Glasgow Math. J. **42** (2000) 345–350. © Glasgow Mathematical Journal Trust 2000. Printed in the United Kingdom

SPECTRAL AND STRUCTURAL PROPERTIES OF LOG-HYPONORMAL OPERATORS

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Dedicated to Professor Nobuo Aoki on his 60th birthday

(Received 1 October, 1998)

Abstract. In this paper we study fundamental properties of spectra of log-hyponormal operators on a Hilbert space. In particular we show that log-hyponormal operators are normaloid and quasi-similar log-hyponormal operators have the same spectra and essential spectra.

1991 *Mathematics Subject Classification*. 47B20.

1. Introduction. Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. For $T \in B(\mathcal{H})$, we denote the spectrum, the point spectrum and the approximate point spectrum of T by $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$, respectively. An operator T is called *log-hyponormal* if T is invertible and satisfies $\log(T^*T) \geq \log(TT^*)$. An operator T is called *hyponormal* and *semi-hyponormal* if $T^*T \geq TT^*$ and $(T^*T)^{\frac{1}{2}} \geq (TT^*)^{\frac{1}{2}}$, respectively. Let $T = U|T|$ be the polar decomposition of T. If $T = U|T|$ is log-hyponormal, then the operator U is unitary. For a log-hyponormal operator $T = U|T|$, we can find a positive number c such that log $|cT| \geq 0$. We use the following results.

THEOREM A. (Lemma 6 of [**10**]). *Let $T = U|T|$ be log-hyponormal. If* $\log|T| \geq 0$, *then $S = U\log|T|$ is semi-hyponormal and*

$$\sigma(T) \ = \ \{(e^r)e^{i\theta} : re^{i\theta} \ \in \sigma(S)\}.$$

THEOREM B. (Proposition 3 of [**8**] or Theorem 1 of [**4**]). *Let $T = U|T|$ be log-hyponormal. If $R = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, then R is semi-hyponormal.*

THEOREM C. (Berberian [**2**]). *For a given Hilbert space $\mathcal{H}$, there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and an isometric into *-isomorphism $B(\mathcal{H}) \ \to \ B(\mathcal{K}) : A \ \to \ A^\circ$ preserving order such that $\sigma(A) = \sigma(A^\circ)$ and $\sigma_a(A) = \sigma_a(A^\circ) = \sigma_p(A^\circ)$.*

2. Spectra of log-hyponormal operators. In this section we show fundamental properties of log-hyponormal operators.

This research is partially supported by Grant-in-Aid Scientific Research (No. 09640229)

https://doi.org/10.1017/S0017089500030032 Published online by Cambridge University Press

THEOREM 1. *Let $T = U|T|$ be log-hyponormal. Then the eigenspaces of U are invariant for $|T|$.*

Proof. Let $\mathcal{M} = \{x \in \mathcal{H} : \ Ux = ax\}$. Let a positive number c satisfy $\log|cT| \geq 0$. We let $S = U\log|cT|$. Then, by Theorem A, the operator S is semi-hyponormal and by [**12**, (5), p. 8] $\mathcal{M}$ is invariant under $\log|cT|$. Hence, for every polynomial f, we have

$$f(\log|cT|)(\mathcal{M}) \subset \mathcal{M}.$$

Therefore, we have $e^{\log|cT|}(\mathcal{M}) \subset \mathcal{M}$ and $|cT|(\mathcal{M}) \subset \mathcal{M}$. Since $c > 0$, it follows that $|T|(\mathcal{M}) \subset \mathcal{M}$.

THEOREM 2. *Let $T = U|T|$ be log-hyponormal. If $\sigma(U) \neq \{z \in \mathbf{C} : |z| = 1\}$, then the eigenspaces of $|T|$ are reducing for U.*

Proof. Let r be an eigenvalue of $|T|$ and $\mathcal{N} = \{x \in \mathcal{H} : \ |T|x = rx\}$. Also for some positive number c, which satisfies $\log|cT| \geq 0$, we let $S = U\log|cT|$. If $x \in \mathcal{N}$, we have

$$(\log|cT|)x = (\log cr)x.$$

Since S is semi-hyponormal, by [**12**, (6), p. 8] the eigenspace of the eigenvalue $\log cr$ is invariant under U. Hence, for every polynomial f, we have

$$f(\log|cT|)Ux \ = \ f(\log cr)Ux.$$

Therefore, we have $(|cT|)Ux = crUx$ and $|T|Ux = rUx$. Hence the space $\mathcal{N}$ is invariant under U. Next, since the eigenspace of the eigenvalue $\log cr$ is invariant under U^* by [**12**, (6), p. 8], we can prove that the eigenspace $\mathcal{N}$ is invariant under U^* in a similar way.

