{aligned} \int_{S^1} I_{\alpha-\epsilon}(\mu_\omega)\, d\lambda(\omega) &= \int_{S^1} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{d\mu_\omega(y_1) d\mu_\omega(y_2)}{|y_1 - y_2|^{\alpha-\epsilon}} d\lambda(\omega) \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{S^1} \frac{d\lambda(\omega)}{|\omega \cdot \frac{x_1 - x_2}{|x_1 - x_2|}|^{\alpha-\epsilon}} \frac{d\mu(x_1) d\mu(x_2)}{|x_1 - x_2|^{\alpha-\epsilon}} \\ &\le C(\epsilon) I_{\alpha-\epsilon}(\mu) \end{aligned}$$

is due to Kaufman [2]. Refining an earlier result of Marstrand [3], it shows that if $E \subset \mathbb{R}^2$ has dimension $\beta \le 1$ and $p_\omega(E)$ is the projection of E onto the line through the origin in the direction of ω, then

$$\dim\{\omega \in S^1 : \dim p_\omega(E) < \alpha\} \le \alpha, \tag{1.5}$$

whenever $\alpha \le \beta$. (In this note "dim" stands for Hausdorff dimension.) In particular,

$$\dim\{\omega \in S^1 : \dim p_\omega(E) < \beta\} \le \beta. \tag{1.6}$$

The next theorem, whose analog for Minkowski dimension is trivial, complements Kaufman's results (1.5) and (1.6):

Theorem 1.2 *If* $\dim E = \beta \le 1$, *then*

$$\dim\{\omega \in S^1 : \dim p_\omega(E) < \beta/2\} = 0. \tag{1.7}$$

The estimates (1.6) and (1.7) lead naturally to the conjecture that if $\alpha \le \beta \le 1$, then

$$\dim\{\omega \in S^1 : \dim p_\omega(E) < (\alpha + \beta)/2\} \le \alpha. \tag{1.8}$$

One may view this conjecture as an analog of the following conjecture of Furstenberg: for $0 < \alpha < 1$, a Furstenberg α-set is a Borel subset of $\mathbb{R}^2$ which contains for each $\omega \in S^1$ an α-dimensional subset of some line parallel to ω. Furstenberg's conjecture is that Furstenberg α-sets have dimension at least $(3\alpha+1)/2$. It is shown in [5] that there exist Furstenberg α-sets having dimension $(3\alpha+1)/2$ and that any Furstenberg α-set has dimension at least $\min\{2\alpha, (\alpha+1)/2\}$. In the analogy between the two conjectures, (1.5) is the analog of the 2α lower bound for the dimension of Furstenberg sets and (1.7) is the analog of the $(\alpha+1)/2$ lower bound. Inequality (1.8) with $\beta = 1$ would imply the Furstenberg conjecture for a certain class of model Furstenberg sets. The link between Theorems 1.1 and 1.2 is the fact that formally $\mu_\omega = R\mu(\,\cdot\,,\omega)$.

2 Proof of Theorem 1.1

The lines bounding the regions defined by (1.2) and (1.4) intersect at

$$\Big(\frac{1}{p}, \frac{1}{s}\Big) = \Big(\frac{1}{1+\alpha}, \frac{1-\alpha}{1+\alpha}\Big).$$

Then equality in (1.3) gives $\frac{1}{q} = \frac{1}{1+\alpha}$, so the important estimate is an

$$L^{1+\alpha} \to L^{1+\alpha}(L^{(1+\alpha)/(1-\alpha)})$$

estimate. Easy estimates combined with an interpolation argument show that Theorem 1.1 will follow if we establish (1.1) for $f = \chi_E$ and a collection of triples (p, q, r) which are arbitrarily close to $(1+\alpha, 1+\alpha, (1+\alpha)/(1-\alpha))$. Standard arguments then show that it is enough to prove that if $R\chi_E(t,\omega) \geq \mu$ for

