16

Noncommutative gauge theories

We have seen in the previous chapter that the twisted reduced models reproduce planar graphs of the d-dimensional quantum field theories as $N \to \infty$. However, the twisted reduced models make sense order by order in $1/N^2$. For the continuum twisted reduced models, the topological expansion goes in the parameter $\det(B_{\mu\nu})$.

At finite $B_{\mu\nu}$, the twisted reduced models are mapped [CDS98, AII00] into quantum field theories on noncommutative space characterized by a (dimensional) parameter of noncommutativity $\theta_{\mu\nu} = B^{-1}_{\mu\nu}$. The noncommutative gauge field is no longer matrix-valued as in Yang–Mills theory but noncommutativity of matrices in the reduced models is transformed into noncommutativity of coordinates in the noncommutative gauge theory. The planar limit of ordinary Yang–Mills theory is reproduced at large noncommutativity parameter $\theta_{\mu\nu} \to \infty$, while ordinary quantum electrodynamics is reproduced as $\theta_{\mu\nu} \to 0$.

Noncommutative gauge theories possess a number of remarkable properties. The noncommutative extension of Maxwell's theory is interacting and asymptotically free. The group of noncommutative gauge symmetry is very large and incorporates space-time symmetries, in particular, translation, Lorentz transformation, parity reflection. This restricts a set of observables in noncommutative gauge theory which are built out of both closed and open Wilson loops. At rational values of a (dimensionless) noncommutativity parameter, noncommutative gauge theories on a torus are equivalent to ordinary Yang–Mills theories on a smaller torus with twisted boundary conditions representing the 't Hooft flux.

We begin this chapter by mapping the twisted reduced models into noncommutative theories. Then we discuss some properties of noncommutative scalar and gauge theories including their lattice regularization.

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16.1 The noncommutative space

As we have seen in the previous chapter, the twisted reduced models make sense order by order of the topological expansion in the parameter $\det(B_{\mu\nu})$. We start this section by showing how the twisted reduced models are mapped into noncommutative quantum field theories. We simply repeat the consideration of Sect. 15.2 using the continuum operator notation of Sect. 15.4.

Substituting the expansion (15.100) into the action (15.104) of the continuum twisted reduced model and using the orthogonality condition,

$$\frac{(2\pi)^{d/2}}{\mathrm{Pf}\,(B_{\mu\nu})}\,\mathrm{tr}_{\mathcal{H}}\,\boldsymbol{J}_k\boldsymbol{J}_q^\dagger \;=\; (2\pi)^d\delta^{(d)}(k-q)\,, \tag{16.1}$$

we obtain for the kinetic term

$$\begin{aligned} S^{(2)} &= \frac{1}{2}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\left(k^2+m^2\right)\varphi(-k)\varphi(k) \\ &= \frac{1}{2}\int\mathrm{d}^dx\Big\{\left[\partial_\mu\varphi(x)\right]^2+m^2\varphi^2(x)\Big\}. \end{aligned} \tag{16.2}$$

Here $\varphi(x)$ is given by Eq. (15.101), i.e. it is related to the operator $\tilde{\boldsymbol{\varphi}}$ by the Weyl transformation. The RHS of Eq. (16.2) is simply the free action for a scalar field in d dimensions.

Let us now repeat the calculation for the cubic self-interaction. Using Eq. (16.1), we find

$$\frac{(2\pi)^{d/2}}{\mathrm{Pf}\,(B_{\mu\nu})}\,\mathrm{tr}_{\mathcal{H}}\,\tilde{\boldsymbol{\varphi}}^3 \;=\; \int\frac{\mathrm{d}^dp}{(2\pi)^d}\int\frac{\mathrm{d}^dq}{(2\pi)^d}\varphi(-p-q)\varphi(p)\varphi(q)\,\mathrm{e}^{-\mathrm{i}p\theta q/2}, \tag{16.3}$$

where $\theta_{\mu\nu}=B^{-1}_{\mu\nu}$. This is a continuum analog of Eq. (15.35).