THEOREM 3. *Let T be log-hyponormal. Then T^{-1} is also log-hyponormal.*

Proof. Since T is log-hyponormal,

$$\begin{aligned} \log((T^{-1})^*T^{-1}) \ &= \ \log(TT^*)^{-1} \ = \ -\log(TT^*) \\ \geq \ -\log(T^*T) \ &= \ \log(T^*T)^{-1} \ = \ \log(T^{-1}(T^{-1})^*). \end{aligned}$$

Hence T^{-1} is log-hyponormal.

Next let π be the mapping on $\mathbf{C}$ given by $\pi(z) = |z|$.

THEOREM 4. *Let $T = U|T|$ be log-hyponormal. Then*

$$\pi(\partial\sigma(T)) \subset \sigma(|T|) \subset \pi(\sigma(T)).$$

Proof. Let a positive number c satisfy $\log|cT| \geq 0$ and $S = U\log|cT|$. Then S is semi-hyponormal. To prove the first containment we assume that $r \in \pi(\partial\sigma(T))$. Then we have

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$$r \in \pi(\partial\sigma(T)) \Longleftrightarrow \log cr \in \pi(\partial\sigma(S)) \Longrightarrow \log cr \in \sigma(|S|) \Longleftrightarrow r \in \sigma(|T|).$$

To prove the second containment we assume that $r \in \sigma(|T|)$. Since, by Lemma VI.3.10 of [**12**] we have

$$\sigma(\log|cT|) \subset \pi(\sigma(S)),$$

it follows that $\log cr \in \pi(\sigma(S))$. Hence, by Theorem A, $r \in \pi(\sigma(T))$ and so the proof is complete.

Theorem 5. *Let $T = U|T|$ be log-hyponormal. Then*

$$\{m, M\} \subset \pi(\sigma_a(T)),$$

where m and M are the minimum and maximum of $\sigma(|T|)$, respectively. In addition, if $\sigma(U) \neq \mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$, then

$$\pi(\sigma_a(T)) = \sigma(|T|).$$

Proof. Let a positive number c satisfy $\log|cT| \geq 0$ and $S = U\log|cT|$. Since S is semi-hyponormal, by Theorems A and II.3.3 of [**12**] we have $\log cM \in \pi(\sigma_a(S))$. Therefore, it follows that $M \in \pi(\sigma_a(T))$ by Lemma 6 of [**10**]. By Theorem 3 we have $m \in \sigma(|T|)$, similarly. Next we assume that $\sigma(U) \neq \mathbf{T}$. Then by Theorem 2 we have $\sigma(|T|) \subset \pi(\sigma_a(T))$, and, by Theorem 11 of [**8**], we have $\pi(\sigma_a(T)) \subset \sigma(|T|)$. The proof is complete.

Theorem 6. *If $T = U|T|$ is a log-hyponormal operator, then T is normaloid; i.e., the spectral radius $r(T) = ||T||$.*

Proof. Let the maximum value of the set $\sigma(|T|)$, the maximum value of the set $\pi(\sigma_a(T))$, and the maximum value of the set $\pi(\sigma(T))$ be M, M', and M'', respectively. Then it is enough to show that $r(T) \geq ||T||$. Indeed, from Theorem 5 above, we have

$$||T|| = |||T||| = M \leq M' \leq M'' = r(T).$$

Next we consider the Aluthge transformation for a log-hyponormal operator $T = U|T|$; i.e., $R = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Then, by Theorem B, the operator R is semi-hyponormal.

