CONVOLUTION OPERATORS ASSOCIATED WITH VECTOR MEASURES

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Abstract. In this note, we present a thorough investigation of convolution operators that are naturally associated to vector measures. We characterize those convolution operators that are weakly compact and compact on $\mathcal{L}^1(G)$ and $\mathcal{C}(G)$ as well as those that are p-summing, $(1 \leq p \leq \infty)$ and nuclear on $\mathcal{C}(G)$.

1. Preliminaries. Let G be a compact metrizable abelian group with λ a normalized Haar measure and let X be a Banach space. For $1 \leq p < \infty$, let $\mathcal{L}^P(G, X)$ (resp. $\mathcal{L}^\infty(G, X)$) denote the Banach space of (all classes of) λ-Bochner p-integrable (resp. (all classes of) λ-measurable functions that are essentially bounded) X-valued functions defined on G. We also let $\mathcal{C}(G, X)$ denote the Banach space of all continuous X-valued functions defined on G. If X is the field of scalars, we simply write $\mathcal{L}^P(G)$ (resp. $\mathcal{L}^\infty(G)$) and $\mathcal{C}(G)$. For an X-valued measure of bounded variation μ, the formula $f * \mu(t) = \int_G f(t - s)d\mu(s)$ defines a bounded linear operator $T_\mu : \mathcal{L}^1(G) \to \mathcal{L}^1(G, X)$ (resp. $T_\mu : \mathcal{C}(G) \to \mathcal{C}(G, X)$) by $T_\mu(f) = f * \mu$ for every $f \in \mathcal{L}^1(G)$ (resp. $\mathcal{C}(G)$). Such an operator T_μ is called a *convolution operator associated to the measure μ*.

For the case where X is reduced to the scalars, convolution operators associated to regular Borel scalar measures on compact groups have been well studied and understood. For example, one has the following theorem:

THEOREM 1.1. *Let G be a compact metrizable abelian group with Haar measure λ. Let μ be a regular Borel scalar measure on G. Let $T_\mu : \mathcal{L}^1(G) \to \mathcal{L}^1(G)$ be the convolution defined by $T_\mu(f) = f * \mu$. Each of the following statements implies all the others:*

(a) *the operator T_μ is compact;*
(b) *the operator T_μ is weakly compact;*
(c) *the operator T_μ is representable;*
(d) *the measure μ is λ-continuous.*

Proofs of these results can be found in [4], [8] and [11] among others. We simply notice that the implication (b) $\Rightarrow$ (c) follows from the fact that every weakly compact operator on $\mathcal{L}^2(G)$ is representable; this is of course due to N. Dunford and B. J. Pettis. The equivalence (c)$\Longleftrightarrow$(d) is due to A. Costé, a proof can be found in [4, p. 90–91]. The implication (d) $\Rightarrow$ (a) is a simple exercise and a proof can be inferred from the vector valued case which will be treated in Section 2. We will see that these equivalences remain true for the vector valued case.

Moreover, each one of the assertions of Theorem 1.1 is equivalent to every one of the following theorem.

THEOREM 1.2. *Let G be a compact metrizable abelian group with Haar measure λ. Let μ be a regular Borel scalar measure on G. Let $T_\mu : C(G) \to C(G)$ be the convolution defined by $T_\mu(f) = f * \mu$. Then the following assertions are equivalent:*

(a′) *the operator $T_\mu : C(G) \to C(G)$ is compact;*
(b′) *the operator $T_\mu : C(G) \to C(G)$ is weakly compact;*
(c′) *the measure μ is λ-continuous.*

Proof. (a′)⇒(b′) is of course obvious. The implication (b′)⇒(c′) follows from a result of [**6**, p. 278] and a proof can be inferred from the vector valued case which will be treated in Section 3.

The implication (c′)⇒(a′) follows from the fact that trigonometric polynomials are dense in $\mathcal{L}^1(G)$. To see this, note that if $\mu = f \cdot \lambda$, where $f \in \mathcal{L}^1(G)$, then these exists a sequence $\{p_n\}$ of trigonometric polynomials such that

$$\lim_{n \to \infty} \|f - p_n\|_1 = 0.$$

Since $\|T_\mu - T_{p_n}\| \le \|f - p_n\|_1$, and since each T_{p_n} is a finite rank operator, it follows that T_μ is a compact operator. □

More details and related results of this theorem can be seen in [**8**] and [**11**]. Unlike the case of convolution operators on $\mathcal{L}^1(G)$, we will see in Section 3 that the results of Theorem 1.2 do not hold for their vector valued counterparts. Also in Section 3 we shall give a complete answer to the following natural question.

QUESTION 1. For what Banach space X is the following true?

Given a compact metrizable abelian group G and an X-valued measure of bounded variation μ on G, then the convolution operator $T_\mu : C(G) \to C(G, X)$ is compact whenever T_μ is weakly compact.

Finally, we shall study in Section 4 the extension to the vector valued case of the following result.

THEOREM 1.3. *Let G be a compact metrizable abelian group with Haar measure λ. Let μ be a regular Borel scalar measure on G. Let $T_\mu : C(G) \to C(G)$ be the convolution defined by $T_\mu(f) = f * \mu$. Then*

(a) *the operator T_μ is 1-summing if and only if $\mu = \phi \cdot \lambda$ where $\phi \in \mathcal{L}^\infty(G)$;*
(b) *the operator T_μ is nuclear if and only if $\mu = \phi \cdot \lambda$ where $\phi \in C(G)$.*

First, let us fix some notations and terminologies.

NOTATIONS 1. If X and Y are Banach spaces, a bounded linear operator $T : X \to Y$ is *compact* (resp. *weakly compact*), if T maps the unit ball of X into a relatively compact (resp. a relatively weakly compact) subset of Y.

If G is a compact metrizable abelian group, we will denote by $\hat{G}$ its dual group and by λ a normalized Haar measure on G. If X is a Banach space and $f \in \mathcal{L}^1(G, X)$, we denote by $\hat{f}$ its *Fourier transform*, defined as follows:

$$\hat{f}(\gamma) = \int_G f(t)\bar{\gamma}(t)d\lambda(t), \text{ for each } \gamma \in \hat{G}.$$

More generally, if μ is an X-valued vector measure of bounded variation on G, we denote by $\hat{\mu}$ its *Fourier transform* defined as follows:

$$\hat{\mu}(\gamma) = \int_G \bar{\gamma} d\mu, \text{ for each } \gamma \in \hat{G}.$$

Here one should note that when $f \in \mathcal{L}^1(G, X)$ and or more generally, when μ is an X-valued measure, $\hat{f}(\gamma)$ and $\hat{\mu}(\gamma)$ are elements of X for each γ in $\hat{G}$. Also if $f \in \mathcal{L}^1(G)$, and if $t \in G$, we denote by f_t the element of $\mathcal{L}^1(G)$ defined by

$$f_t(x) = f(x - t), \text{ for each } x \in G.$$

The σ-field of Borel subsets of G will be denoted by $\mathcal{B}(G)$. If A is a Borel subset of G, then χ_A will stand for the characteristic function of A, i.e.