The RHS of Eq. (16.3) involves the phase factor $\mathrm{e}^{-\mathrm{i}p\theta q/2}$ representing noncommutativity of the generators $\boldsymbol{J}_k$. Relabeling the operators by introducing

$$\boldsymbol{x}_\mu \;=\; -\,\theta_{\mu\nu}\boldsymbol{P}_\nu\,, \tag{16.4}$$

which obey

$$[\boldsymbol{x}_\mu,\boldsymbol{x}_\nu] \;=\; \mathrm{i}\,\theta_{\mu\nu}\boldsymbol{1}\,, \tag{16.5}$$

we obtain

$$\boldsymbol{J}_k \;=\; \mathrm{e}^{\mathrm{i}k\boldsymbol{x}} \tag{16.6}$$

and

$$\boldsymbol{J}_k\boldsymbol{J}_q \;=\; \boldsymbol{J}_{k+q}\,\mathrm{e}^{-\mathrm{i}k\theta q/2} \tag{16.7}$$

according to the Baker–Campbell–Hausdorff formula (15.98).

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In order to represent the multiplication rule (16.7) by the Fourier-basis functions e^{ikx}, we introduce a *noncommutative* product of functions:

$$f_1(x) \star f_2(x) \stackrel{\text{def}}{=} f_1(x) \exp\left(\frac{\mathrm{i}}{2} \overleftarrow{\partial}_\mu \theta_{\mu\nu} \partial_\nu\right) f_2(x) . \tag{16.8}$$

Here $\overleftarrow{\partial}_\mu$ acts on $f_1(x)$ and ∂_ν acts on $f_2(x)$. It is noncommutative but associative, i.e.

$$\Big[f_1(x) \star f_2(x)\Big] \star f_3(x) = f_1(x) \star \Big[f_2(x) \star f_3(x)\Big], \tag{16.9}$$

similarly to the product of matrices (operators).

The product (16.8) is called the *star product* or the *Moyal product*. It becomes the ordinary product when $\theta_{\mu\nu} \to 0$ since

$$f_1(x) \star f_2(x) = f_1(x) f_2(x) + \frac{\mathrm{i}}{2} \theta_{\mu\nu} \left[\partial_\mu f_1(x)\right] \left[\partial_\nu f_2(x)\right] + \mathcal{O}\big(\theta^2\big) \tag{16.10}$$

to the linear order in θ.

Given the star product (16.8), the function $f(x)$ can be viewed as a coordinate-space representation of the operator $\boldsymbol{f}$ to which it is related by the Weyl transform. Whenever we have a product of two operators, its coordinate-space representation is given by the star product of the two functions associated with the operators by the Weyl transform.

In particular, the function x is the coordinate-space representation of the operator $\boldsymbol{x}$ and the commutation relation (16.5) has the coordinate-space representation

$$x_\mu \star x_\nu - x_\nu \star x_\mu = \mathrm{i}\, \theta_{\mu\nu} . \tag{16.11}$$

Equation (16.11) holds as a result of the definition (16.8) with $f_1 = x_\mu$ and $f_2 = x_\nu$.

Similarly, we have

$$\mathrm{e}^{ikx} \star \mathrm{e}^{iqx} = \mathrm{e}^{\mathrm{i}(k+q)x}\, \mathrm{e}^{-\mathrm{i}k\theta q/2} \tag{16.12}$$

reproducing the coordinate-space representation of Eq. (16.7).

