CONVEX STRUCTURES AND CONTINUOUS SELECTIONS

ERNEST MICHAEL

Introduction. This paper continues the study of continuous selections begun in **(13**; **14**; **15)** and the expository paper **(12)**.[1] The purpose of these papers, which is described in detail in the introduction to **(13)**, can be summarized here as follows. If X and Y are topological spaces, and ϕ a function (called a *carrier*) from X to the space 2^Y of non-empty subsets of Y, then a *selection* for ϕ is a continuous $f: X \to Y$ such that $f(x) \in \phi(x)$ for every $x \in X$. For reasons which are explained in **(13)**, we restrict our attention to carriers which are *lower semi-continuous* (l.s.c.), in the sense that, whenever U is open in Y, then $\{x \in X \mid \phi(x) \cap U \neq \emptyset\}$ is open in X. Our purpose in these papers is to find conditions for the existence and extendability of selections.

The principal purpose of *this* paper is to generalize the following result, which is half of the principal theorem (Theorem 3.2″) of **(13)** (and is repeated as Theorem I of **(12)**).

THEOREM A. *Let S be paracompact, Y a Banach space, and $\mathfrak{S}$ the family of closed, convex, non-empty subsets of Y. Then every lower semi-continuous carrier $\phi: X \to \mathfrak{S}$ admits a selection.*

In this paper, the Banach space Y is replaced by a complete metric space carrying an axiomatically defined *convex structure* (Definition 1.1) which permits one to take "convex combinations" of some (but not necessarily all) ordered n-tuples of points in Y, in a suitably continuous fashion. With *convex* sets defined in the obvious way (Definition 1.4), the generalization of Theorem A which is thus obtained is given in Theorem 1.5 (a). The significance of this generalization is illustrated by the following example.

Example A. G is a metrizable group, and H a closed subgroup which is isomorphic to the additive group of a Banach space. Convex combinations can be taken of n-tuples lying in the same right coset of H. The right cosets of H are then convex sets. By applying Theorem 1.5 (a) to this situation, it is shown in Corollary 7.3 that there exists a cross-section.[2] (For G a Banach

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[1]Except for § 8, no previous knowledge of continuous selections is necessary to read this paper, although some acquaintance with the first three sections of **(13)** will be helpful.

[2]If G is a topological group, H a closed subgroup, u the canonical map from G to G/H, and if $\phi: G/H \to G$ is defined by $\phi(x) = u^{-1}(x)$, then a *cross-section* (resp. *local cross section*) is a selection for ϕ (resp. $\phi|U$ for some non-empty open $U \subset G/H$).

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space and H a linear subspace, this already follows from Theorem A, and was first proved by Bartle and Graves in **(2)**.)

If the family of sets $\mathfrak{S}$, instead of having convex elements, is only what we shall call (Definition 1.4) *equi-locally convex*, then the global Theorem 1.5 (a) changes into the local Theorem 1.5 (b). This is illustrated by the following example.

Example B. If Example A is modified only by assuming H to be *locally* isomorphic to the additive group of a Banach space, then only "close together" n-tuples in the same right coset of H admit convex combinations, the collection of right cosets of H is equi-locally convex, and Theorem 1.5 (b) implies that there exists a *local* cross-section[3] (Corollary 7.3).

Both parts of Theorem 1.5 will be obtained as simple consequences of a single Theorem, 1.3, which also generalizes Theorem A. The statements of these theorems, as well as the basic definitions, are found in § 1. § 2 contains some preliminary results which are not directly concerned with selections. § 3, the core of this paper, contains the proof of Theorem 1.3. This proof parallels that of Theorem A in **(13)**, by obtaining the desired function f as the limit of a uniformly convergent sequence of functions f_n, However, while in the proof of Theorem A the f_n's are *continuous* with $f_n(x)$ *near* $\phi(x)$ for all x, in the proof of Theorem 1.3 the f_n's may be *discontinuous* with $f_n(x)$ *in* $\phi(x)$ for all x; this new approach is necessitated by the inability to take convex combinations of n-tuples in Y not lying in an element of $\mathfrak{S}$. The simple derivation of Theorem 1.5 from Theorem 1.3 is contained in § 4.

An important source of convex structures on a metric space Y is provided by so-called *geodesic structures* on Y, which permit one to take convex combinations of certain *pairs* of points of Y in an appropriate fashion. Geodesic structures are studied in § 5, where it is shown (Proposition 5.3) how they inductively generate convex structures in a canonical fashion. In § 6, a recent theorem of Nijenhuis **(17)** is used to show that the usual geodesic segments on a Riemannian manifold give rise to a geodesic structure, and hence to a convex one.

The last two sections are devoted to applications of Theorems 1.3 and 1.5 and, except when dealing with Lie groups, are independent of §§ 5 and 6. § 7 deals with *locally convex groups*, which are, essentially, topological groups with an invariant convex structure defined on a neighbourhood of the identity. Theorem 1.5 is used to prove a cross-section theorem (Theorem 7.2) for such groups, which implies (Corollary 7.3) the results given in Examples A and B above, as well as Gleason's cross-section theorem for arbitrary Lie groups. § 8, finally, uses Theorem 1.3 to prove a theorem of the covering homotopy type, which is then applied to fibre spaces in the sense of Hurewicz **(7)**.

In conclusion, it should be noted that the results of this paper are of interest only if the domain X of the functions to be defined is *infinite* dimen-

[3] *Ibid.*

sional; for *finite* dimensional X, better results were already obtained in **(14)** and, with reference to § 8, in **(15)**.

1. Principal theorems and definitions. Throughout this paper, P_n denotes the unit simplex in Euclidean n-space R^n; that is,

$$P_n = \left\{ t \in R^n | 0 \leqslant t_i \leqslant 1, \qquad (i = 1, \ldots, n), \sum_{i=1}^{n} t_i = 1 \right\}.$$

If E is any set, then E^n will denote the n-fold Cartesian product of E, and if $i \leqslant n$, then $\partial_i: E^n \to E^{n+1}$ is defined by $\partial_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

Definition 1.1. A *convex structure* on a metric space E with metric ρ assigns to each positive integer n a subset M_n of E^n, and a function $k_n: M_n \times P_n \to E$, such that

(a) If $x \in M_1$, then $k_1(x, 1) = x$.

(b) If $x \in M_n$ $(n \geqslant 2)$ and $i \leqslant n$, then $\partial_i x \in M_{n-1}$ and, for any $t \in P_n$ with $t_i = 0$, $k_n(x, t) = k_{n-1}(\partial_i x, \partial_i t)$.

(c) If $x \in M_n$ $(n \geqslant 2)$ with $x_i = x_{i+1}$ for some $i < n$, and if $t \in P_n$, then $k_n(x, t) = k_{n-1}(\partial_i x, t^*)$, where $t^* = (t_1, \ldots, t_{i-1}, t_i, + t_{i+1}, t_{i+2}, \ldots, t_n)$.

(d) If $x \in M_n$, then the map $t \to k_n(x, t)$, from P_n to E, is continuous.

(e) For all $\epsilon > 0$ there exists a neighbourhood V_ϵ of the diagonal in $E \times E$ such that, for all n and all $x, y \in M_n$, $(x_i, y_i) \in V_\epsilon$ for $i = 1, \ldots, n$ implies $\rho(k_n(x, t), k_n(y, t)) < \epsilon$ for all $t \in P_n$.

Note that conditions (a) and (c) together imply that, if $x \in M_n$ with $x_1 = \ldots = x_n$, then $k_n(x, t) = x_1$ for all $t \in P_n$.

Definition 1.2. A subset S of a space E with convex structure is *admissible* if $S^n \subset M_n$ for all n. If S is admissible, then the *convex hulk of* S, denoted by $\operatorname{conv}(S)$, is

$$\{k_n(x, t) \mid x \in S^n, \qquad t \in P_n, \qquad n = 1, 2, \ldots\}.$$

THEOREM 1.3. *Let Y be a complete metric space with a convex structure, and let $\mathfrak{S}$ be the family of non-empty admissible subsets of Y. Let X be paracompact, and $\phi: X \to \mathfrak{S}$ lower semi-continuous. Then there exists a continuous $f: X \to Y$ such that*

$$f(x) \in [\operatorname{conv}(\phi(x))]^-$$

for all $x \in X$.