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

If X is a Banach space and μ is an X-valued measure on G that is of bounded variation, we say that μ has *relatively compact range* if $\{\mu(A) : A \in \mathcal{B}(G)\}$ is a relatively compact subset of X.

Finally, all notions not defined can be found in [3], [4] or [11].

2. Convolution operators on $\mathcal{L}^1(G)$.

Our first result extends the assertions of Theorem 1.1 to the vector case.

THEOREM 2.1. *Let G be a compact metrizable abelian group with Haar measure λ. Let X be a Banach space and let μ be an X-valued measure of bounded variation on G. Let $T_\mu : \mathcal{L}^1(G) \to \mathcal{L}^1(G, X)$ be the convolution defined by $T_\mu(f) = f * \mu$ for all $f \in \mathcal{L}^1(G)$. Then the following properties are equivalent:*

    (a) *the operator T_μ is compact;*
    (b) *the operator T_μ is weakly compact;*
    (c) *the operator T_μ is representable;*
    (d) *the measure $\mu = h \cdot \lambda$, where $h \in \mathcal{L}^1(G, X)$.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are immediate.

(c) $\Rightarrow$ (d). The following argument is due to A. Costé ([4, p. 90–91]) for the corresponding scalar case, we shall include his argument which extends to the vector case for the sake of completeness. Suppose that the convolution operator T_μ is representable and let ψ be the element of $\mathcal{L}^\infty(G, \mathcal{L}^1(G, X))$ such that $T_\mu(f)(\cdot) = \int_G f(s)\psi(s)(\cdot)d\lambda(s)$, for every $f \in \mathcal{L}^1(G)$. The function ψ can be considered as function of two variables in the following manner:

$$\phi(u, v) = \psi(u)(v),$$

for $(u, v) \in G \times G$.

Define $h \in \mathcal{L}^1(G, X)$ by $h(t) = \int_G \phi(u - t, u)d\lambda(u)$. For each $f \in \mathcal{L}^1(G)$, a simple change of variable in tandem with Fubini's Theorem establishes the following:

$$
\begin{aligned}
f * h(s) &= \int_G f(t)h(s - t)d\lambda(t) \\
&= \int_G f(t) \int_G \phi(u + t - s, u)d\lambda(u)d\lambda(t) \\
&= \int_G \int_G f(t)\phi(v + t, v + s)d\lambda(v)d\lambda(t) \\
&= \int_G \left(\int_G f(t)\phi(v + t, v + s)d\lambda(t) \right) d\lambda(v)
\end{aligned}
\tag{2.1}
$$

Again for each $v \in G$ and each $t \in G$, if one denotes $f_v(t) = f(t - v)$, then one notices that

$$
\begin{aligned}
\int\int_{G \times G} f(t)\phi(v + t, v + s)d\lambda(t)d\lambda(v) &= \int_G \int_G f_v(t + v)\phi(v + t, v + s)d\lambda(t)d\lambda(v) \\
&= \int_G T_\mu(f_v)(v + s)d\lambda(v) \\
&= \int_G f_v * \mu(v + s)d\lambda(v) \\
&= \int_G \int_G f_v(v + s - t)d\mu(t)d\lambda(v) \\
&= \int_G \int_G f(s - t)d\mu(t)d\lambda(v) \\
&= \int_G f * \mu(s)d\lambda(v) = f * \mu(s),
\end{aligned}
\tag{2.2}
$$

it then follows that $f * h = f * \mu$, for each $f \in \mathcal{L}^1(G)$. This shows that $\mu = h \cdot \lambda$.

   (d) $\Rightarrow$ (a) Suppose $\mu = g \cdot \lambda$, where $g \in \mathcal{L}^1(G, X)$. Let $(i_n)_{n \in \mathbb{N}}$ be a good approximate identity on the group G [6]. Consider the sequence of convolution operators (T_{i_n}) associated to the sequence of functions $\{i_n\}$, that is $T_{i_n}(f) = f * i_n$, for each $f \in \mathcal{L}^1(G)$, where $f * i_n(t) = \int_G f(t - s)i_n(s)d\lambda(s)$, for each $f \in \mathcal{L}^1(G)$. By Theorem 1.1, since the i_n are in $\mathcal{L}^1(G)$ the operators T_{i_n} are all compact. It follows that for each $n \geq 1$ the operator $T_\mu T_{i_n}$ is also compact. Since for every $f \in \mathcal{L}^1(G)$, we have

$$
\|T_\mu T_{i_n}(f) - T_\mu(f)\| = \|f * i_n * g - f * g\| \leq \|f\|_1 \|i_n * g - g\|_1,
\tag{2.3}
$$

and since it is easily seen that the sequence $(i_n * g)$ converges to g in $\mathcal{L}^1(G, X)$[7], the inequality (2.3) then implies that the sequence $\{T_\mu T_{i_n}\}$ converges to the operator T_μ in the operator norm and hence it shows that the operator T_μ is compact. $\qquad\square$

As a consequence, this result sets up the following characterization of Banach spaces with the Radon-Nikodým property (RNP); for more details on the RNP see [4].

COROLLARY 2.2. *Let G be a compact metrizable abelian group with Haar measure λ. A Banach space X has RNP if and only if for every λ-continuous X-valued countably additive measure of bounded variation μ on G, the convolution operator $T_\mu : \mathcal{L}^1(G) \to \mathcal{L}^1(G, X)$ defined by $T_\mu(f) = f * \mu$ for all $f \in \mathcal{L}^1(G)$ is compact.*

3. Compact and weakly compact convolution operators on $\mathcal{C}(G)$.

The study of convolution operators associated to vector measure on $\mathcal{C}(G)$ spaces turned out to be very interesting. Unlike their behavior on $\mathcal{L}^1(G)$ spaces where results in the scalar case easily extended to the vector case, the analogue of Theorem 1.2 does not extend to the vector case. In what follows, we present two easy examples that illustrate such failure.

EXAMPLE 1. Let $\mathbb{T}$ be the circle group. Let μ be the c_0-valued measure defined by

$$\mu(A) = \left(\frac{1}{2\pi} \int_{\mathbb{T}} \chi_A(s) e^{-ins} ds \right)_{n \geq 1,}$$

for every Borel subset A of $\mathbb{T}$. Then the convolution operator T_μ associated to the measure μ is a weakly compact operator on $\mathcal{C}(\mathbb{T})$ that is not compact.