Theorem 7. *Let $T = U|T|$ be log-hyponormal and $R = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Let $R = V|R|$ be the polar decomposition of R. Then $\sigma(U) = \sigma(V)$.*

Proof. Let $e^{i\theta} \in \sigma(U)$. Since $e^{i\theta} \in \sigma_a(U)$, by the Berberian technique and Theorem 1, there exists a number $r > 0$ such that $re^{i\theta} \in \sigma(T)$. Since $\sigma(T) = \sigma(R)$, we have $re^{i\theta} \in \sigma(R)$. Let $\rho = \max\{a : ae^{i\theta} \in \sigma(R)\}$. Since then $\rho e^{i\theta}$ is a boundary point of $\sigma(R)$, we have $\rho e^{i\theta} \in \sigma_a(R)$. Since also R is semi-hyponormal by Theorem B, from Theorem I.2.3 of [**12**] we have $e^{i\theta} \in \sigma(V)$. Conversely, if $e^{i\theta} \in \sigma(V)$, then there exists a number $r > 0$ such that $re^{i\theta} \in \sigma(R)$. Since $re^{i\theta} \in \sigma(T)$, we can choose a positive number μ such that $\mu e^{i\theta} \in \sigma_a(T)$. Hence we have $e^{i\theta} \in \sigma(U)$ by Theorem 11 of [**9**], similarly.

THEOREM 8. *Let* $T = U|T|$ *be log-hyponormal and* $R = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. *If* $\sigma(U) \neq \mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$, *then*

$$\sigma(|T|) = \sigma(|R|).$$

Proof. Let $r \in \sigma(|T|)$. By Theorem 2 and the Berberian technique, there exists $e^{i\theta} \in \mathbf{C}$ and a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that

$$(|T| - r)x_n \to 0 \text{ and } (U - e^{i\theta})x_n \to 0 \text{ as } n \to \infty.$$

Therefore, we have $re^{i\theta} \in \sigma_a(R)$. Since R is semi-hyponormal, we have $r \in \sigma(|R|)$. Next we assume that $s \in \sigma(|R|)$. Using the Berberian technique it follows that there exists a number $e^{i\theta} \in \mathbf{C}$ such that $se^{i\theta} \in \sigma_a(R)$. Since by Theorem 4 of [**4**]

$$Tx = zx \text{ if and only if } Rx = zx,$$

by the Berberian technique also we have $se^{i\theta} \in \sigma_a(T)$. Hence it follows that $s \in \sigma(|T|)$ and so the proof is complete.

REMARK. If $\sigma(U) = \mathbf{T}$, then Theorem 8 does not hold in general even if $T = U|T|$ is hyponormal. The following result is due to B. P. Duggal, M. Chō and T. Huruya [**7**]. Let $\mathcal{H}$ be the Hilbert space of all two-way (bilateral) square-summable sequences. The elements of $\mathcal{H}$ are written in the form

$$(\cdots, x_{-2}, x_{-1}, x_0, x_1, x_2, \cdots).$$

Let U be the bilateral shift on $\mathcal{H}$. Take a sequence $\{a_n\}$ as $a_n = 0$ $(n < 0)$, $a_0 = \frac{1}{2}$ and $a_n = 1$ $(n > 0)$. We define an operator $|T|$ as follows: for a vector $\mathbf{x} = (\cdots, x_n, \cdots) \in \mathcal{H}$, set

$$|T|\mathbf{x} = (\cdots, a_n x_n, \cdots).$$

Let $T = U|T|$. Then we can easily verify that the operator T is hyponormal,

$$\sigma(|T|) = \{0, \frac{1}{2}, 1\} \text{ and } \sigma(|S|) = \{0, \frac{1}{\sqrt{2}}, 1\},$$

where $S = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ and $|S|$ is the positive part of S. This example is due to S. Li (cf. [**12**, p. 219]).

3. Quasi-similarity of log-hyponormal operators. For an operator T, let $T = U|T|$ be the polar decomposition of T. Then $R = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is called the *Aluthge transformation* of T. Also if T is log-hyponormal and semi-hyponormal, then R is semi-hyponormal and hyponormal, respectively ([**1**],[**4**],[**9**]). Let $R = V|R|$ be the polar decomposition of R. The operator $\tilde{T}$ is defined by $\tilde{T} = |R|^{\frac{1}{2}}V|R|^{\frac{1}{2}}$. Hence, if T is log-hyponormal, then $\tilde{T}$ is hyponormal. By the Aluthge transformation, such operators belong to a nicer class. The following theorem shows that the various spectra of a log-hyponormal operator T can be completely determined by the corresponding spectra of a hyponormal operator $\tilde{T}$ induced by its Aluthge transform R, as in the case of p-hyponormal operators [**6**].