Definition 1.4. Let E be a metric space with convex structure. Then a subset S of E is *convex* if it is admissible and $\operatorname{conv}(S) \subset S$. A family $\mathfrak{S}$ of subsets is *equi-locally convex* if there exists an open covering $\mathfrak{B}$ of E such that, whenever $S \in \mathfrak{S}$ and $B \in \mathfrak{B}$, then $S \cap B$ is admissible and $\operatorname{conv}(S \cap B) \subset S$.

Note that every family of convex sets is equi-locally convex.

THEOREM 1.5. *Let X be paracompact, Y a metric space with convex structure, $\mathfrak{S}$ a family of non-empty, complete subsets of Y, and $\phi: X \to \mathfrak{S}$ lower semi-continuous. Let $A \subset X$ be closed and let g be a selection for $\phi|A$. Then*

(a) *If every $S \in \mathfrak{S}$ is convex, then g can be extended to a selection for ϕ.*

(b) *If $\mathfrak{S}$ is equi-locally convex, then g can be extended to a selection for $\phi|U$ for some open $U \supset A$.*

No discussion of spaces with a convex structure would be complete without mentioning their extension properties. For convenience, let us call a metric space E *convex* if it admits a convex structure making E itself a convex set. As a special case of Theorem 1.5, we see that a convex complete metric space is an AE (absolute extensor) for paracompact spaces (in the sense of **(10)**); a somewhat better result is obtainable directly as a straightforward generalization of Dugundiji's extension theorem **(5**, Theorem 4.1**)**. A more interesting fact is a partial converse: *Every compact metric space E, which is an* AE *for metric spaces, is convex.* To see this, embed E in a Banach space B **(8)**, and let F be the closed convex hull of E in B. Then there exists a retraction $r: F \to E$, and we define $k_n: E^n \times P_n \to E$ by

$$k_n(x, t) = r\left(\sum_{i=1}^{n} t_i x_i\right).$$

That conditions (a) — (d) of Definition 1.1 are satisfied is clear, while (e) follows from the factthat F is compact **(9)** and hence r uniformly continuous. It is not know whether compactness can be replaced by a weaker condition in the above result.

2. Two lemmas.

Our first lemma deals with the following very elementary concept.

Definition 2.1. If X is a topological space, (Y, ρ) a metric space, and $a \geqslant 0$, then a function $f: X \to Y$ is *a-continuous at $x_0 \in X$* if to every $\epsilon > 0$ there corresponds a neighbourhood U of x_0 such that

$$\rho(f(x), \quad f(x_0)) < a + \epsilon$$

for every x in U. Moreover, f is *a-continuous* if it is a-continuous at every $x_0 \in X$.

Note that clearly f is 0-continuous (at x_0) if, and only if, f is continuous (at x_0).

LEMMA 2.2. *Le X be a topological space, (Y, ρ) a metric space, and let $f_n: X \to Y$ $(n = 1, 2, \ldots)$ be a sequence of functions which converges uniformly to a function $f: X \to Y$. Let $x_0 \in X$, and suppose that to each $a > 0$ and positive integer N corresponds an $n > N$ such that f_n is a-continuous at x_0. Then f is continuous at x_0.*

Proof. Let $\epsilon > 0$. Pick an n such that $\rho(f_n(x), f(x)) < \frac{1}{3}\epsilon$ for all $x \in X$, and

f_n is $\frac{1}{3}\epsilon$-continuous at x_0. Pick a neighbourhood U of x_0 such that, if $x \in U$, then $\rho(f_n(x), f_n(x_0)) < \frac{1}{3}\epsilon$. Then, if $x \in U$,

$$\rho(f(x_0), f(x)) \leqslant \rho(f(x_0), f_n(x_0)) + \rho(f_n(x_0), f_n(x) + \rho(f_n(x), f(x)) < \epsilon.$$

This completes the proof.

Our second lemma concerns partitions of unity.[4] In it, as well as later in the paper, we denote the set of points where a real-valued function f is positive by $P(f)$. An indexed family of real-valued functions $\{p_\alpha\}_{\alpha \in A}$ will be called *locally-finite* if the indexed family of sets $\{P(p_\alpha)\}_{\alpha \in A}$ is locally finite.[5]

LEMMA 2.3. *Let X be paracompact, Y a topological space, $\phi: X \to 2^Y$ lower semi-continuous, and $\{\sigma_n\}_{n=1}^{\infty}$ a sequence of continuous pseudometrics on Y. Then for each integer $n > 0$ there exists an index set A_n, a locally finite partition of unity $\{p_\alpha\}_{\alpha \in A_n}$ on X, points $y_\alpha(x) \in \phi(x)$ whenever $\alpha \in A_n$ and $x \in P(p_\alpha)$, and a map $\pi_n: A_{n+1} \to A_n$ onto, such that the following conditions are satisfied for all n.*

(a) *If $\alpha \in A_n$ and $x, x' \in P(p_\alpha)^-$, then $\sigma_n(y_\alpha(x), y_\alpha(x')) < 1$.*

(b) *If $\alpha \in A_n$, $\beta \in \pi_n^{-1}(\alpha)$, and $x \in P(p_\beta)^-$, then $\sigma_n(y_\alpha(x), y_\beta(x)) < 1$.*

(c) *If $\alpha \in A_n$, then $p_\alpha(x) = \Sigma\{p_\beta(x) | \beta \in \pi_n^{-1}(\alpha)\}$ for all $x \in X$.*

Proof. This theorem follows easily from **(16**, Lemma 2.1**)**. The statement of that result parallels that of our theorem, with the following differences: Instead of a partition of unity $\{p_\alpha\}_{\alpha \in A_n}$ on X, **(16**, Lemma 2.1**)** only provided us with an open covering $\{U_\alpha\}_{\alpha \in A_n}$ of X, the sets $P(p_\alpha)^-$ in (a) and (b) were replaced by $\bar{U}_\alpha$, and (c) was replaced by

(c′) *If $\alpha \in A_n$, then $U_\alpha = \bigcup \{U_\beta | \beta \in \pi_n^{-1}(\alpha)\}$.*

To obtain our partitions of unity, we proceed by induction, in such a way that

(d) $P(p_\alpha)^- \subset U_\alpha \qquad \alpha \in A_n,\ n = 1, 2, \ldots.$

For $n = 1$, simply pick an open covering $\{V_\alpha\}_{\alpha \in A_1}$ of X such that $\bar{V}_\alpha \subset U_\alpha$ for all $\alpha \in A_1$, and let $\{p_\alpha\}_{\alpha \in A_1}$ be a partition of unity subordinated[6] to $\{V_\alpha\}_{\alpha \in A_1}$. Suppose we have $\{p_\alpha\}_{\alpha \in A_n}$, and let us construct $\{p_\beta\}_{\beta \in A_{n+1}}$. For convenience, we shall write π for π_n.

Let $\alpha \in A_n$. Then $\{U_\beta \cap P(p_\alpha)^-\}_{\beta \in \pi^{-1}(\alpha)}$ is a relatively open covering of $P(p_\alpha)^-$ by (c′) and (d), and hence has a relatively open refinement $\{V_\beta\}_{\beta \in \pi^{-1}(\alpha)}$ with $\bar{V}_\beta \subset U_\beta$ for all $\beta \in \pi^{-1}(\alpha)$. Let $\{q_\beta\}_{\beta \in \pi^{-1}(\alpha)}$ be a partition of untiy on

[4] *A partition of unity* in a topological space is an indexed family $\{p_\alpha\}_{\alpha \in A}$ of functions from X to [0, 1] such that $\Sigma_{\alpha \in A} p_\alpha(x) = 1$ for all $x \in X$.

[5] An indexed family of sets $\{U_\alpha\}_{\alpha \in A}$ is called *locally finite* if each $x \in X$ has a neighbourhood intersecting U_α for only finitely many $a \in A$.

[6] $\{p_\alpha\}_{\alpha \in A}$ is *subordinated to* $\{U_\alpha\}_{\alpha \in A}$ if p_α vanishes outside U_α for all α. Every open covering of a paracompact space has a partition of unty subordinated to it.