Proof. To see that the convolution operator T_μ is weakly compact on $\mathcal{C}(\mathbb{T})$, we first consider T_μ as a convolution operator on $\mathcal{L}^1(\mathbb{T})$. As such T_μ a priori takes its values in $\mathcal{L}^1(\mathbb{T}, c_0)$. We claim that T_μ as actually a bounded linear operator from $\mathcal{L}^1(\mathbb{T})$ into $\mathcal{C}(\mathbb{T}, c_0)$. This will show that T_μ as a convolution operator on $\mathcal{C}(\mathbb{T})$, can be factored through $\mathcal{L}^1(\mathbb{T})$, and hence it is weakly compact, since the canonical injection from $\mathcal{C}(\mathbb{T})$ into $\mathcal{L}^1(\mathbb{T})$ is weakly compact [3].

To justify the claim, notice that for $f \in \mathcal{L}^1(\mathbb{T})$, and for each $t \in \mathbb{T}$, one has

$$f * \mu(t) = \int_{\mathbb{T}} f(t - s) d\mu(s)$$

$$= \left(\frac{1}{2\pi} \int_{\mathbb{T}} f(t - s) e^{-ins} ds \right)_{n \geq 1}$$

$$= \left(\frac{1}{2\pi} \int_{\mathbb{T}} \bar{f}_t(s) e^{-ins} ds \right)_{n \geq 1}$$

$$= \left(\hat{\bar{f}}_t(n) \right)_{n \geq 1},$$

where $\bar{f}(s) = f(-s)$ for every $s \in \mathbb{T}$. It is a well known fact [7] and easily seen that the map $t \mapsto f_t$, and hence the map $t \mapsto \bar{f}_t$ is continuous from the group $\mathbb{T}$ into $\mathcal{L}^1(\mathbb{T})$. On the other hand, since $\sup_n |\hat{f}(n)| \leq \|f\|_1$ for every $f \in \mathcal{L}^1(\mathbb{T})$, the Fourier transform $\widehat{(\cdot)}$ considered as a linear operator from $\mathcal{L}^1(\mathbb{T})$ into c_0 is bounded. Hence, (3.1) shows that the map $t \mapsto f * \mu(t)$ is continuous, hence $f * \mu \in \mathcal{C}(\mathbb{T}, c_0)$ for each $f \in \mathcal{L}^1(\mathbb{T})$. It also follows from (3.1) that

$$\|f * \mu\|_{\mathcal{C}(\mathbb{T}, c_0)} = \sup_{t \in \mathbb{T}} \sup_n |\hat{\bar{f}_t}(n)| = \sup_{t \in \mathbb{T}} \sup_n |\hat{f}_t(n)| \leq \|f_t\|_1 = \|f\|_1.$$

This proves our claim.

On the other hand, if the convolution operator T_μ were compact as an operator from $\mathcal{C}(\mathbb{T})$ into $\mathcal{C}(\mathbb{T}, c_0)$, then so would the operator $S = \delta_0 T_\mu$, where $\delta_0 : \mathcal{C}(\mathbb{T}, c_0) \to c_0$ is the evaluation operator at 0, i.e. $\delta_0(f) = f(0)$, for every $f \in \mathcal{C}(\mathbb{T}, c_0)$. This cannot be the case since S is nothing but

$$S : \mathcal{C}(\mathbb{T}) \to c_0$$

$$f \mapsto \left(\hat{\bar{f}}(n) \right)_{n > 1}.$$

S is not compact: indeed the characters form a bounded set in $\mathcal{C}(\mathbb{T})$, but clearly, for any two different positive integers n and n', a simple computation shows that

$$\|S(e^{in(\cdot)}) - S(e^{in'(\cdot)})\| = 1.$$

This finishes the proof.   □

EXAMPLE 2. Let G be a compact metrizable abelian group with Haar measure λ, and let μ be the $\mathcal{L}^1(G)$-valued measure defined by

$$\mu(A) = \chi_A,$$

for every Borel subset A of G. Then the convolution operator T_μ associated to the measure μ is a non compact weakly compact operator on $\mathcal{C}(G)$.

Proof. As in the proof of the above example, we show that the convolution operator $T_\mu : \mathcal{C}(G) \to \mathcal{C}(G, \mathcal{L}^1(G))$ factors through $\mathcal{L}^1(G)$. It is clear that the measure μ defines a convolution operator from $\mathcal{L}^1(G)$ into $\mathcal{L}^1(G, \mathcal{L}^1(G))$, still denoted by T_μ. For each $f \in \mathcal{L}^1(G)$, and for each $t \in \mathcal{L}^1(G)$, we have

$$f * \mu(t) = \int_G f(t - s) d\mu(s) = \int_G \bar{f}_t(s) d\mu(s) = \bar{f}_t,$$

where $\bar{f}$ is the function defined by $\bar{f}(s) = f(-s)$, for every $s \in G$. The continuity of the map $t \mapsto \bar{f}_t = f * \mu(t)$ shows that $f * \mu$ is indeed in $\mathcal{C}(G, \mathcal{L}^1(G))$. That is the range of T_μ considered as a convolution operator from $\mathcal{L}^1(G)$ into $\mathcal{L}^1(G, \mathcal{L}^1(G))$ is contained in $\mathcal{C}(G, \mathcal{L}^1(G))$. We also have for each $f \in \mathcal{C}(G)$

$$\|f * \mu\|_{\mathcal{C}(G, \mathcal{L}^1(G))} = \sup_{t \in G} \|f * \mu(t)\|_1$$

$$= \sup_{t \in G} \|\bar{f}_t\| = \|f\|_1.$$

This shows that $T_\mu : C(G) \to C(G, \mathcal{L}^1(G))$ factors through $\mathcal{L}^1(G)$, and hence it is weakly compact [3].

On the other hand, if $T_\mu : C(G) \to C(G, \mathcal{L}^1(G))$ were compact, then so would the operator $S = \delta_0 T_\mu : C(G) \to \mathcal{L}^1(G)$ defined by $Sf = f * \mu(0) = \bar{f}$. The operator S is not compact since it is clearly seen to be a bounded linear operator from $C(G)$ onto $C(G)$. The proof is complete.