$$(t,\omega) \in F = \{(t,\omega) : \omega \in A, t \in B(\omega) \subset [-1,1]\},$$

where there is some B such that $B \leq m_1(B(\omega)) \leq 2B$ for $\omega \in A$, then

$$\mu^p \lambda(A)^{p/q} B^{p/s} \leq C(\delta)\, m_2(E)$$

if

$$p = \frac{\alpha + \delta\alpha + 1}{\delta\alpha + 1}, \quad q = \alpha + \delta\alpha + 1, \quad s = \frac{\alpha + \delta\alpha + 1}{\delta\alpha + 1 - \alpha}$$

for small $\delta > 0$.

For each $\omega \in A$ let $E(\omega) = \{t\omega + s\omega^\perp \in E : t \in B(\omega), s \in [-1,1]\}$. Since $R\chi_E(t,\omega) \geq \mu$ and $m_1(B(\omega)) \geq B$, it follows that

$$m_2(E(\omega)) \geq \mu B. \tag{2.1}$$

Using the change of coordinates $x \mapsto (x\cdot\omega_1, x\cdot\omega_2)$, one can check that

$$m_2\big(E(\omega_1) \cap E(\omega_2)\big) \lesssim \frac{B^2}{|\omega_1 - \omega_2|}. \tag{2.2}$$

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We will bound $m_2(E)$ from below by using

$$(2.3)\quad m_2(E) \geq m_2\Big(\bigcup_{j=1}^{N} E(\omega_j)\Big) \geq \sum_{j=1}^{N} m_2(E(\omega_j)) - \sum_{1\leq j<k\leq N} m_2\big(E(\omega_j)\cap E(\omega_k)\big)$$

for appropriately chosen $\omega_j \in A$. Fix, for the moment, a small positive number η and consider a partitioning of S^1 into intervals of length about η. Since $\lambda(B(x,r)) \lesssim r^\alpha$, the λ-measure of each of these intervals is $\lesssim \eta^\alpha$. So at least, roughly, $\eta^{-\alpha}\lambda(A)$ of them must intersect A. Thus it is possible to choose $N \sim \eta^{-\alpha}\lambda(A)$ points $\omega_j \in A$ with $|\omega_j - \omega_k| \gtrsim \eta\,|j-k|$. Then for any $\delta > 0$

$$\sum_{1\leq j<k\leq N} \frac{1}{|\omega_j - \omega_k|} \lesssim \eta^{-1} \sum_{1\leq j<k\leq N} \frac{1}{|j-k|} \lesssim \eta^{-1} N^{1+\delta}$$

and so, by (2.2),

$$(2.4)\qquad \sum_{1\leq j<k\leq N} m_2\big(E(\omega_1)\cap E(\omega_2)\big) \leq CB^2\eta^{-1}N^{1+\delta} \leq C_1 B^2 N^{1+\delta+1/\alpha}\lambda(A)^{-1/\alpha},$$

where we have used $N \sim \eta^{-\alpha}\lambda(A)$. We would now like to choose N such that

$$(2.5)\qquad 2\,C_1\,B^2 N^{1+\delta+1/\alpha}\lambda(A)^{-1/\alpha} \leq N\mu B \leq 3\,C_1\,B^2 N^{1+\delta+1/\alpha}\lambda(A)^{-1/\alpha}$$

or

$$(2.6)\quad 3^{-\alpha/(1+\delta\alpha)}\Big(\frac{\mu\,B^{-1}\lambda(A)^{1/\alpha}}{C_1}\Big)^{\alpha/(\delta\alpha+1)} \leq N \leq 2^{-\alpha/(1+\delta\alpha)}\Big(\frac{\mu\,B^{-1}\lambda(A)^{1/\alpha}}{C_1}\Big)^{\alpha/(\delta\alpha+1)}.$$