$P(p_\alpha)^-$ which is subordinated to $\{V_\beta\}_{\beta\epsilon\pi^{-1}(\alpha)}$. For each $\beta \in \pi^{-1}(\alpha)$, define $p_\beta: X \to [0, 1]$ by

$$p_\beta(x) = 0 \qquad\qquad x \notin P(p_\alpha),$$
$$p_\beta(x) = p_\alpha(x) \,.\, q_\beta(x) \qquad\qquad x \in P(p_\alpha)^-.$$

This function p_β is well defined, and is continuous on X since it is continuous on two closed sets whose union is X. Moreover, $\{p_\beta\}_{\beta\epsilon \mathrm{A}_{n+1}}$ is now clearly a partition of unity on X which satisfies (a) – (d). This completes the proof.

3. Proof of Theorem 1.3. We begin by observing that conditions (c) and (e) of Definition 1.1 are equivalent to the following apparently stronger conditions (c′) and (e′). In fact, (c′) follows from (c) by straightforward induction, while (e′) follows from (e) by means of a standard result (see, for instance (**11**, Corollary 2.3)).

1.1 (c′). Let m and m' be positive integers with $m \leqslant m'$, $y \in M_m$, $t \in P_m$, $y' \in M_{m'}$, and $t' \in P_{m'}$. Suppose that, for some order-preserving map $\pi: \{1, \ldots, m'\} \to \{1, \ldots, m\}$, we have

$$y'_i = y_{\pi(i)} \qquad\qquad i = 1, \ldots, m',$$
$$t_j = \sum_{i\epsilon\pi-1} (j)t'_i \qquad\qquad j = 1, \ldots, m.$$

Then

$$k_m(y, t) = k_{m'}(y', t').$$

1.1 (e′). For all integers $n > 0$ there exsits a continuous pseudometric σ_n on E with the following property: If $x, y \in M_m$ for some m, then $\sigma_n(x_i, y_i) < 1$ for $i = 1, \ldots, m$ implies $\rho(k_m(x, t), k_m(y, t)) < 2^{-n}$ for all $t \in P_m$.

The main step in the roof of Theorem 1.3 will be the following lemma, whose proof, in turn, rests heavily on Lemma 2.3.

LEMMA 3.1. *With X, Y, and $\phi: X \to 2^Y$ as in Theorem* 1.3, *there exists a sequence of functions $f_n: X \to Y$ $(n = 1, 2, \ldots)$ such that, for all n,*

(a) *$f_n(x) \in \operatorname{conv}(\phi(x))$ for all $x \in X$,*
(b) *f_n is 2^{-n}-continuous,*
(c) *$\rho(f_n(x), f_{n+1}(x)) < 2^{-n}$ for all $x \in X$.*

Proof. We begin by applying Lemma 2.3, with σ_n $(n = 1, 2, \ldots)$ as in 1.1 (e′) at the beginning of this section. Let A_n, p_α, $y_\alpha(x)$, and π_n be as in Lemma 2.3. Let us also suppose that the sets A_n are well ordered in such a way that each π_n is order preserving; this is easily done by induction.

For each n and $x \in X$, let

$$\mathrm{A}_n(x) = \{\alpha \in \mathrm{A}_n \mid x \in P(p_\alpha)\},$$
$$\bar{A}_n(x) = \{\alpha \in \mathrm{A}_n \mid x \in P(p_\alpha)^-\}.$$

Note that $\mathrm{A}_n(x) \subset \bar{A}_n(x)$, and that both are finite. Let $\alpha_1, \ldots, \alpha_m$ be the

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elements of $\bar{A}_n(x)$ in order. Remembering that $\phi(x)$ is admissible, we can now define $f_n(x)$ by

$$(1) \qquad f_n(x) = k_m((y_{\alpha_1}(x), \dots, y_{\alpha_m}(x)), (p_{\alpha_1}(x), \dots, p_{\alpha_m}(x))).$$

Note that, since $p_\alpha(x) = 0$ if $\alpha \notin \mathrm{A}_n(x)$, it follows from Definition 1.1(b) that $f_n(x)$ is not changed by the omission from (1) of terms with index $\alpha \notin \mathrm{A}_n(x)$.

Let us check that our conditions are satisfied.

(a) This follows from the definitions.

(b) We will show that f_n is 2^{-n}-continuous at a given $x_0 \in X$. Let

$$V = \bigcap\{P(p_\alpha)|\alpha \in \mathrm{A}_n(x_0)\} - \bigcup\{P(p_\alpha)^-|\alpha \in (\mathrm{A}_n - \bar{A}_n(x_0)\}.$$

Then V is a neighbourhood of x_0, since $\{P(p_\alpha)\}_{\alpha \in \mathrm{A}_n}$ is locally finite. Now let $x \in V$. Then

$$\mathrm{A}_n(x_0) \subset \mathrm{A}_n(x) \subset \bar{A}_n(x) \subset \bar{A}_n(x_0).$$

So if $\alpha_1, \dots, \alpha_m$ are the elements of $\bar{A}_n(x)$ in order, we have, by the remark following (1) above,

$$f_n(x) = k_m((y_{\alpha_1}(x), \dots, y_{\alpha_m}(x)), (p_{\alpha_1}(x), \dots, p_{\alpha_m}(x))),$$
$$f_n(x_0) = k_m((y_{\alpha_1}(x_0), \dots, y_{\alpha_m}(x_0)), (p_{\alpha_1}(x_0), \dots, p_{\alpha_m}(x_0))).$$

Let

$$h_n(x, x_0) = k_m((y_{\alpha_1}(x_0), \dots, y_{\alpha_m}(x_0)), (p_{\alpha_1}(x), \dots, p_{\alpha_m}(x))).$$

Clearly

$$(2) \quad \rho(f_n(x), f_n(x_0)) \leqslant \rho(f_n(x), h_n(x, x_0)) + \rho(h_n(x, x_0), f_n(x_0)).$$

Now

$$x, x_0 \in P(p_{\alpha_i})^- \ (i = 1, \dots, m),$$

whence

$$\sigma_n(y_{\alpha_i}(x), y_{\alpha_i}(x_0)) < 1$$

by 2.4(a), and hence, by 1.1(e′) at the beginning of this section,

$$(3) \qquad \rho(f_n(x), h_n(x, x_0)) < 2^{-n}.$$

On the other hand, it follows from Definition 1.1(d) and the continuity of the functions $p_{\alpha_1}, \dots, p_{\alpha_m}$ that to every $\epsilon > 0$ there corresponds a neighbourhood W of x_0 such that

$$(4) \qquad \rho(h_n(x, x_0), f_n(x_0)) < \epsilon \qquad \text{if } x \in W.$$

Combining (2), (3), and (4), we conclude

$$\rho(f_n(x), f_n(x_0)) < 2^{-n} + \epsilon \qquad \text{if } x \in V \cap W,$$

and hence f_n is 2^{-n}-continuous at x_0.

(c) Pick a fixed n and $x \in X$. Letting $\alpha_1, \dots, \alpha_m$ be the elements of $\mathrm{A}_n(x)$ in order, we have

$$f_n(x) = k_m((y_{\alpha_1}(x), \dots, y_{\alpha_m}(x)), (p_{\alpha_1}(x), \dots, p_{\alpha_m}(x)));$$

letting $\gamma_1, \ldots, \gamma_m$ be the elements of $A_{n+1}(x)$ in order, we have

$$f_{n+1}(x) = k_{m'}((y_{\gamma_1}(x), \ldots, y_{\gamma_{m'}}(x)), (p_{\gamma_1}(x), \ldots, p_{\gamma_{m'}}(x))).$$

Now let

$$h_{n+1}(x) = k_{m'}((y_{\theta_n(\gamma_1)}(x), \ldots, y_{\theta_n(\gamma_{m'})}(x)), (p_{\gamma_1}(x), \ldots, p_{\gamma_{m'}}(x))).$$

To prove (c), it will suffice to show that

$$(5) \qquad f_n(x) = h_{n+1}(x),$$

$$(6) \qquad \rho(f_{n+1}(x), h_{n+1}(x)) < 2^{-n}.$$

Let us prove (5). From lemma 2.3 (c) we see that, for each $i \leqslant m'$, $\pi_n(\gamma_i) = \alpha_j$ for some $j \leqslant m$. We can, therefore, define a map $\pi: \{1, \ldots, m'\} \to \{1, \ldots, m\}$ by

$$\pi(i) = j \qquad \text{if, and only if,} \qquad \pi_n(\gamma_i) = \alpha_j.$$

It follows, again from Lemma 2.3(c), that

$$p_{\alpha_j}(x) = \sum_{i \epsilon \pi} - 1_{(j)} p_{\gamma_i}(x) \qquad j = 1, \ldots, m.$$

Remembering that π_n, and hence π, is order-preserving, we now use Definition 1.1 (c′) (at the beginning of this section), with

$$y_j = y_{\alpha_j}(x), t_j = p_{\alpha_j}(x), y'_i = y_{\pi_n(\gamma_i)}(x),$$

and

$$t_i = p_{\gamma_i}(x) \; (j = 1, \ldots, m; i = 1, \ldots, m'),$$

to conclude (5).