The next result characterizes those convolution operators that are weakly compact on $C(G)$. Before we state the result, we need the following lemma:

LEMMA 3.1. *Let G be a compact metrizable abelian group with Haar measure λ. Let X be a Banach space, and let μ be an X-valued countably additive measure of bounded variation defined on the group G. Let $T_\mu : C(G) \to C(G, X)$ be the convolution operator defined by $T_\mu(f) = f * \mu$ for all $f \in C(G)$ and let $\mathfrak{F}$ be its representing measure. Then $\mathfrak{F}$ is given by $\mathfrak{F}(A) = \chi_A * \mu$, for every $A \in \mathcal{B}(G)$.*

Proof. We first notice that since the group G is compact and metrizable, the class of Baire subsets and the class of Borel subsets of G coincide. Thus we are done if we prove that $\mathfrak{F}(A) = \chi_A * \mu$, for every compact G_δ subset A of G.

To see this, let A be a compact G_δ subset of the group G. Let (φ_n) be a sequence of continuous functions on G converging pointwise to χ_A and such that $0 \le \varphi_n \le 1$. For each $t \in G$, one can appeal to the Lebesgue Dominated Convergence Theorem to get

$$\lim_n \varphi_n * \mu(t) = \lim_n \int_G \varphi_n(t-y)d\mu(y)$$

$$= \int_G \chi_A(t-y)d\mu(y)$$

$$= \chi_A * \mu(t).$$

Now let ν be in the dual of $C(G, X)$. Then ν can be considered as an X^*-valued vector measure of bounded variation (see [4, p. 182]). Then there exists a function $g : G \to X^*$ having the following properties (see [2]):

1. $\|g(t)\| = 1$ $|\nu|$-almost everywhere;
2. $\int_G f d\nu = \int_G \langle g(t), f(t)\rangle d|\nu|(t)$, for $f \in \mathcal{L}^\infty(\mathcal{G}, \mathcal{X})$.

Thus since for each n, $\varphi_n * \mu$ belongs to $C(G, X)$, we have

$$\int_G \varphi_n * \mu d\nu = \int_G \langle g(t), \varphi_n * \mu(t)\rangle d|\nu|(t).$$

We also notice that

$$|\langle g(t), \varphi_n * \mu(t)\rangle| \le \|g(t)\|\|\varphi_n * \mu(t)\| \le \|g(t)\|\|\varphi_n * \mu\|_\infty \le \|\mu\|.$$

Therefore, one can once again appeal to the Lebesgue Dominated Convergence Theorem to see that

$$\lim_n \int_G \varphi_n * \mu \, dv = \lim_n \int_G \langle g(t), \varphi_n * \mu(t) \rangle d|v|(t)$$

$$= \int_G \langle g(t), \chi_A * \mu(t) \rangle d|v|(t)$$

$$= \int_G \chi_A * \mu \, dv.$$

On the other hand, one notices that for each v in the dual of $C(G, X)$,

$$\langle \mathfrak{F}(A), v \rangle = \langle T_\mu^{**}(\chi_A), v \rangle$$

$$= \lim_n \langle T_\mu(\varphi_n), v \rangle$$

$$= \lim_n \int_G \varphi_n * \mu \, dv.$$

Hence we see that $\mathfrak{F}(A) = \chi_A * \mu$ as desired. This completes the proof.   □

We are now ready to characterize those convolution operators that are weakly compact on $C(G)$.

THEOREM 3.2. *Let G be a compact metrizable abelian group with Haar measure λ. Let X be a Banach space, and let μ be an X-valued countably additive measure of bounded variation. Let $T_\mu : C(G) \to C(G, X)$ be the convolution operator defined by $T_\mu(f) = f * \mu$ for all $\in C(G)$. Then the following properties are equivalent:*

(a) *the operator T_μ is weakly compact;*
(b) *the measure μ is absolutely continuous with respect to λ.*

Proof. Suppose the convolution operator T_μ is weakly compact. We first notice that by Lemma 3.1, the representing measure $\mathfrak{F}$ of T_μ is given by $\mathfrak{F}(A) = \chi_A * \mu$. Since T_μ is weakly compact, it follows that $\chi_A * \mu \in C(G, X)$ for each Borel subset A. Therefore, for every character γ on the group G, one has

$$\widehat{\mathfrak{F}(A)}(\gamma) = \widehat{\chi_A * \mu}(\gamma) = \int_G \chi_A * \mu(t) \bar{\gamma}(t) d\lambda(t) = \int_G \int_G \chi_A(t - s) \bar{\gamma}(t) d\mu(s) d\lambda(t)$$

$$= \int_G \int_G \chi_A(u) \bar{\gamma}(s + u) d\lambda(u) d\mu(s)$$

$$= \left(\int_G \chi_A(u) \bar{\gamma}(u) d\lambda(u) \right) \left(\int_G \bar{\gamma}(s) d\mu(s) \right)$$

$$= \widehat{\chi_A}(\gamma) \hat{\mu}(\gamma).$$

It follows that $\widehat{\mathfrak{F}(A)}(\gamma) = 0$, whenever $\lambda(A) = 0$. That is $\mathfrak{F}(A) = 0$ almost everywhere whenever $\lambda(A) = 0$. Since $\mathfrak{F}(A) = \chi_A * \mu$ is a continuous function, we simply have $\mathfrak{F}(A) = 0$ whenever $\lambda(A) = 0$. It follows that $\mathfrak{F}$ is λ-continuous. Since for each Borel subset A of G,

$$\mu(A) = \chi_{-A} * \mu(0) = \mathfrak{F}(-A)(0),$$

one sees that μ is also λ-continuous.

For the converse, define for each $t \in G$, the set-function $\mathfrak{F}_t : \mathcal{B}(G) \to X$ by

$$\mathfrak{F}_t(A) = \mathfrak{F}(A)t = \chi_A * \mu(t) = \mu(t - A).$$

We claim that $\lim_{\lambda(A) \to 0} \|\mathfrak{F}_t(A)\| = 0$ uniformly in t. Indeed fix $\varepsilon > 0$. Since by hypothesis μ is λ-continuous, there is a $\delta > 0$ such that

$$\lambda(A) < \delta \text{ implies } \|\mu(A)\| < \varepsilon.$$

Let t be in G. By the translation invariance property of the Haar measure λ, one has if $\lambda(A) < \delta$ then $\lambda(t - A) < \delta$, and thus

$$\|\mathfrak{F}_t(A)\| = \|\mu(t - A)\| < \varepsilon.$$

That is for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\sup_t \|\mathfrak{F}(A)(t)\| < \varepsilon$, whenever $\lambda(A) < \delta$. This proves our claim, and shows that $\mathfrak{F}$ is absolutely continuous with respect to λ.