THEOREM 10. *If T is a log-hyponormal operator, then*

$$\sigma_*(T) = \sigma_*(\tilde{T}),$$

where σ_* denotes each one of the following: the spectrum σ, the point spectrum σ_p, the approximate point spectrum σ_a, the essential spectrum σ_e, and the Weyl spectrum σ_w, respectively.

Proof. For the proof it is enough to observe that there exists an invertible operator $X = |R|^{\frac{1}{2}}|T|^{\frac{1}{2}}$ such that $T = X^{-1}\tilde{T}X$; (i.e., T is similar to $\tilde{T}$). It is well known that similar operators have isomorphic lattices of invariant subspaces and the similarity preserves the spectral picture (i.e., the spectrum, the essential spectrum, and the index function). Thus the proof is complete by combining these facts.

COROLLARY 11 (Theorem 11 of [**9**] and Lemma 3 of [**10**]). *Let T be log-hyponormal. Then we have*

$$\sigma_p(T) = \sigma_{np}(T) \text{ and } \sigma_a(T) = \sigma_{na}(T),$$

where $\sigma_{np}(*)$ and $\sigma_{na}(*)$ denote the normal point spectrum and the normal approximate point spectrum, respectively.

Proof. Since T is similar to $\tilde{T}$, T^* is similar to $\tilde{T}^*$. Thus, by Theorem 10 above and the hyponormality of $\tilde{T}$, we have

$$\sigma_p(T) = \sigma_p(\tilde{T}) \subset \overline{\sigma_p(\tilde{T}^*)} = \overline{\sigma_p(T^*)} \text{ and } \sigma_a(T) = \sigma_a(\tilde{T}) \subset \overline{\sigma_a(\tilde{T}^*)} = \overline{\sigma_a(T^*)}.$$

COROLLARY 12 (Theorem 7 of [**4**]). *Let T be log-hyponormal. Then Weyl's theorem holds for T; that is,*

$$\sigma(T) - \sigma_w(T) = \pi_{00}(T).$$

Proof. Let $\pi_{00}(T)$ and $\pi_{00}(\tilde{T})$ denote the set of all isolated eigenvalues of finite multiplicity of T and $\tilde{T}$, respectively. Then the similarity of T and $\tilde{T}$ implies that $\pi_{00}(T) = \pi_{00}(\tilde{T})$. Thus from the well-known fact that Weyl's theorem holds for hyponormal operators we have

$$\sigma(T) - \sigma_w(T) = \sigma(\tilde{T}) - \sigma_w(\tilde{T}) = \pi_{00}(\tilde{T}) = \pi_{00}(T).$$

We say $A, B \in B(\mathcal{H})$ are *quasi-similar*, denoted $A \sim B$, if there exist quasi-affinities X and Y (i.e., operators that are each injective with dense range) such that $XA = BX$ and $AY = YB$.

THEOREM 13. *If log-hyponormal operators A and B are quasi-similar, then*

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$$\sigma(A) = \sigma(B) \text{ and } \sigma_e(A) = \sigma_e(B).$$

Proof. Since quasi-similarity is an equivalence relation, if A and B are quasi-similar, then $\tilde{A}$ and $\tilde{B}$ are quasi-similar. By Theorem 2 of S. Clary [**5**] and Theorem 2.10 of L. Yang [**11**] we have

$$\sigma(\tilde{A}) = \sigma(\tilde{B}) \text{ and } \sigma_e(\tilde{A}) = \sigma_e(\tilde{B}), \text{ respectively.}$$

Also, the facts that $\tilde{A}$ is similar to A and $\tilde{B}$ is similar to B imply that

$$\sigma(\tilde{A}) = \sigma(A), \ \sigma(\tilde{B}) = \sigma(B), \ \sigma_e(\tilde{A}) = \sigma_e(A), \ \sigma_e(\tilde{B}) = \sigma_e(B).$$

Thus we have

$$\sigma(A) = \sigma(B), \ \sigma_e(A) = \sigma_e(B).$$

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