It remains to establish (6). But this is easy since

$$\sigma_n(y_{\gamma_j}(x), y_{\pi(\gamma_j)}(x) < 1 \qquad j = 1, \ldots, m'$$

by Lemma 2.3 (b), and hence (6) follows from Definition 1.1 (e′) at the beginning of this section. This completes the proof of the lemma.

Theorem 1.3 is now a simple consequence of Lemma 3.1 and Lemma 2.2. In fact, the sequence

$$\{f_n\}_{n=1}^{\infty}$$

in Lemma 3.1 is uniformly Cauchy by 3.1 (c), and hence, by the completeness of Y, converges uniformly to an $f: X \to Y$. This f is continuous by Lemma 3.1 (b) and Lemma 2.2, and Lemma 3.1 (a) implies that $f(x) \in (\text{conv}(\phi(x)))^-$ for every $x \in X$. Thus f has all the required properties.

4. Proof of Theorem 1.5 from Theorem 1.3. Let X, $A \subset X$, Y, $\mathfrak{S} \subset 2^Y$, $\phi: X \to \mathfrak{S}$, and $g: A \to Y$ be as in Theorem 1.5. Since all our assumptions remain true if Y is replaced by its completion, we may as well assume that Y is complete.

(a) Assume that every $S \in \mathfrak{S}$ is convex. Define $\psi: X \to 2^Y$ by

(1) $$\psi(x) = \{g(x)\} \quad \text{if} \quad x \in A$$ $$\psi(x) = \phi(x) \quad \text{if} \quad x \notin A.$$

This ψ is lower semi-continuous by **(13**, Example 1.3***)**, and $\psi(x)$ is convex and closed for each $x \in X$ by the assumptions on ϕ and the remark following Definition 1.1. But Theorem 13 implies that such a ψ must have a selection f, and this f is the required extension of g.

(b) Assume that $\mathfrak{S}$ is equi-locally convex. Let $Y' = \cup \mathfrak{S}$. If $\mathfrak{B}$ is as in Definition 1.4, use **(11**, Corollary 3.2**)** to obtain a metric τ on Y', agreeing with the topology, such that the family of all τ-spheres of radius 1 in Y' is a refinement of $\mathfrak{B}$. We thus have

(2) *If* $S \in \mathfrak{S}$, *and* $A \subset S$ *has* τ*-diameter* < 1, *then* A *is admissible, and* conv(A) $\subset S$, *whence* $(\text{conv}(A))^- \subset \bar{S} = S$.

Now let E be any Banach space, with metric d, containing (Y', τ) isometrically **(8)**, let $h: X \to E$ be a continuous extension of g **(1**, Theorem 1.4**)**, and let

$$W = \{x \in X \mid d(h(x), \phi(x)) < \tfrac{1}{2}\}.$$

Then obviously $W \subset A$, and W is open by the lower semi-continuity of ϕ. Pick an open $U \subset X$ such that $A \subset U \subset \bar{U} \subset W$. Let ψ be as in (1) above, and define $\theta: \bar{U} \to 2^Y$ by[7]

$$\theta(x) = \psi(x) \cap S_{\frac{1}{2}}(h(x)).$$

This θ is lower semi-continuous by **(13**, Proposition 2.5**)**. Moreover, by (2) $\theta(x)$ is admissible and $[\text{conv}(\theta(x))]^- \subset \phi(x)$ for all $x \in X$, and by the remark following Definition 1, $\text{conv}(\theta(x)) = \{g(x)\}$ for all $x \in A$. Our conclusion, therefore, follows from Theorem 1.3, with X replaced by $\bar{U}$, and ϕ by θ.

5. Geodesic structures. In this section, we define geodesic structures, and show how they generate convex structures. The closed unit interval will always be denoted by I.

Definition 5.1. A geodesic structure on a metric space E with metric ρ is a function $k: M \times I \to E$ (where $M \subset E \times E$) satisfying the following conditions:

(a) If $(x, x) \in M$, then $k(x, x, t) = x$ for all $t \in I$.

(b) If $(x_1, x_2) \in M$, then $k(x_1, x_2, 0) = x_1$ and $k(x_1, x_2, 1) = x_2$.

(c) If $(x_1, x_2) \in M$, $t \in I$, and $(k(x_1, x_2, t), x_2) \in M$, then $k(k(x_1, x_2, t), x_2; s) = k(x_1, x_2, t + s(1 - t))$ for all $s \in I$.

(d) For all $(x_1, x_2) \in M$, the map $t \to k(x_1, x_2, t)$, from I to E, is continuous.

(e) For each $\epsilon > 0$ there exist neigbourhoods $W_\epsilon \subset N_\epsilon$ of the diagonal in $E \times E$ which are small[8] of order ϵ, such that if $(x_1, x_2) \in M$ and $(y_1, y_2) \in M$,

[7]We use $S_r(y)$ to denote the open r-sphere about y.

[8]A neighbourhood U of the diagonal is *small of order* ϵ if $(x, y) \in U$ implies $\rho(x, y) < \epsilon$.

then $(x_1, y_1) \in N_\epsilon$ and $(x_2, y_2) \in W_\epsilon$ implies that $(k(x_1, x_2, t), k(y_1, y_2, t)) \in N_\epsilon$ for all $t \in I$.

Notice that Definition 5.1 is quite similar to Definition 1.1 with $n = 2$, but that 5.1 (e) is distinctly stronger than 1.1 (e) with $n = 2$. This extra strength is needed to carry out the inductive proof of Proposition 5.3 below.

Definition 5.2. Let E be a metric space with a geodesic structure. A subset S of E is *geodesic* if, whenever $x_1, x_2 \in S$, then $(x_1, x_2) \in M$ and $k(x_1, x_2, t) \in S$ for all $t \in I$. A family $\mathfrak{S}$ of subsets of E is *equi-locally geodesic* if there exists an open covering $\mathfrak{A}$ of E, such that, if $S \in \mathfrak{S}$, $A \in \mathfrak{A}$, and $x_1, x_2 \in (S \cap A)$, then $(x_1, x_2) \in M$ and $k(x_1, x_2, t) \in S$ for all $t \in I$.

Note that any family of geodesic sets is equi-locally geodesic.

PROPOSITION 5.3. *If E is a metric space with geodesic structure (M, k), then there exists a convex structure $\{M_n, k_n\}_{n=1}^{\infty}$ on E such that*

(a) *Every geodesic set $S \subset E$ is convex.*

(b) *Every equi-locally geodesic family of subsets of E is equi-locally convex.*

Proof. We define the sets $M_n \subset E^n$ and functions $k_n: M_n \times P_n \to E$ inductively as follows. First, let

$$M_1 = E, \qquad k_1(x, 1) = 1.$$

Suppose M_n and k_n have been defined, and let us define M_{n+1} and k_{n+1}. We introduce the following notation for the rest of this proof.

(1) If $x \in E^{n+1}$, then $\tilde{x} \in E^n$ is defined by

$$\tilde{x}_i = x_i \qquad (i = 1, \ldots, n).$$

(2) If $t \in P_{n+1}$ and $t_{n+1} \neq 1$, then $\tilde{t} \in P_n$ is defined by

$$\tilde{t}_i = \frac{t_i}{1 - t_{n+1}} \qquad (i = 1, \ldots, n).$$

Now let

(3) $M_{n+1} = \{x \in E^{n+1} | \tilde{x} \in M_n, (k_n(\tilde{x}, t), x_{n+1}) \in M \text{ for all } t \in P_n\}$,

(4) $$k_{n+1}(x, t) = \begin{cases} x_{n+1} & \textit{if } t_{n+1} = 1 \\ k(k_n(\tilde{x}, \tilde{t}), x_{n+1}, t_{n+1}) & \text{if } t_{n+1} \neq 1. \end{cases}$$

Let us check that the conditions of Definition 1.1 are satisfied.