On the other hand, we also have $\|\mathfrak{F}(A)\|_\infty \leq \|\mu\|$, for each $A \in \mathcal{B}(G)$. That is

$$\sup\{\|\mathfrak{F}(A)\|_\infty, A \in \mathcal{B}(G)\} < \infty.$$

This means by [**4**, Proposition I.1.11], that the measure $\mathfrak{F}$ is of bounded semivariation. A corollary of the Carathéodory-Hahn-Kluvanek Extension Theorem (see [**4**, p. 27–28]) now ensures that $\mathfrak{F}$ is strongly additive. Thus by [**4**, Theorem VI.1.1.], the operator T_μ is weakly compact. This finishes the proof. □

Our next result characterizes those convolution operators that are compact on $\mathcal{C}(G)$.

THEOREM 3.3. *Let G be a compact metrizable abelian group with Haar measure λ. Let X be a Banach space, and let μ be an X-valued countably additive measure of bounded variation. Let $T_\mu : \mathcal{C}(G) \to \mathcal{C}(G, X)$ be the convolution operator defined by $T_\mu(f) = f * \mu$ for all $f \in \mathcal{C}(G)$. Then the following properties are equivalent:*

(a) *the operator T_μ is compact;*
(b) *the measure μ is absolutely continuous with respect to λ and has relatively compact range.*

Proof. Suppose the convolution operator T_μ is compact. Then in particular T_μ is weakly compact, and thus by Theorem 3.2, μ is λ-continuous. According to Lemma 3.1, the representing measure $\mathfrak{F}$ of T_μ is given by $\mathfrak{F}(A) = \chi_A * \mu$. On the other hand, since the operator T_μ is compact, it is a well known result (see for example [**4**, p. 161]) that the representing

measure $\mathfrak{F}$ has relatively compact range. Thus the set $\left\{ \chi_A * \mu : A \in \mathcal{B}(G) \right\}$ is relatively compact in $\mathcal{C}(G, X)$. We notice that $\mu(A) = \chi_{-A} * \mu(0) = \mathfrak{F}(-A)(0)$. To finish the proof of the implication (a) $\Rightarrow$ (b) it suffices to notice that the evaluation operator $\delta_0 : \mathcal{C}(G, X) \to X; f \mapsto f(0)$ is a continuous linear operator.

Conversely, suppose that the measure μ is absolutely continuous with respect to Haar measure λ and suppose μ has relatively compact range. We want to show that the operator T_μ is compact.

Consider the operator $S : \mathcal{C}(G) \to X$ defined as follows: for each $f \in \mathcal{C}(G)$,

$$Sf = \int\limits_G f(s) d\mu(s).$$

Since the measure μ has relatively compact range, the operator S is compact [4]. Therefore, its adjoint $S^* : X^* \to (\mathcal{C}(G))^*$ is also compact. Since for each $x^* \in X^*$, and for every $f \in \mathcal{C}(G)$, we have

$$\langle S^* x^*, f \rangle = \langle x^*, Sf \rangle = x^* \left(\int\limits_G f(s) d\mu(s) \right) = \int\limits_G f(s) dx^* \mu(s),$$

it follows that $S^* x^* = x^* \mu$. Since the measure μ is absolutely continous with respect to λ, we see that the operator S^* takes its values in $\mathcal{L}^1(G)$. Moreover, the compactness of the operator S^* implies that the set $K = \{x^* \mu : \|x^*\| \le 1\}$ is compact in $\mathcal{L}^1(G)$.

Now for each $t \in G$ and for every $x^* \in X^*$, consider the element $x^* \otimes \delta_t$ of $(\mathcal{C}(G, X))^*$ given by $\langle x^* \otimes \delta_t, \varphi \rangle = x^* \varphi(t)$ for every $\varphi \in \mathcal{C}(G, X)$. Then we observe that for each $f \in \mathcal{C}(G)$,

$$\langle T_\mu^*(x^* \otimes \delta_t), f \rangle = \langle x^* \otimes \delta_t, T_\mu f \rangle = x^* f * \mu(t) = \int\limits_G f dx^* \mu_t$$

where μ_t is the measure defined by $\mu_t(A) = \mu(t - A)$. Thus $T_\mu^*(x^* \otimes \delta_t) = x^* \mu_t$. Since μ is λ-continuous, so is the scalar measure $x^* \mu_t$ for each $x^* \in X^*$ and for every $t \in G$. That is the set $T_\mu^*(\mathcal{E})$ lies in $\mathcal{L}^1(G)$, where $\mathcal{E} = \{x^* \otimes \delta_t : \|x^*\| \le 1, t \in G\}$. We are going to show that in fact, the set $T_\mu^*(\mathcal{E})$ is compact.

To this end, we first notice that the map

$$\phi : G \times \mathcal{L}^1(G) \to \mathcal{L}^1(G)$$
$$(t, f) \mapsto \bar{f}_t,$$

where $\bar{f}_t(s) = f(t - s)$, is continuous. Indeed if (t_n, f_n) converges to (t, f) in $G \times \mathcal{L}^1(G)$ then

$$\| \bar{f}_t - \bar{f}_{nt_n} \|_1 \le \int\limits_G |f(t - u) - f(t_n - u)| d\lambda(u)$$

$$+ \int\limits_G |f(t_n - u) - f_n(t_n - u)| d\lambda(u)$$

$$= \| \bar{f}_t - \bar{f}_{t_n} \|_1 + \| f - f_n \|_1.$$

For every $f \in \mathcal{L}^1(G)$, it is easily checked that the map $t \mapsto \bar{f}_t$ is continuous (see for example [7]). One then deduces the continuity of ϕ. It follows that the set $T_\mu^*(\mathcal{E})$ is compact as the image by ϕ of the compact $G \times K$. This proves our claim.

Now for simplicity, let B denote the unit ball of the dual of $C(G, X)$, and let ext B be the set of its extreme points. Thus we have

$$T_\mu^*(B) \subset \overline{T_\mu^* \overline{co}(\text{ext } B)}^{\text{weak}*} \subset \overline{\overline{co}\left(\overline{T_\mu^*(\text{ext } B)}\right)}^{\text{weak}*}.$$

It is a well known fact [10] that ext B is contained in $\mathcal{E}$. This implies that $\overline{T_\mu^*(\text{ext } B)}$ is compact and by Mazur's Theorem (see [4]), so is $\bar{co}(\overline{T_\mu^*(\text{ext } B)})$. It follows that in particular $\bar{co}(\overline{T_\mu^*(\text{ext } B)})$ is weakly compact and thus it is weak* closed. Therefore

$$\overline{\overline{co}\left(\overline{T_\mu^*(\text{ext } B)}\right)}^{\text{weak}*} = \overline{co}\left(\overline{T_\mu^*(\text{ext } B)}\right).$$

Accordingly,

$$T_\mu^*(B) \subset \bar{co}\left(\overline{T_\mu^*(\text{ext } B)}\right).$$

This shows that $T_\mu^*(B)$ has compact closure and hence terminates our proof. □

As promised in the introduction, we are now ready to give a characterization of those Banach spaces X where for every X-valued measure of bounded variation, the convolution operator $T_\mu : C(G) \to C(G, X)$ is compact as soon as it is weakly compact. First let us recall that a Banach space X has the *complete continuity property* (CRP) if for any X-valued measure of bounded variation μ defined on a group G, which is absolutely continuous with respect to Haar measure, the measure μ has a relatively compact range. For more on the CRP property see [5] or [9]. Our next Corollary gives a complete answer to Question 1.