With the aid of the star product, we can represent Eq. (16.3) in the coordinate space as

$$\frac{(2\pi)^{d/2}}{\mathrm{Pf}\,(B_{\mu\nu})} \mathrm{tr}_{\mathcal{H}}\, \tilde{\boldsymbol{\varphi}}^3 = \int \mathrm{d}^d x\, \varphi(x) \star \varphi(x) \star \varphi(x) \tag{16.13}$$

and similarly for higher interaction terms

$$\frac{(2\pi)^{d/2}}{\mathrm{Pf}\,(B_{\mu\nu})} \mathrm{tr}_{\mathcal{H}}\, \tilde{\boldsymbol{\varphi}}^n = \int \mathrm{d}^d x\, \overbrace{\varphi(x) \star \cdots \star \varphi(x)}^{n} \tag{16.14}$$

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so that

$$\frac{(2\pi)^{d/2}}{\mathrm{Pf}\,(B_{\mu\nu})}\,\mathrm{tr}_{\mathcal{H}}\,\widetilde{V}(\tilde{\boldsymbol{\varphi}}) \;=\; \int \mathrm{d}^d x\, V(\star\varphi(x))\,. \tag{16.15}$$

The prescription for writing down the action of the noncommutative theory that comes from the mapping of the operator action (15.104) of the continuum twisted reduced model is obvious. We simply replace products of operators by the star products of their Weyl transforms and substitute the trace over the Hilbert space by the integral over coordinate space according to Eq. (15.106). In fact, the star product (16.8) is defined precisely in the way needed for this prescription to be valid!

One could ask why there is a usual product rather than the star product in the kinetic term (16.2)? The point is that it does not matter what product we write for the integral of a product of two functions: the ordinary product or the star product. It is easy to show that

$$\int \mathrm{d}^d x\, f_1(x)\star f_2(x) \;=\; \int \mathrm{d}^d x\, f_1(x) f_2(x) \;=\; \int \mathrm{d}^d x\, f_2(x)\star f_1(x) \tag{16.16}$$

for functions decreasing with their derivatives at infinity as a consequence of the definition (16.8). This is a counterpart of the cyclic symmetry of the trace.*

Finally we obtain the following action:

$$S[\varphi] \;=\; \int \mathrm{d}^d x \Big[\frac{1}{2}\partial_\mu\varphi(x)\star\partial_\mu\varphi(x) + V(\star\varphi(x))\Big]. \tag{16.17}$$

The parameter of noncommutativity $\theta_{\mu\nu}$ enters the action via the star product (16.8).

The action (16.17) is associated with a *noncommutative scalar theory.* In the limit of $\theta_{\mu\nu}\to 0$, it reproduces the ordinary theory of a single scalar field. In the opposite limit of $\theta_{\mu\nu}\to\infty$, only planar graphs survive and the noncommutative scalar theory is equivalent to the theory of a matrix-valued scalar field with the action (14.20) at $N=\infty$. This can be easily shown directly using the theorem of Sect. 15.2 which was considered for noncommutative quantum field theories in [Fil96], where the phase factor associated with a generic nonplanar diagram was calculated.

Problem 16.1 Prove associativity of the star product (16.8).

Solution Equation (16.9) can easily be proven by expanding in θ. To the quadratic order in θ, we are to verify that

$$\theta_{\mu\nu}\theta_{\lambda\rho}\,(\partial_\mu f_1)\,(\partial_\lambda\partial_\nu f_2)\,(\partial_\rho f_3) \;=\; \theta_{\mu\nu}\theta_{\lambda\rho}\,(\partial_\mu f_1)\,(\partial_\nu\partial_\lambda f_2)\,(\partial_\rho f_3) \tag{16.18}$$

which is true since ∂_μ commute. It is similar to the next orders.

* To avoid confusion, let us mention that Eq. (16.16) is *not* valid when f_1 and f_2 are not decreasing at infinity, say, for $f_1 = x_\mu$ and $f_2 = x_\nu$. The trace of a commutator is then reproduced as a surface term.

If ∂_μ were noncommutative derivatives (say, like covariant derivatives in an external electromagnetic field), the star product would not be, generally speaking, associative.