1.1 (a). This follows from our definition of k_1.

1.1 (b). For $n = 2$, this follows from the definitions. Suppose that it holds for n, and let us prove it for $n + 1$. Let $x \in M_{n+1}$, $i \leqslant n + 1$, and $t \in M_{n+1}$ with $t_i = 0$. If i $= n + 1$, then $\partial_i x = \tilde{x}$, which is in M_n by (3), and $k_{n+1}(x, t) = k(k_n(\tilde{x}, \tilde{t}), x_{n+1}, 0) = k_n(\tilde{x}, \tilde{t}) = k_n(\partial_i x, \partial_i t)$, which is what we had to show.

Suppose, then, that $i \neq n + 1$, and let us first show that $\partial_i x \in M_n$. By (3), we must show that $(\partial_i x)^{\sim} \in M_{n-1}$, and that $(k_{n-1}((\partial_i x)^{\sim}, s), x_{n+1}) \in M$ for all $s \in P_{n-1}$. Now $(\partial_i x)^{\sim} = \partial_i \tilde{x}$, and this is in M_{n-1} by the inductive hypothesis. As for the other requirement, pick an $s' \in P_n$ such that $s_i' = 0$ and

$\partial_i s' = s$; then $(k_{n-1}((\partial_i x)^{\sim}, s), x_{n+1}) = (k_{n-1}(\partial_i \tilde{x}, \partial_i s'), x_{n+1}) = (k_n(\tilde{x}, s'), x_{n+1})$ by the inductive hypothesis, and this last pair is in M by (3). Finally, let $t \in M_{n+1}$ with $t_i = 0$. If $t_{n+1} = 1$, then both sides of the equation in 1.1 (b) are x_{n+1}, and hence the equation is satisfied. If $t_{n+1} \neq 1$, then (by the induction hypothesis)

$$\begin{aligned} k_{n+1}(x, t) &= k(k_n(\tilde{x}, \tilde{t}), x_{n+1}, t_{n+1}) \\ &= k(k_{n-1}(\partial_i \tilde{x}, \partial_i \tilde{t}), x_{n+1}, t_{n+1}) \\ &= k(k_{n-1}((\partial_i x)^{\sim}, (\partial_i t)^{\sim}, x_{n+1}, t_{n+1}) \\ &= k_n(\partial_i x, \partial_i t), \end{aligned}$$

which is what we had to show.

1.1 (c). In the presence of 1.1 (b), 1.1 (c) is equivalent to the following condition, which we are going to verify: *If $x \in M_n$ $(n \geqslant 2)$ with $x_i = x_{i+1}$ for some $i < n$, and if $t, t' \in P_n$ with $t_j = t_j'$ for $j \neq i, i+1$, then $k_n(x, t) = k_n(x, t')$.*

For $n = 2$, this follows from 5.1 (a) and the definition of k_1. Suppose it is true for n, and let us prove it for $n + 1$.

Case 1. $i < n$. In this case, $t_{n+1} = t_{n+1}'$. If $t_{n+1} = 1$, then $k_{n+1}(x, t) = k_{n+1}(x, t') = x_{n+1}$. If $t_{n+1} \neq 1$, it suffices to show that $k_n(\tilde{x}, \tilde{t}) = k_n(\tilde{x}, \tilde{t}')$. But $\tilde{t}_j = \tilde{t}_j'$ for $j \neq i, i+1$, so this follows from the inductive hypothesis.

Case 2. $i = n$. In this case, $x_n = x_{n+1}$, and

$$\sum_{j=1}^{n-1} t_j = \sum_{j=1}^{n-1} t_j'.$$

If this sum is 0, then $k_{n+1}(x, t) = k_2((x_n, x_n), (t_n, t_{n+1}))$ (by 1.1 (b)) $= k(x_n, x_n, t_{n+1})$ (by (4)) $= x_n$ (by (5.1); similarly $k_{n+1}(x, t') = x_n$. If this sum is not zero, then $t_n + t_{n+1} = t_n' + t_{n+1}' < 1$, and $\tilde{t} = \tilde{t}'$. Applying (4) twice we get

$$\begin{aligned} k_{n+1}(x, t) &= k(k(k_{n-1}(\tilde{\tilde{x}}, \tilde{\tilde{t}}), x_n, t_n(1 - t_{n+1})^{-1}), x_n, t_{n+1}) \\ &= k(k_{n-1}(\tilde{\tilde{x}}, \tilde{\tilde{t}}), x_n, t_n + t_{n+1}) \quad \text{by 5.1 (c).} \end{aligned}$$

Similarly

$$k_{n+1}(x, t') = k(k_{n-1}(\tilde{\tilde{x}}, \tilde{\tilde{t}}'), x_n, t_n' + t_{n+1}').$$

But $t_n + t_{n+1} = t_n' + t_{n+1}'$, and $\tilde{\tilde{t}} = \tilde{\tilde{t}}'$, as already observed. Hence $k_{n+1}(x, t) = k_{n+1}(x, t')$, which is what had to be shown.

1.1 (e). Proof by induction. The result is clear for k_1. Suppose it is true for k_n, and let us prove it for k_{n+1}. Pick a fixed $y \in M_n$ and $s \in P_n$, and let us show that the map $t \to k_{n+1}(y, t)$, from P_{n+1} to E, is continuous at s.

Case 1. $s_{n+1} \neq 1$. In this case, for some neighbourhood U of s in P_{n+1}, $t \in U$ implies $t_{n+1} \neq 1$. Hence, for $t \in U$, $k_{n+1}(y, t)$ is given by the second formula in (4), and the required continuity follows from 5.1 (d) and our inductive hypothesis.

Case 2. $s_{n+1} = 1$. In this case $k_{n+1}(y, s) = y_{n+1}$. We will show that $k_{n+1}(y, t)$ is close to y_{n+1} if t_{n+1} is close to 1. To do this, let $A = \{k(y, t)|t \in P_n\}$ and, for each $\tau \in I$, define $f_\tau: A \to E$ by

$$f_\tau(a) = k(a, y_{n+1}, \tau).$$

Then

$$\begin{aligned} k_{n+1}(y, t) &= f_{t_{n+1}}(a_t) \quad \text{for some } a_t \in \text{A} \quad \text{if} \quad t_{n+1} \neq 1 \\ k_{n+1}(y, t) &= y_{n+1} \quad \text{if} \quad t_{n+1} = 1. \end{aligned}$$

It follows that it is sufficient to show that $f \to f_1$ uniformly as $\tau \to 1$. That $f_\tau \to f_1$ *pointwise* follows from 5.1 (d). Moreover A is compact, and $\{f_\tau\}_{\tau \in I}$ is equicontinuous by 5.1 (e). Hence (see, for instance, (**3**, p. 34, Prop. 14)), the topologies of pointwise convergence and uniform convergence coincide on $\{f_\tau\}_{\tau \in I}$, and hence $f_\tau \to f_1$ uniformly as $\tau \to 1$.

1.1 (f). For $\epsilon > 0$, let $V_\epsilon = W_\epsilon$, where W_ϵ is as in 5.1 (e). Let us show that this works. For k_1 this is clear. Suppose it works for k_n, and let us prove it for k_{n+1}. If $t_{n+1} = 1$, then $k_{n+1}(x, t) = x_{n+1}$ and $k_{n+1}(y, t) = y_{n+1}$, and we know that $\rho(x_{n+1}, y_{n+1}) < \epsilon$ since $(x_{n+1}, y_{n+1}) \in W_\epsilon$. If $t_{n+1} \neq 1$, then

$$\begin{aligned} k_{n+1}(x, t) &= k(k_n(\tilde{x}, \tilde{t}), x_{n+1}, t_{n+1}) \\ k_{n+1}(y, t) &= k(k_n(\tilde{y}, \tilde{t}), y_{n+1}, t_{n+1}). \end{aligned} \tag{5}$$

Now, since $(x_i, y_i) \in W_\epsilon$ for $i = 1, \ldots, n + 1$, we have $(\tilde{x}_i, \tilde{y}_i) \in W_\epsilon$ for $i = 1, \ldots, n$, and hence our inductive hypothesis yields

$$k_n(\tilde{x}, \tilde{t}), k_n(\tilde{y}, \tilde{t})) \in N_\epsilon.$$

But $(x_{n+1}, y_{n+1}) \in W_\epsilon$ by assumption, and hence, by (5) and 5.1(e),

$$(k_{n+1}(x, t), k_{n+1}(y, t)) \in N_\epsilon,$$

and now the desired conclusion follows from the fact that N_ϵ is small of order ϵ.