COROLLARY 3.4. *Let G be a compact metrizable abelian group with Haar measure λ. A Banach space X has the complete continuity property if and only if for every X-valued measure μ of bounded variation, the convolution operator $T_\mu : C(G) \to C(G, X)$ defined by $T_\mu f = f * \mu$, for $f \in C(G)$, is compact whenever it is weakly compact.*

4. p-summing and nuclear convolution operators on $C(G)$. In this section, we shall study the vector valued counterpart of Theorem 1.3. More precisely, we shall examine those convolution operators that are either p-summing or nuclear. Recall that a bounded linear operator T from a Banach space X into a Banach space Y is said to be *p-summing* $(1 \leq p < \infty)$, if there exists a constant $C > 0$ such that for any $x_1, x_2, \ldots, x_n$ in X

$$\left(\sum_{i=1}^n \|Tx_i\|^p\right)^{1/p} \leq C \sup\left\{\left(\sum_{i=1}^n |x^*(x_i)|^p\right)^{1/p} : x^* \in X^*, \|x^*\| \leq 1\right\}. \tag{4.1}$$

We denote by $\pi_p(T)$ the p-summing norm of T, which is the smallest constant C such that the inequality (4.1) holds. The operator T is said to be *nuclear* if there exist sequences $\{x_n\}_{n\geq 1}$ in X and $\{x_n^*\}_{n\geq 1}$ in X^* such that

$$Tx = \sum_{n=1}^{\infty} x_n^*(x)x_n, \text{ for each } x \in X$$

and

$$\sum_{n=1}^{\infty} \|x_n^*\|\|x_n\| < \infty.$$

For an extensive study of p-summing and nuclear operators, we refer the reader to [3]. We first consider the case of the p-summing convolution operators for $1 \leq p < \infty$.

First let $1 \leq q \leq \infty$ and $1/p + 1/q = 1$, and let μ be a vector measure defined on a compact metrizable abelian group G with values in a Banach space X. For every measureable set $A \subset G$, let $\mathcal{S}^p(A)$ be the set of all the step functions $f = \Sigma_i a_i \chi_{A_i} \in \mathcal{L}^p(G)$ where the A_i are disjoint Borel subsets of G contained in A, and such that $\|f\|_p \leq 1$. For a Borel subset of G, the *q-semivariation of μ at A* is defined as follows:

$$\tilde{\mu}_q(A) = \sup_{\Sigma_i a_i \chi_{A_i} \in \mathcal{S}^p(A)} \| \sum a_i \mu(A_i)\|_X.$$

A vector measure μ valued in a Banach space X is said to be of *bounded q-semivariation* if $\tilde{\mu}_q(G) < \infty$, and the measure is said to be of *bounded average range* if there exists a constant $C > 0$ such that $\|\mu(A)\| \leq C\lambda(A)$, for each Borel subset A of G.

The following fact is well known [2] and will be used later.

REMARK 4.1. *A vector measure μ is of bounded average range if and only if it is of bounded ∞-semivariation, i.e. $\tilde{\mu}_\infty(G) < \infty$.*

Proof. Clearly, we have

$$\sup_{A\in\mathcal{B}(G)} \frac{\|\mu(A)\|}{\lambda(A)} \leq \sup_{\Sigma_i a_i \chi_{A_i} \in \mathcal{S}^1(G)} \| \sum a_i \mu(A_i)\| = \tilde{\mu}_\infty(G).$$

So if $\tilde{\mu}_\infty(G) < \infty$, then $\|\mu(A)\| \leq \tilde{\mu}_\infty(G)\lambda(A)$ for each Borel subset A of G.

Conversely, if $f = \Sigma_i a_i \chi_{A_i} \in \mathcal{S}^1(G)$, then since the A_i are disjoint Borel subsets of G we notice that $\|f\|_1 = \|\Sigma|a_i|\chi_{A_i}\|_1$, and therefore

$$\|\Sigma a_i \mu(A_i)\| \leq \Sigma|a_i|\|\mu(A_i)\|$$
$$= \Sigma|a_i|\frac{\|\mu(A_i)\|}{\lambda(A_i)}\lambda(A_i)$$
$$= \int_G \Sigma|a_i|\chi_{A_i}\Sigma\chi_{A_j}\frac{\|\mu(A_j)\|}{\lambda(A_j)}d\lambda$$
$$\leq \|\Sigma|a_i|\chi_{A_i}\|_1 \left\|\Sigma\chi_{A_i}\frac{\|\mu(A_i)\|}{\lambda(A_i)}\right\|_\infty$$
$$\leq \|f\|_1 \sup_{B\subset G}\frac{\|\mu(B)\|}{\lambda(B)}.$$

This of course shows that $\tilde{\mu}_\infty(G) < \infty$ if μ is of bounded average range. $\qquad\square$

The reader is referred to [2] for the following result.

THEOREM 4.2. *Let G be a compact metrizable abelian group with Haar measure λ. Let $1 \le p \le \infty$ and $1/p + 1/q = 1$. There is an isomorphism between the set of all bounded linear mappings $U : \mathcal{L}^p(G) \to X$ and the set of all the measures $\nu : \mathcal{B}(G) \to X$ of bounded q-semi-variation given by the equality*

$$U(f) = \int\limits_G f d\nu, \ for f \in \mathcal{L}^p(G),$$

and in particular

$$\nu(A) = U(\chi_A),$$

for every Borel subset A of G. Moreover, if U and ν are in such correspondence, then $\|U\| = \tilde{\nu}_q(G)$.

Before stating our next result, we shall include for the sake of completeness the following well known result (see [3, p. 56] for a more general statement).