This will be possible unless $\mu B^{-1}\lambda(A)^{1/\alpha} \lesssim 1$, in which case

$$\mu^{\alpha/(\delta\alpha+1)}B^{-\alpha/(\delta\alpha+1)}\lambda(A)^{1/(\delta\alpha+1)} \lesssim 1,$$

so that the desired inequality

$$(2.7)\qquad m_2(E) \gtrsim \mu^{(\alpha+\delta\alpha+1)/(\delta\alpha+1)}\lambda(A)^{1/(\delta\alpha+1)}B^{(\delta\alpha+1-\alpha)/(\delta\alpha+1)}$$

follows from $m_2(E) \geq \mu B$ unless F is empty. Now (with N chosen so that (2.5) and (2.6) are valid), (2.3), (2.1), (2.4), and the left member of (2.5) give $m_2(E) \gtrsim N\mu B$. Then the left member of (2.6) gives (2.7) again. ∎

3 Proof of Theorem 1.2

For $\rho > 0$, let K_ρ be the kernel defined on $\mathbb{R}^d$ by $K_\rho(x) = |x|^{-\rho}\chi_{B(0,R)}(x)$, where $R = R(d)$ is positive. Suppose that the finite nonnegative Borel measure ν is a γ-dimensional measure on $\mathbb{R}^d$ in the sense that $\nu(B(x,\delta)) \le C(\nu)\delta^\gamma$ for all $x \in \mathbb{R}^d$ and $\delta > 0$. If $\rho < \gamma$, it follows that $\nu * K_\rho \in L^\infty(\mathbb{R}^d)$. Also $\nu * K_\rho \in L^1(\mathbb{R}^d)$ so long as $\rho < d$. Thus, for $\epsilon > 0$,

$$\nu * K_\rho \in L^p(\mathbb{R}^d), \rho = \gamma + \frac{1}{p}(d-\gamma) - \epsilon \tag{3.1}$$

by interpolation. The following lemma is a weak converse of this observation.

Lemma 3.1 *If (3.1) holds with $\epsilon = 0$ and $p > 1$, then ν is absolutely continuous with respect to Hausdorff measure of dimension $\gamma - \epsilon$ for any $\epsilon > 0$. Thus the support of ν has Hausdorff dimension at least γ.*

Proof Recall from [1, p. 140] that for $s \in \mathbb{R}$ and $1 \le p, q \le \infty$ the norm $\|f\|^s_{p,q}$ of a distribution f on $\mathbb{R}^d$ in the Besov space $B^s_{p,q}$ can be defined by

$$\|f\|^s_{pq} = \|\psi * f\|_{L^p(\mathbb{R}^d)} + \Big(\sum_{k=1}^{\infty}(2^{sk}\,\|\phi_k * f\|_{L^p(\mathbb{R}^d)})^q\Big)^{1/q}$$

for certain fixed $\psi \in \mathcal{S}(\mathbb{R}^d)$, $\phi \in C_c^\infty(\mathbb{R}^d)$, and where $\phi_k(x) = 2^{kd}\phi(2^k x)$. If $\nu * K_\rho \in L^p(\mathbb{R}^d)$, then $\|\nu * \chi_{B(0,\delta)}\|_{L^p(\mathbb{R}^d)} \lesssim \delta^\rho$. It follows that $\|\nu\|^s_{pq} < \infty$ if $s < \rho - d = (\gamma - d)/p'$. Now for $t > 0$ and $1 < p', q' < \infty$, the Besov capacity $A_{t,p',q'}(K)$ of a compact $K \subset \mathbb{R}^d$ is defined by

$$A_{t,p',q'}(K) = \inf\{\|f\|^t_{p',q'} : f \in C_c^\infty(\mathbb{R}^d),\ f \ge \chi_K\}.$$