We have now completed the proof that the sets M_n and functions k_n defined in (3) and (4) are a convex structure on E. To complete the proof, we must check conditions (a) and (b) of our proposition.

5.2 (a). Let $S \subset E$ be geodesic, and let $x \in S^n$. We must show that $x \in M_n$, and that $k_n(x, t) \in S$ for all $t \in P_n$. This is clear for $n = 1$. Let us therefore assume it for n, and prove it for $n + 1$. Now observe that, by the inductive hypothesis, $\tilde{x} \in M_n$ and $k(\tilde{x}, s) \in S$ for all $s \in P_n$. Since S is convex, it follows that $x \in M_{n+1}$ by (3), and $k_{n+1}(x, t) \in S$ for all $t \in P_{n0+1}$ by (4).

5.2 (b). Let $\mathfrak{S}$ be an equi-locally geodesic family of subsets of E, and let $\mathfrak{A}$ be as in definition of this concept in Definition 5.2. For each $x \in E$, pick an $A_x \in \mathfrak{A}$ which contains x, and then pick $r(x) > 0$ such that $S_{r(x)}(x) \subset A_x$. Let $B_x = W_{r(x)}(x)$, with W as in 5.1 (e), and let $\mathfrak{B} = \{B_x\}_{x \in E}$. Let us show that this $\mathfrak{B}$ works.

Let $S \in \mathfrak{S}$, $x_0 \in E$, and let $T = S \cap B_{x_0}$. We must show that if $x \in T^n$,

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then $x \in M_n$, and $k_n(x, t) \in S$ for all $t \in P_n$. We will prove this inductively and, to keep the induction going, we will also show that, for all n,

(6) $$(k_n(x, t), x_0) \in N_{r(x_0)} \quad \text{for all} \quad t \in P_n,$$

where $N_{r(x_0)}$ is as in 5.1(e). Now for $n = 1$, all this is clear. Suppose it is true for n, and let us prove it for $n + 1$.

By the inductive hypothesis, $\tilde{x} \in M_n$ and

$$k_n(\tilde{x}, s) \in (S \cap A_{x_0})$$

for all $s \in P_n$. Since also

$$x_{n+1} \in (S \cap A_{x_0}),$$

it follows that $x \in M_{n+1}$ by (3), and $k_{n+1}(x, t) \in S$ for all $t \in P_{n+1}$ by (4). It remains to check (6) for $n + 1$. If $t_{n+1} = 1$, this is clear, since then $k_n(x, t) = x_{n+1}$. If $t_{n+1} \neq 1$, note that

$$(k_n(\tilde{x}, \tilde{t}), x_0) \in N_{r(x_0)}$$

by the inductive hypothesis, and

$$(x_{n+1}, x_0) \in W_{r(x_0)}$$

by assumption. Hence

$$(k_n(x, t), x_0) = (k(k_n(\tilde{x}, \tilde{t}), x_{n+1}, t_{n+1}), x_0) \in N_{r(x_0)}$$

by (4) and 5.1 (c), and that is what had to be shown. This completes the proof.

To conclude this section, let us record the following consequence of Proposition 5.3.

THEOREM 5.4. *Theorem* 1.5 *remains true if "(locally) convex" is replaced by "(locally) geodesic".*

6. Riemannian manifolds. Let E be a Riemannian manifold with Riemannian metric ρ. If $x_1, x_2 \in X$ have a unique shortest geodesic joining them, this geodesic is called a *segment.* Let

(1) $$L = \{(x_1, x_2) \in E \times E | x_1 \text{ and } x_2 \text{ are joined by a segment}\}.$$

If $(x_1, x_2) \in L$, then the segment from x_1 to x_2 is given by a continuous

$$g_{x_1, x_2}: I \to E.$$

Define $h: L \times I \to E$ by

(2) $$h(x_1, x_2, t) = g_{x_1, x_2}(t).$$

The elementary properties of geodesics imply that (L, h) satisfies conditions (a) – (d) of Definition 5.1. In general, of course, 5.1 (e) is not satisfied. However we have the following theorem of Nijenhuis **(17)**.

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THEOREM. *Every $p \in E$ has a spherical neighbourhood $V(p)$ with the following property:*

(a) *If $x_1, x_2 \in V(p)$, then $(x_1, x_2) \in L$, and $h(x_1, x_2, t) \in V(p)$ for all $t \in I$.*

(b) *If $\epsilon > 0$, and $x_1, x_2, y_1, y_2 \in V(p)$ with $\rho(x_1, y_1) < \epsilon$ and $\rho(x_2, y_2) < \frac{1}{2}\epsilon$, then $\rho(h(x_1, x_2, t), h(y_1, y_2, t)) < \epsilon$ for all $t \in I$.*

It follows immediately from the above theorem that, if $M = V(p) \times V(p)$ and $k = h|M$, then (M, h), in addition to satisfying conditions (a) – (d) of Definition 5.1, also satisfies condition (e) (by taking N_ϵ to be the ϵ-neighbourhood of the diagonal, and W_ϵ the $\frac{1}{2}\epsilon$-neighbourhood), and hence we conclude

PROPOSITION 6.1. *Every point p of a Riemannian manifold E has a spherical neighbourhood $V(p)$ such that every pair of points in $V(p)$ is joined by a unique geodesic segment, and these geodesic segments generate a geodesic structure on $V(p)$ under which $V(p)$ is a geodesic set.*

While the above proposition is sufficient for our application to Lie groups in the next section, let us conclude this section by proving the following more "global" result.

PROPOSITION 6.2. *On every compact Riemannian manifold E there exists a geodesic structure (M, k) such that*

(a) *If L and h are as in* (1) *and* (2), *then $M \subset L$ and $k = h|M$.*

(b) *Every $p \in E$ has a neighbourhood $U(p)$ such that $U(p) \times U(p) \subset M$.*

Proof. For each $p \in E$, let $S(p)$ be the open sphere about p whose radius is half the radius of the sphere $V(p)$ in Nijenhuis's theorem. Then $\{S(p)\}_{p \in E}$ is an open covering of E, and by compactness there exsits a finite subcovering $\{S(p_i)\}_{i=1}{}^n$. With L and h as in (1) and (2), let

$$(3) \qquad \begin{aligned} M &= \{(x, y) \in L | x, y \in S(p_i) \quad \text{for some } i = 1, \ldots, n\} \\ k &= h|M \end{aligned}$$

The only requirement that needs further checking is that (M, k) satisfies condition (e) of Definition 5.1.

Let $\epsilon > 0$, and let us define the required neighbourhoods W_ϵ and N_ϵ of the diagonal. For each i, let r_i be the radius of $S(p_i)$, and let

$$\gamma = \min(\min\{r_i | i = 1, \ldots, n\}, \epsilon).$$

Now let

$$(4) \qquad \begin{aligned} N_\epsilon &= \{(x, y) \in E \times E | \rho(x, y) < \gamma\}, \\ W_\epsilon &= \{(x, y) \in E \times E | \rho(x, y) < \tfrac{1}{2}\gamma\}. \end{aligned}$$

To see that this works, suppose that $(x_1, x_2) \in M$, $(y_1, y_2) \in M$, and that $(x_1, y_1) \in N_\epsilon$ and $(x_2, y_2) \in W_\epsilon$. Since $(x_1, x_2) \in M$, it follows from (3) that there exists a positive integer $k \leqslant n$ such that

$$x_1, x_2 \in S(p_k) \subset V(p_k).$$

Since $\rho(x_i, y_i) < \gamma < r_k$ for $i = 1, 2$, it follows that

$$y_1, y_2 \in V(p_k).$$

Finally, observe that $\rho(x_1, y_1) < \gamma$, while $(x_2, y_2) < \frac{1}{2}\gamma$. We therefore apply Nijenhuis's theorem to $V(p_k)$ to conclude that, for all $t \in I$, $\rho(k(x_1, y_1, t), k(x_2, y_2, t)) < \gamma \leqslant \epsilon$, which is what we had to show.