LEMMA 4.3. *Let G be a compact metrizable abelian group. Let X be a Banach space, and let μ be an X-valued measure of bounded variation. Let $T_\mu : \mathcal{C}(G) \to \mathcal{C}(G, X)$ be the convolution operator defined by $T_\mu(f) = f * \mu$ for all $f \in \mathcal{C}(G)$. Suppose that T_μ is p-summing, for $1 \le p < \infty$. Then T_μ factors through $\mathcal{L}^p(G)$.*

Proof. First for every $f \in \mathcal{C}(G)$, consider $f_{-t}(y) = f(y + t)$. Then clearly $f_{-t} \in \mathcal{C}(G)$ and a simple calculation shows that

$$\|f_{-t} * \mu\| = \|f * \mu\|. \tag{4.2}$$

Now suppose T_μ is p-summing. One can find a non-negative Pietsch measure [3, p. 48] σ on the group G such that

$$\|T_\mu f\|^p \le \left(\pi_p(T_\mu)\right)^p \int\limits_G |f(s)|^p d\sigma(s).$$

Here $\pi_p(T_\mu)$ denotes the p-summing norm of the operator T_μ. Define $\sigma_t \in \mathcal{C}(G)^*$ by

$$\langle \sigma_t, f \rangle = \langle \sigma, f_{-t} \rangle,$$

for all $f \in \mathcal{C}(G)$ and for every $t \in G$. It then follows from (4.2) that

$$\|T_\mu(f)\|^p = \|T_\mu(f_{-t})\|^p \le \pi_p^p(T_\mu) \int\limits_G |f_{-t}(s)|^p d\sigma(s) = \pi_p^p(T_\mu) \int\limits_G |f(s)|^p d\sigma_t(s). \tag{4.3}$$

This shows that σ_t is also a Pietsch measure for the operator T_μ. It is easy to show that the map $G \to \mathcal{C}(G)^*: t \mapsto \sigma_t$ is continuous when $\mathcal{C}(G)^*$ is endowed with the weak* topology. Thus

the weak* integral $\int_G \sigma_t d\lambda(t)$ exists. Therefore there exists a measure $\nu \in \mathcal{C}(G)^*$ such that for every $f \in \mathcal{C}(G)$

$$\langle \nu, f \rangle = \int_G \langle \sigma_t, f \rangle d\lambda(t).$$

Thus ν is a non-negative regular Borel measure and since

$$\nu(G) = \int_G \sigma_t(G) d\lambda(t) = \int_G \chi_G d\lambda = 1,$$

we see that ν is a probability measure. Thus by integrating both sides of (4.3) one has, for each $f \in \mathcal{C}(G)$,

$$\|T_\mu f\|^p \le \pi_p^p(T_\mu) \int_G \langle \sigma_t, |f|^p \rangle d\lambda(t) = \pi_p^p(T_\mu)\langle \nu, |f|^p \rangle.$$

This shows that ν is also a Pietsch measure for T_μ. To finish the proof we only need to show that ν is translation invariant, and so must coincide with the Haar measure λ on G. To this end, note that for any $t_0 \in G$ and $f \in \mathcal{C}(G)$, if we use Fubini's theorem and the translation invariance property of the Haar measure we have

$$
\begin{aligned}
\langle \nu, f_{-t_0} \rangle &= \int_G \langle \sigma_t, f_{-t_0} \rangle d\lambda(t) = \int_G \int_G f_{-t_0}(s) d\sigma_t(s) d\lambda(t) \\
&= \int_G \int_G f(s + t_0) d\sigma_t(s) d\lambda(t) = \int_G \int_G f(s + t_0 + t) d\sigma(s) d\lambda(t) \\
&= \int_G \int_G f(s + t_0 + t) d\lambda(t) d\sigma(s) = \int_G \int_G f(s + t) d\lambda(t) d\sigma(s) \\
&= \int_G \int_G f(s + t) d\sigma(s) d\lambda(t) = \int_G \int_G f(s) d\sigma_t(s) d\lambda(t) \\
&= \int_G \langle \sigma_t, f \rangle d\lambda(t) = \langle \nu, f \rangle. \qquad \square
\end{aligned}
$$

We now state our next result which characterizes those convolution operators which are p-summing on $\mathcal{C}(G)$.

THEOREM 4.4. *Let G be a compact metrizable abelian group with Haar measure λ. Let X be a Banach space, and let μ be an X-valued measure of bounded variation. Let $T_\mu : \mathcal{C}(G) \to \mathcal{C}(G, X)$ be the convolution operator defined by $T_\mu(f) = f * \mu$ for all $f \in \mathcal{C}(G)$. Let $1 \le p < \infty$ and $1/p + 1/q = 1$. Then the following properties are equivalent:*

    (a) *the operator T_μ is p-summing;*
    (b) *the measure μ is of bounded q-semivariation.*

Proof. Suppose the convolution operator T_μ is p-summing. By Lemma 4.3 there exists a bounded linear map $U : \mathcal{L}^p(G) \to \mathcal{C}(G, X)$ such that $Uj_p = T_\mu$ where $j_p : \mathcal{C}(G) \to \mathcal{L}^p(G)$ is the canonical injection. Let $\delta_0 : \mathcal{C}(G, X) \to X$ denote the evaluation operator defined by $\delta_0(f) = f(0)$. Consider the operator

$$U_0 = \delta_0 U : \mathcal{L}^p(G) \to X$$
$$f \mapsto Uf(0).$$

Let ν be the measure of bounded q-semivariation associated to the operator U_0. By Theorem 4.2 we have $U_0(f) = \int_G f d\nu$ for every $f \in \mathcal{L}^p(G)$. We then notice that for every $f \in \mathcal{C}(G)$,

$$\int_G f(t) d\nu(t) = U_0(f) = U(f)(0) = f * \mu(0) = \int_G f(-t) d\mu(t).$$

That is $\mu = \nu'$, where $\nu'(A) = \nu(-A)$. This shows that μ is of bounded q-semivariation.

Conversely suppose the measure μ is of bounded q-semivariation. Let $U : \mathcal{L}^p(G) \to X$ be the bounded linear operator associated to μ as in Theorem 4.2. Then $U(f) = \int_G f d\mu$, for $f \in \mathcal{L}^p(G)$. One has

$$\|U(f)\| = \| \int_G f d\mu \| \leq \|f\|_p \bar{\mu}_q(G) = \|f\|_p \|U\|.$$

It follows that for every $f \in \mathcal{L}^p(G)$, and for every t and t' in G

$$\|f * \mu(t) - f * \mu(t')\|_X = \| \int_G f(t - s) - f(t' - s) d\mu(s)\|_X \leq \|\bar{f}_t - \bar{f}_{t'}\|_p \|U\|$$

where $\bar{f}(s) = f(-s)$ for every $s \in G$. This shows that the convolution operator $U_\mu : \mathcal{L}^p(G) \to \mathcal{L}^p(G, X); f \mapsto f * \mu$ actually is a bounded operator into $\mathcal{C}(G, X)$, and that $\|U_\mu\| \leq \|U\|$. It is plain that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{C}(G) & \overset{T_\mu}{\longrightarrow} & \mathcal{C}(G, X) \\
{}_{j_p} \searrow & & \nearrow {}_{U_\mu} \\
& \mathcal{L}^p(G) &
\end{array}
$$

where j_p is the canonical injection. This diagram clearly shows that the operator T_μ is p-summing, and completes the proof. $\square$

In view of Remark 4.1, we have the following result.