Remark on the Weyl transformation

The Weyl transformation from operators to functions and vice versa can be written conveniently with the aid of the operator-valued function

$$\boldsymbol{\Delta}(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \mathrm{e}^{-\mathrm{i}kx} \boldsymbol{J}_k = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \mathrm{e}^{\mathrm{i}k(\boldsymbol{x}-x)} \tag{16.19}$$

which is an operator counterpart of Eq. (15.40).

We then represent the Weyl transformation by

$$\boldsymbol{f} = \int \mathrm{d}^d x\, \boldsymbol{\Delta}(x)\, f(x) \tag{16.20}$$

and

$$f(x) = \frac{(2\pi)^{d/2}}{\mathrm{Pf}\,(B_{\mu\nu})} \,\mathrm{tr}_{\mathcal{H}}\Big[\boldsymbol{f}\,\boldsymbol{\Delta}(x)\Big]. \tag{16.21}$$

Remember that

$$\frac{1}{\mathrm{Pf}\,(B_{\mu\nu})} = \mathrm{Pf}\,(\,\theta_{\mu\nu}\,)\,. \tag{16.22}$$

Note that $\boldsymbol{\Delta}(x)$ becomes an ordinary delta-function as $\theta \to 0$ when $\boldsymbol{x}_\mu$ commute.

Problem 16.2 Derive Eq. (16.8) by calculating the Weyl transform of the product of two operators.

Solution The star product can be defined via the Weyl transform of the product of two operators:

$$f_1(x) \star f_2(x) \stackrel{\mathrm{def}}{=} \frac{(2\pi)^{d/2}}{\mathrm{Pf}\,(B_{\mu\nu})} \,\mathrm{tr}_{\mathcal{H}}\Big[\boldsymbol{f}_1\,\boldsymbol{f}_2\,\boldsymbol{\Delta}(x)\Big]. \tag{16.23}$$

Inserting Eq. (16.20) and using Eqs. (16.7) and (16.1), we obtain

$$\begin{aligned} f_1(x) \star f_2(x) &= \int \mathrm{d}^d y\, \mathrm{d}^d z\, f_1(y) f_2(z) \frac{(2\pi)^{d/2}}{\mathrm{Pf}\,(B_{\mu\nu})} \,\mathrm{tr}_{\mathcal{H}}\Big[\boldsymbol{\Delta}(x)\boldsymbol{\Delta}(y)\boldsymbol{\Delta}(z)\Big] \\ &= \int \mathrm{d}^d y\, \mathrm{d}^d z\, f_1(y) f_2(z) \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d}\, \mathrm{e}^{\mathrm{i}(x-y)k + \mathrm{i}(x-z)q - \mathrm{i}k\theta q/2} \\ &= \int \mathrm{d}^d y\, \mathrm{d}^d z\, f_1(y) f_2(z) \frac{1}{\pi^d \,|\, \det\theta\,|}\, \mathrm{e}^{-2\mathrm{i}(x-y)\theta^{-1}(x-z)}. \end{aligned} \tag{16.24}$$

This is just the representation of the operator in Eq. (16.8) via a kernel.

The associativity of the star product is clear from the integral representation (16.24) as a consequence of the associativity of the matrix (operator) multiplication.

Remark on the Moyal bracket

The term linear in θ in Eq. (16.10) for the star product is in $d = 2$ just the Poisson bracket. It is a "semiclassical" limit of the Moyal bracket of the two functions:

$$\{f, g\}_{\mathrm{MB}} \stackrel{\mathrm{def}}{=} -\mathrm{i}\,(f \star g - g \star f)\,. \tag{16.25}$$

The Moyal bracket represents the Weyl transform of the commutator of two operators. It allows one to construct quantum mechanics without operators using instead functions on noncommutative phase space. It is known as the Weyl–Wigner–Moyal approach [Wey27, Wig32, Moy49] to quantum mechanics. Generically, the Moyal bracket appears in various physical problems whenever the large-N limit of a matrix commutator is represented by functions. It was associated [FZ89] with the commutator (15.55) and discussed in early works [Li96, Far97] on the star product in matrix theory.