7. Locally convex groups. The reader is reminded that, as elsewhere in this paper, A^n always (even in a group) denotes the n-fold Cartesian product of A for any positive integer n. Moreover, if G is a group, $x \in G^n$, and $z \in G$, then xz will denote $(x_1 z, \ldots, x_n z) \in G^n$, while if $K \subset G^n$ and $z \in G$, then Kz will denote $\{xz | x \in K\} \subset G^n$.

Definition 7.1. A metrizable topological group H with right-invariant metric d is *locally convex* if there exists a convex structure $\{L_n, h_n\}_{n=1}^{\infty}$ on H, and a neighbourhood W of the identity, such that the following conditions are satisfied for all n.

(a) $L_n = W^n$.

(b) h_n is right-invariant; that is, if $x \in W^n$, $z \in H$, and $xz \in W^n$, then $h_n(xz, t) = (h_n(x, t))z$ for all $t \in P_n$.

(c) The left and right uniform structures on H coincide on W.

(d) Condition (e) of Definition 1.1 is satisfied with V_ϵ of the special form $\{(x, y) \in E \times E | d(x, y) < \delta(\epsilon)\}$ for some $\delta(\epsilon) > 0$.

If we can take $W = H$, then H is called a *convex* group.

Note that this definition does not depend on which right-invariant metric is used, since any two such metrics are uniformly equivalent.

The following are some examples of locally convex groups.

(1) A locally convex metrizable topological linear space, with the usual convex structure. (This group is, in fact, convex.)

(2) A Lie group. A Lie group can be made into a Riemannian manifold with right-invariant Riemannian metric. By Proposition 6.1, there exists a neighbourhood U of e on which the segments generate a geodesic structure (M, k) making U into a geodesic set. Since the metric is right-invariant, so is k. By Proposition 5.3, this geodesic structure generates a convex structure $\{M_n, k_n\}_{n=1}^{\infty}$ on U which makes U a convex set; hence $M_n = U^n$. From the way k_n is defined in terms of k in the proof of Proposition 5.3, it follows that each k_n is right-invariant along with k. Now let W be a compact subneighbourhood of U, and let $h_n = k_n | W^n$. Then all requirements are satisfied, the compactness of W taking care of 7.1 (c) and (d).

(3) The multiplicative group H_0 of invertible elements in a Banach algebra B with unit e. Let W be the sphere of radius $\frac{1}{2}$ about e. For the convex structure, use the ordinary linear one. The right and left uniform structures on H coincide on W with the uniform structure induced by the norm. More generally, one can take any group of the form $H_0 \cap (I + e)$, where I is an ideal in B.

(4) A topological group which is locally isomorphic to a locally convex group.

(5) A finite Cartesian product of locally convex groups.

THEOREM 7.2. *If G is a metrizable group, and H a closed, complete (locally), convex subgroup, then there exists a (local) cross-section.*[9]

Proof. Let ρ be a *left* invariant metric on G. We shall construct a convex structure $\{M_n, k_n\}_{n=1}^{\infty}$ on (G, ρ) such that the collection $\mathfrak{H}$ of *right* cosets of H becomes equi-locally convex (and such that each right coset of H becomes a convex set in case H is convex.) The theorem then follows from Theorem 1.5, which applies because the natural map $u: G \to G/H$ is open, and hence the carrier $\phi: G/H \to 2^G$, defined by $\phi(x) = u^{-1}(x)$, is lower semi-continuous **(13,** Example 1.3***)**.

By 1.1 (e) and the remark following Definition 1.1, there exists a neighbourhood W_1 of the identity e such that, whenever $x \in W_1^n$, then $h_n(x, t) \in W$ for all $t \in P_n$. Pick a symmetric neighbourhood W_2 of e such that $W_2 . W_2 \subset W_1$ Now let

$$M_n' = \bigcup\{W_1^n z | z \in G\},$$
$$M_n = \bigcup\{W_2^n z | z \in G\}.$$

Define $k_n': M_n' \times P_n \to G$ by

$$k_n'(x, t) = (h_n(xz^{-1}, t))\, z,$$

where z is any element of G such that $x \in W_1^n z$, and let

$$k_n = k_n' | (M_n \times P_n).$$

It follows from Definition 7.1 (b) that the definition of k_n', and hence that of k_n, does not depend on the choice of z. That $\{M_n, k_n\}_{n=1}^{\infty}$ satisfies conditions (a) – (d) of Definition 1.1 follows from the fact that $\{L_n, h_n\}_{n=1}^{\infty}$ satisfies them. It remains to verify that $\{M_n, k_n\}_{n=1}^{\infty}$ satisfies condition (e) of Definition 1.1.

Observe first that, for any topological group G and any $z \in G$, right translation by z ($z \to xz$) is a uniformity isomorphism for both the right and left uniform structures on G. For the right one this is clear. For the left one, it follows from the fact that our map is a composition of $x \to z^{-1}xz$ and $y \to zy$; the first of these is a group isomorphism, and the second one is a left translation.

It follows from the previous paragraph, applied to the left uniform structure, that 7.1(c) and (d) imply

(1) *To every $z \in G$ and $\epsilon > 0$ corresponds a $\delta(\epsilon, z) > 0$ such that, for all n, if $x, y \in W_1^n z$ and $\rho(x_i, y_i) < \delta(\epsilon, z)$ for $i = 1, \ldots, n$, then $\rho(k_n'(x, t), k_n'(y, t) < \epsilon$ for all $t \in P_n$.*

[9] See footnote 2, p. 556.

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Now for each $\epsilon > 0$ and $z \in G$, let

(2) $\xi(\epsilon, z) = \frac{1}{4}\delta(\frac{1}{2}\epsilon, z)$.

(3) $V_\epsilon = \{(x, y) \in G \times G | x, y \in S_{\xi(\epsilon, z)}(z) \text{ for some } z \in G\}$.

Let us check that this V_ϵ works. We must show that, if $x, y \in M_n$, and if for each $i = 1, \ldots, n$ there exists a $z_i \in G$ such that

$$x_i, y_i \in S_{\xi(\epsilon, z_i)}(z_i),$$

then

(4) $\rho(k_n(x, t), k_n(y, t)) < \epsilon$

for all $t \in P_n$.

Pick $k \leqslant n$ such that $\xi(\epsilon, z_k) = \max_{i=1}{}^n \xi(\epsilon, z_i)$, and let $a = x_k^{-1}z_k$, $b = y_k^{-1}z_k$. Note that, since ρ is left invariant, $\rho(g, ga) = \rho(x_k, z_k) < \xi(\epsilon, z_k)$ and $\rho(g, gb) = \rho(y_k, z_k) < \xi(\epsilon, z_k)$ for all $g \in G$. For $i = 1, \ldots, n$, let $x_i' = x_i a$, $y_i' = y_i b$. Then

$$\rho(x_i', y_i') \leqslant \rho(x_i', x_i) + \rho(x_i, y_i) + \rho(y_i, y_i') \leqslant 4\xi(\epsilon, z_k) = \delta(\tfrac{1}{2}\epsilon, z_k).$$

Now $x \in M_n$, so $x \in W_2{}^n r$ for some $r \in G$, and hence $x' \in W_2{}^n ra$. Thus, for $i = 1, \ldots, n$, $x_i' \in W_2 ra = W_2(rx_k^{-1})z_k \subset (W_2 . W_2)z_k \subset W_1 z_k$; in other words, $x' \in W_1{}^n z_k$. Similarly, $y' \in W_1{}^n z_k$. We can therefore, apply (1), with $z = z_k$, to conclude that $\rho(k_n'(x', t), k_n'(y', t)) < \frac{1}{2}\epsilon$ for all $t \in P_n$.