COROLLARY 4.5. *Let G be a compact metrizable abelian group with Haar measure λ. Let X be a Banach space, and let μ be an X-valued measure of bounded variation. Let $T_\mu : \mathcal{C}(G) \to \mathcal{C}(G, X)$ be the convolution operator defined by $T_\mu(f) = f * \mu$ for all $f \in \mathcal{C}(G)$. Then the following properties are equivalent:*

(a) *the operator T_μ is 1-summing;*
(b) *the measure μ is of bounded average range.*

REMARK 4.6. It follows from Theorem 4.4 and Corollary 4.5 that if X is a Banach space that has the Radon-Nikodym property (RNP), and μ is an X-valued measure of bounded variation, then $T_\mu : C(G) \to C(G, X)$ is p-summing ($1 \leq p < \infty$) iff there exists $\phi \in \mathcal{L}^p(G, X)$ such that $\mu = \phi \cdot \lambda$, in particular T_μ is 1-summing iff $\mu = \phi \cdot \lambda$ for some $\phi \in \mathcal{L}^\infty(G, X)$.

We finish this section by studying those convolution operators that are nuclear on the space of continuous functions $C(G)$.

THEOREM 4.7. *Let G be a compact metrizable abelian group with Haar measure λ. Let X be a Banach space, and let μ be an X-valued measure of bounded variation. Let $T_\mu : C(G) \to C(G, X)$ be the convolution operator defined by $T_\mu(f) = f * \mu$ for all $f \in C(G)$. Then the following properties are equivalent:*

 (a) *the operator T_μ is nuclear;*
 (b) *there exists $h \in C(G, X)$ such that $\mu = h \cdot \lambda$.*

Proof. Suppose the operator T_μ is nuclear. Let $\mathfrak{F}$ be the representing measure of T_μ. Then $\mathfrak{F}$ is a countably additive $C(G, X)$-valued measure of bounded variation and there exists $g \in \mathcal{L}^1(G, |\mathfrak{F}|, C(G, X))$ such that $\mathfrak{F} = g \cdot |\mathfrak{F}|$ (see for example [2, p. 173]). Here $\mathcal{L}^1(G, |\mathfrak{F}|, C(G, X))$ stands for (all classes of) $|\mathfrak{F}|$-Bochner integrable functions on G with values in $C(G, X)$. Moreover, one should note that the function g is actually an essentially bounded function.

Since the operator T_μ is in particular 1-summing, by Corollary 4.5, the measure μ is of bounded average range. It then follows that for every $t \in G$, and each Borel subset A of G and for some constant $C > 0$,

$$\|\mathfrak{F}(A)(t)\| = \|\chi_A * \mu(t)\| = \|\mu(t - A)\| \leq C\lambda(A).$$

Thus since the measure $\mathfrak{F}$ takes its values in $C(G, X)$, one deduces

$$\|\mathfrak{F}(A)\|_\infty \leq C\lambda(A).$$

This shows that $\mathfrak{F}$ is of bounded average range and its variation satisfies $|\mathfrak{F}| \leq C\lambda$. Therefore $\mathfrak{F} = \phi . \lambda$ for some $\phi \in \mathcal{L}^\infty(G, C(G, X))$. The function ϕ can be considered as function of two variables as follows

$$\phi(u, v) = \phi(u)(v), \text{ for each } (u, v) \in G \times G.$$

As in the proof of Theorem 2.1, we define $h \in \mathcal{L}^1(G, X)$ by

$$h(t) = \int_G \phi(u - t, u) d\lambda(u).$$

We claim that h is a continuous function. To see this, we notice that

$$h(s) = \int_G \phi(u - s, u) d\lambda(u)$$

$$= \int_G \phi(v, v + s) d\lambda(v).$$

Therefore,

$$\|h(t) - h(s)\| = \| \int_G \phi(v, v + t) - \phi(v, v + s)d\lambda(v)\|$$

$$\leq \int_G \|\phi(v, v + t) - \phi(v, v, +s)\|d\lambda(v).$$

Since the function $t \mapsto \phi(v, v + t)$ is continuous for each $v \in G$, the integrand in the last integral converges to zero as t tends to s, for each $v \in G$. On the other hand, the following inequality clearly holds

$$\|\phi(v, v + t) - \phi(v, v + s)\| \leq 2\|\phi\|_\infty.$$

An appeal to the Lebesgue Dominated Convergence Theorem now establishes our claim. The same reasoning as in (2.1) and (2.2) now shows that $\mu = h \cdot \lambda$.

Conversely, suppose $\mu = h \cdot \lambda$ where $h \in C(G, X)$. Consider the map $g : G \to C(G, X) :$ $y \mapsto h_y$. Then clearly the function g is continuous and therefore it is λ-Bochner integrable. Now fix $\varepsilon > 0$. According to Lemma 3 of ([4, p. 173]), there is a sequence (ϕ_n) in $C(G, X)$ and a sequence of Borel sets (E_n) such that

$$g = \sum_{n=1}^{\infty} \phi_n \chi_{E_n}$$

λ-almost everywhere, and

$$\int_G \|g\|d\lambda \leq \sum_{n=1}^{\infty} \|\phi_n\|\lambda(E_n) \leq \int_G \|g\|d\lambda + \varepsilon.$$

For $f \in C(G)$, we have

$$T_\mu(f) = f * h = \int_G f(y) \sum_{n=1}^{\infty} \phi_n \chi_{E_n}(y)d\lambda(y)$$

$$= \sum_{n=1}^{\infty} \int_{E_n} f(y)d\lambda(y)\phi_n.$$

Obviously, the map $f \mapsto L_n(f) = \int_{E_n} fd\lambda$ is a bounded linear functional on $C(G)$ and $T_\mu(f) = \sum_{n=1}^{\infty} L_n(f)\phi_n$ and

$$\sum_{n=1}^{\infty} \|L_n(f)\|\|\phi_n\| \leq \int_G \|g\|d\lambda + \varepsilon.$$

This proves that T_μ is nuclear and ends our proof.   □

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