Remark on nonlocality of the star product

A nonlocal structure of the star product is obvious from the integral representation (16.24), while part of the integration region is suppressed by oscillations of the kernel. If f_1 and f_2 has support on a small region of size ϵ, their star product $f_1(x) \star f_2(x)$ is nonvanishing over a larger region of size $|\theta|/\epsilon$. In particular, we find

$$\delta^{(d)}(x) \star \delta^{(d)}(x) \;=\; \frac{1}{\pi^d\,|\det\theta\,|} \tag{16.26}$$

for $\epsilon \to 0$.

Remark on the double scaling limit

It has been recognized recently that the continuum noncommutative quantum field theories can be obtained as a large-N limit of the twisted reduced models.

In the previous chapter we considered the limit of the twisted reduced model when $N \to \infty$ at fixed a. Then $\theta \to \infty$ according to Eq. (15.38) and this limit is associated [EN83, GO83b] with the 't Hooft limit of lattice matrix theory, where only planar diagrams survive.

Alternatively, one can approach the continuum limit of the twisted reduced models keeping θ fixed as $N \to \infty$, which requires $a \sim 1/\sqrt{L} \sim N^{-1/d}$. The period $\ell = aL \sim \sqrt{L} \sim N^{1/d} \to \infty$ in this limit so noncommutative theories on $\mathbb{R}^d$ are reproduced [AII00]. This limit is of the same type as the double scaling limit which is considered in Sect. 13.5 for the matrix models.

In Sect. 16.6 we shall describe how the original construction of [CDS98] for a torus can also be reproduced from the twisted reduced models.

16.2 The $U_\theta(1)$ gauge theory

A noncommutative gauge theory can be constructed from the continuum version of the twisted Eguchi–Kawai model described in Sect. 15.5. We should only remember that the operator $\boldsymbol{A}_\mu$ represents the reduction of the covariant derivative, as has already been pointed out, so we first substitute $\boldsymbol{A}_\mu = -\boldsymbol{P}_\mu + \tilde{\boldsymbol{A}}_\mu$ and then identify the noncommutative gauge field $\mathcal{A}_\mu(x)$ with the Weyl transform (W.t.) of the operator $\tilde{\boldsymbol{A}}_\mu$.

Proceeding in this way, we have

$$\begin{aligned} [\boldsymbol{A}_\mu, \boldsymbol{A}_\nu] + \mathrm{i} B_{\mu\nu}\,\mathbf{1} \quad &= \quad -\Big[\boldsymbol{P}_\mu, \tilde{\boldsymbol{A}}_\nu\Big] + \Big[\boldsymbol{P}_\nu, \tilde{\boldsymbol{A}}_\mu\Big] + \Big[\tilde{\boldsymbol{A}}_\mu, \tilde{\boldsymbol{A}}_\nu\Big] \\ &\xrightarrow{\text{W.t.}} \quad \mathrm{i}\,\mathcal{F}_{\mu\nu}(x)\,, \end{aligned} \tag{16.27}$$

where $\mathcal{F}$ denotes the noncommutative field strength

$$\mathcal{F}_{\mu\nu} \quad = \quad \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - \mathrm{i}\left(\mathcal{A}_\mu \star \mathcal{A}_\nu - \mathcal{A}_\nu \star \mathcal{A}_\mu\right). \tag{16.28}$$

It appeared as the Weyl transform of the LHS of Eq. (16.27).

Using Eqs. (16.27) and (15.106), we rewrite the action (15.110) as

$$S[\mathcal{A}] \quad = \quad \frac{1}{4\lambda}\int \mathrm{d}^d x\, \mathcal{F}^2, \tag{16.29}$$

where $\lambda = g^2 N$ coincides with the 't Hooft coupling of the twisted Eguchi–Kawai model. This action determines the noncommutative $U_\theta(1)$ gauge theory [CR87].