Now note that, for all $t \in P_n$,

$$k_n'(x', t) = h_n(x' z_k^{-1}, t)\, z_k = h_n(x x_k^{-1}, t)\, x_k a = k_n(x, t)a,$$

and hence $\rho(k'_n(x', t), k_n(x, t)) < \xi(\epsilon, z_k)$. Similarly $\rho(k_n'(y', t), k_n(y, t)) < \xi(\epsilon, z_k)$ for all $t \in P_n$. Hence, for all $t \in P_n$,

$$\rho(k_n(x, t), k_n(y, t))$$
$$\leqslant \rho(k_n(x, t), k_n'(x', t)) + \rho(k_n'(x', t), k_n'(y', t)) + \rho(k_n'(y', t), k_n(y, t))$$

which is what we had to show.

To complete the proof, we must show that the family $\mathfrak{H}$ of right cosets of H is equi-locally convex, and that each right coset of H is a convex set in case H is convex. The latter assertion is clear. To prove the former, let d be a *right* invariant metric on G, and pick $r > 0$ such that the d-sphere of radius r about the identity e in H is contained in W_2. Since $x \in M_n$ if, and only if, $x_1, \ldots, x_2$ are all in some right translate of W_2, it follows that $x \in M_n$ whenever $x_1, \ldots, x_n$ is a subset of some member of $\mathfrak{H}$ having d-diameter $<$ r. Hence for $\mathfrak{B}$ we simply pick the family of d-spheres of radius $\frac{1}{2}r$ with centres in G.

Combining Theorem 7.2 with Example (2) yields a new proof—at least in metrizable case—of a cross-section theorem of A. Gleason. When combined with Examples (1) and (4), Theorem 7.2 yields

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COROLLARY 7.3. *Let G be a metrizable group, and H a closed subgroup which is (locally) isomorphic to the additive group of a complete, metrizable, locally convex topological linear space. Then there exists a (local) cross section.*[10]

This corollary generalizes a result of Bartle and Graves (**2**, Theorem 4) (see also (**13**, p. 364)), where G is assumed to be a Banach space and H a closed linear subspace.

8. Homotopy extension, covering homotopy and fibre spaces. A common feature of both the homotopy extension theorem (**6**, Theorem VI, 5) and the covering homotopy theorem (**19**, Theorem 11.7) is that, speaking very roughly, special conditions on the domain permit one to obtain a continuous function globally even though the range is well behaved only locally. In this section we obtain a result (Theorem 8.3) on continuous selections which seems to incorporate the essential aspects of both these theorems. For finite dimensional domains, this was already done in (**15**, Theorem 3.4), and both statement and proof of Theorem 8.3 parallel those of its finite-dimensional analogue. After developing the necessary preliminary concepts, we will thus be able to dispose of the proof of Theorem 8.3 simply by pointing out the obvious modifications which must be made in the proof of (**15**, Theorem 3.4).

Definition 8.1. Let E be a metric space with convex structure. Then a family $\mathfrak{S}$ of subsets of E is *uniformly equi-locally convex* if there exists an $r > 0$ such that, whenever $S \in \mathfrak{S}$ and $A \subset S$ has diameter $< r$, then A is admissible and $\text{conv}(A) \subset S$.

Note that any uniformly equi-locally convex family of sets is equi-locally convex.

Using Theorem 1.3, we now prove

PROPOSITION 8.2. *Let (Y, ρ) be a metric space with convex structure, $\mathfrak{S}$ a uniformly equi-locally convex family of complete, non-empty subsets of Y, and let $r > 0$ be as in Definition 8.1. Let X be paracompact, $\phi: X \to \mathfrak{S}$ lower semi-continuous, and suppose there exists a continuous $g: X \to Y$ such that $\rho(g(x), \phi(x)) < r$ for every $x \in X$. Then there exists a selection for ϕ.*

Proof. Since all assumptions remain unchanged if Y is replaced by its completion, we may assume that Y is complete. Define $\psi: X \to 2^Y$ by

$$\psi(x) = \phi(x) \cap S_r(g(x)).$$

Then ψ is lower semi-continuous by (**13**, Proposition 2.5) and, for every $x \in X$, $\psi(x)$ is admissible and $\text{conv}(\psi(x)) \subset \phi(x)$. Hence Theorem 1.3 asserts the existence of a continuous $f: X \to Y$ such that, for every $x \in X$,

$$f(x) \in [\text{conv}(\psi(x))]^- \subset \phi(x).$$

This f is the required selection, and the proof is complete.

[10]See footnote 2, p. 556.

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The previous theorems in this paper only required the carrier $\phi: X \to 2^Y$ to be lower semi-continuous. In the following theorem, however, just as in [15], we require it to be *continuous*; that is, given $\epsilon > 0$, every $x_0 \in X$ has a neighbourhood U such that

$$\phi(x_0) \subset S_\epsilon(\phi(x)), \quad \phi(x) \subset S_\epsilon(\phi(x_0))$$

for every $x \in U$.

The principal result of this section can now be stated as follows.

THEOREM 8.3. *Let Y be a metric space with convex structure, and $\mathfrak{S}$ a uniformly equi-locally convex family of complete non-empty subsets of Y. Let Z be paracompact, $X = Z \times I$, and $\phi: X \to \mathfrak{S}$ continuous. Finally, let $B \subset Z$ be closed, and define $A \subset X$ by*

$$A = (Z \times \{0\}) \cup (B \times I).$$

Then every selection for $\phi|A$ can be extended to a selection for ϕ.

Proof. This theorem is identical with **(15**, Theorem 3.4**)** except that no dimensional restrictions are placed on Z, and that $\mathfrak{S}$ is uniformly equi-locally convex instead of uniformly equi-LC^n. The proof of our theorem is similarly identical with the first part of the proof of **(15**, Theorem 3.4**)** (the second part takes care of a dimensional difficulty), provided "uniformly equi-LC^n" is replaced by "uniformly equi-locally convex," and the reference to **(14**, Theorem 9.1**)** in the proof of Lemma 3.3 of **(15)** is replaced by a reference to our Proposition 8.2. This is all that need be said, and the proof is thus complete.

Before continuing, let us observe that if every $S \in \mathfrak{S}$ were actually *convex* in Theorem 8.3, then this theorem would be a special case of Theorem 1.5(a). Note also that if ϕ is a constant map, then Theorem 8.3 is simply a homotopy extension theorem, while if B is empty, we get a theorem of the covering-homotopy type.

As mentioned at the beginning of this section, Theorem 8.3 is valid because of the special relation of X to $A \subset X$. However, the relation need not be quite as special as all that. In fact, it is an easy consequence of Theorem 8.3 that it is sufficient to assume that X is paracompact, and that A is a *generalized deformation retract* of X in the following sense: There exists a continuous $r: X \times I \to A$ such that $r(x, 0) = x$ if $x \in X$, $r(x, 1) \in A$ if $x \in X$, and $r(x, t) \in A$ if $x \in A$ and $t \in I$. The simple proof of this generalization from Theorem 8.3 can be omitted, since it is identical with the proof of how Theorem 6.2 follows from Lemma 5.3 in **(15)**.

Just as **(15**, Theorem 3.4**)** was applicable to fibre spaces in the sense of Serre **(18)**, Theorem 8.3 is applicable to fibre spaces in the sense of Hurewicz **(7)**. These are, by definition, triples (E, p, B), where E and B are topological spaces with $p: E \to B$ continuous and onto, satisfying the following condition: If Z is paracompact, $k: Z \times I \to B$ continuous, and if $\phi: Z \times I \to 2^B$ is defined by $\phi(z, t) = p^{-1}(k(z, t))$, then every selection for $\phi|Z \times \{0\}$ can

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be extended to a selection for ϕ. The following result is now an immediate consequence of Theorem 8.3.

COROLLARY 8.4. *Let E, B be metric spaces, $p: E \to B$ continuous and onto, and suppose that*

(a) *each $p^{-1}(x)$ is complete,*

(b) *$\{p^{-1}(x)\}_{x \in B}$ is uniformly equi-locally convex,*

(c) *the carrier $\psi: B \to 2^E$, defined by $\psi(x) = p^{-1}(x)$, is continuous.*

Then (E, p, B) is a fibre space in the sense of Hurewicz **(7)**.

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University of Washington