It is shown in [4, p. 277] that $A_{t,p',q'}(K) \lesssim H_{d-tp'}(K)$. Thus it follows from the duality of $B^s_{p,q}$ and $B^{-s}_{p',q'}$ that

$$\nu(K) \lesssim \|\nu\|^s_{pq} A_{-s,p',q'}(K) \lesssim H_{d+sp'}(K) = H_{\gamma-\epsilon}(K)$$

if $s = (\gamma - d - \epsilon)/p'$. ∎

Now suppose that μ is a nonnegative and compactly supported Borel measure on $\mathbb{R}^2$ which is β-dimensional in the sense that $\mu(B(x,\delta)) \lesssim \delta^\beta$. If the radii $R(1)$ and $R(2)$ (in the definition of K_ρ) are chosen so that $R(1) = 1$ and $R(2)$ is large enough, depending on the support of μ, then one can verify directly that

$$\mu_\omega * K_{(\rho-1)}(t) \lesssim \int_{-2R(2)}^{2R(2)} \mu * K_\rho\,(t\omega + s\omega^\perp)\,ds.$$

If p, q, s are such that (1.1) holds and if $\rho = \beta + (2-\beta)/p - \epsilon$, so that (3.1) implies that $\mu * K_\rho \in L^p(\mathbb{R}^2)$, then a rescaling of (1.1) gives

$$\int_{S_1} \|\mu_\omega * K_{(\rho-1)}\|^q_{L^s(\mathbb{R})}\,d\lambda(\omega) < \infty. \tag{3.2}$$

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If we could take $(p,q,s) = \big(1+\alpha, 1+\alpha, (1+\alpha)/(1-\alpha)\big)$ and $\epsilon = 0$, then (3.2) would yield

$$\int_{S_1} \|\mu_\omega * K_\tau\|^{1+\alpha}_{L^{(1+\alpha)/(1-\alpha)}(\mathbb{R})}\, d\lambda(\omega) < \infty$$

with $\tau = (1-\alpha+\alpha\beta)/(1+\alpha)$. Adjusting for the fact that (3.2) actually holds only for (p,q,s) close to $\big(1+\alpha, 1+\alpha, (1+\alpha)/(1-\alpha)\big)$ and with $\epsilon > 0$, it still follows that

$$\int_{S_1} \|\mu_\omega * K_\tau\|^{1+\alpha-\epsilon}_{L^{(1+\alpha-\epsilon)/(1-\alpha)}(\mathbb{R})}\, d\lambda(\omega) < \infty$$

with $\tau = (1-\alpha+\alpha\beta)/(1+\alpha)-\epsilon$ for any $\epsilon > 0$. With $\nu = \mu_\omega$, $p = (1+\alpha-\epsilon)/(1-\alpha)$, and $d = 1$, Lemma 3.1 then shows that for any $\epsilon > 0$ the Hausdorff dimension of μ_ω's support exceeds $\beta/2-\epsilon$ for λ-almost all ω's. Since this is true for any α-dimensional measure λ and for any $\alpha \in (0,1)$, it follows that

$$\dim\{\omega \in S^1 : \dim p_\omega(E) < \beta/2\} = 0$$

as desired. ■

References

[1] J. Bergh and J. Löfström, *Interpolation Spaces.* Grundlehren der mathematischen Wissenschaften, 223, Springer-Verlag, Berlin, 1976.

[2] R. Kaufman, *On Hausdorff dimension of projections.* Mathematika **15**(1968), 153–155. http://dx.doi.org/10.1112/S0025579300002503

[3] J. Marstrand, *Some fundamental geometrical properties of plane sets of fractional dimension.* Proc. London. Math. Soc. **4**(1954), no. 3, 257–302. http://dx.doi.org/10.1112/plms/s3-4.1.257

[4] B.-M. Stocke, *Differentiability properties of Bessel potentials and Besov spaces.* Arkiv Mat. **22**(1984), no. 2, 269–286. http://dx.doi.org/10.1007/BF02384383

[5] T. Wolff, *Recent work connected with the Kakeya problem.* In: Prospects in Mathematics, American Mathematical Society, Providence, RI, 1999, pp. 129–162.

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