The action (16.29) involves cubic and quartic interactions of $\mathcal{A}_\mu$. This is quite the same as Yang–Mills theory! For this reason the noncommutative gauge theory is often called the noncommutative Yang–Mills theory (NCYM).

The action (16.29) is invariant under the gauge transformation

$$\mathcal{A}_\mu \quad \xrightarrow{\text{g.t.}} \quad \Omega \star \mathcal{A}_\mu \star \Omega^* + \mathrm{i}\,\Omega \star \partial_\mu \Omega^* \tag{16.30}$$

which is related to the Weyl transform of the gauge transformation (14.56) for the twisted Eguchi–Kawai model, where $\Omega(x)$ is the Weyl transform of $\widetilde{\boldsymbol{\Omega}}$. Correspondingly, the complex conjugate function $\Omega^*(x)$ is the Weyl transform of $\widetilde{\boldsymbol{\Omega}}^\dagger$. The transformation (16.30) is often termed the *star gauge* transformation.

Note that

$$(f_1 \star f_2)^* = f_1^* \exp\left(-\frac{\mathrm{i}}{2}\overleftarrow{\partial}_\mu \theta_{\mu\nu} \partial_\nu\right) f_2^* = f_2^* \star f_1^* \tag{16.31}$$

owing to the definition (16.8) of the star product. This represents the rule for Hermitian conjugation of the product of two operators.

The function $\Omega(x)$ in Eq. (16.30) is *star unitary*, i.e. it obeys

$$\Omega \star \Omega^* = 1 = \Omega^* \star \Omega\,. \tag{16.32}$$

This is, of course, just the Weyl transform of the unitarity condition for the operator $\widetilde{\boldsymbol{\Omega}}$.

A star unitary function can be constructed via the *star exponential*

$$\Omega(x) = \mathrm{e}_\star^{\mathrm{i}\alpha(x)}, \qquad \Omega^*(x) = \mathrm{e}_\star^{-\mathrm{i}\alpha(x)}, \tag{16.33}$$

where

$$\mathrm{e}_\star^{\mathrm{i}\alpha(x)} \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} \overbrace{\alpha(x) \star \cdots \star \alpha(x)}^{n} \tag{16.34}$$

is defined via the Taylor expansion with the ordinary product substituted by the star product and α is real. This is simply the Weyl transform of the exponential of i times a Hermitian operator $\boldsymbol{\alpha}$.

Problem 16.3 Prove that the action (16.29) is invariant under the star gauge transformation (16.30).

Solution The noncommutative field strength (16.28) is changed under the star gauge transformation (16.30) as

$$\mathcal{F}_{\mu\nu} \stackrel{\text{g.t.}}{\longrightarrow} \Omega \star \mathcal{F}_{\mu\nu} \star \Omega^*. \tag{16.35}$$

Correspondingly, we have

$$\int \mathrm{d}^d x\, \mathcal{F}^2 \stackrel{\text{g.t.}}{\longrightarrow} \int \mathrm{d}^d x\, \Omega \star \mathcal{F}^2 \star \Omega^* = \int \mathrm{d}^d x\, \Omega^* \star \Omega \star \mathcal{F}^2 = \int \mathrm{d}^d x\, \mathcal{F}^2 \tag{16.36}$$

as a result of Eqs. (16.16) and (16.32).

Note that only the integral of $\mathcal{F}^2_{\mu\nu}$ over space is star gauge invariant rather than $\mathcal{F}^2_{\mu\nu}$ itself.

The group of the star gauge transformations (16.30) of the noncommutative gauge theory is much larger than the group of the gauge transformations (5.4) of ordinary Yang–Mills theory and contains some of the space-time symmetries.

Let us illustrate this statement by the simplest example of a star gauge transformation given by the star unitary function

$$\Omega(x) = \mathrm{e}^{-\mathrm{i}\eta\theta^{-1}x}. \tag{16.37}$$

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We have

$$\Omega(x) \star \varphi(x) \star \Omega^*(x) \;=\; \varphi(x+\eta) \tag{16.38}$$

which means that the star gauge transformation with the function (16.37) results in a translation by the d-vector η_μ.

We have considered here the star gauge transformation of a field $\varphi(x)$ which is uniformly transformed. This could be a scalar field (in the adjoint representation), the field strength $\mathcal{F}_{\mu\nu}(x)$ or the covariant derivative $\mathrm{i}\partial_\mu + \mathcal{A}_\mu(x)$.

A similar formula can be written down for a Lorentz rotation of a scalar field in a noncommutative plane, say, the $(1,2)$-plane. It is generated by the star gauge transformation with the star unitary function

$$\Omega(x) \;=\; \sqrt{1+\alpha^2\theta^2}\,\mathrm{e}^{\mathrm{i}\alpha(x_1^2+x_2^2)}, \tag{16.39}$$

where $\theta = \theta_{12}$. Then we obtain

$$\Omega(x_1,x_2) \star \varphi(x_1,x_2,\ldots) \star \Omega^*(x_1,x_2) \;=\; \varphi\big(x_1',x_2',\ldots\big) \tag{16.40}$$

with

$$\left.\begin{aligned} x_1' &= \cos\gamma\, x_1 + \sin\gamma\, x_2\,, \\ x_2' &= -\sin\gamma\, x_1 + \cos\gamma\, x_2\,, \end{aligned}\right\} \tag{16.41}$$

which is a rotation in the $(1,2)$-plane through the angle $\gamma = 2\arctan\alpha\theta$.

Finally, the parity reflection is represented by the star gauge transformation with the star unitary function

$$\Omega(x) \;=\; \pi^{d/2}\,\mathrm{Pf}\,(\theta_{\mu\nu})\,\delta^{(d)}(x) \tag{16.42}$$

(cf. Eq. (16.26)). It acts as

$$\Omega(x) \star \varphi(x) \star \Omega^*(x) \;=\; \varphi(-x)\,. \tag{16.43}$$

We shall see later in Sect. 16.5 how these properties of the star gauge transformation restrict observables in noncommutative gauge theory.

Problem 16.4 Prove Eqs. (16.40) and (16.43) for the Ωs given, respectively, by Eqs. (16.39) and (16.42).

Solution It is convenient to use the integral representation (16.24) of the star product. For the star product of three functions, we have

$$\begin{aligned} & f_1(x) \star f_2(x) \star f_3(x) \\ &\quad = \frac{1}{\pi^d\,|\det\theta\,|}\int \mathrm{d}^d\xi\,\mathrm{d}^d\eta\,\mathrm{e}^{-2\mathrm{i}\xi\theta^{-1}\eta} f_1(x+\xi) f_2(x+\xi+\eta) f_3(x+\eta)\,. \end{aligned} \tag{16.44}$$

Substituting $f_1 = f_3 = \Omega$ given by Eq. (16.42) into Eq. (16.44), we obtain Eq. (16.43). Choosing there $\varphi(x) = 1$, we prove that $\Omega(x)$ given by Eq. (16.42) is star unitary (cf. Eq. (16.26)).

Analogously, it is easy to derive Eq. (16.40) substituting for $\varphi(x)$ a Fourier decomposition (15.101) and performing the Gaussian integral over ξ and η in Eq. (16.44) for $f_1 = \Omega$, $f_3 = \Omega^*$ with Ω given by Eq. (16.39). Again, it is easy to show that this $\Omega(x)$ is star unitary choosing $\varphi(x) = 1$.

Remark on the Lorentz invariance

The presence of a tensor $\theta_{\mu\nu}$ in Eq. (16.5) superficially breaks the Lorentz invariance for $d > 2$. We can always represent $\theta_{\mu\nu}$ in a canonical skew-diagonal form

$$\theta_{\mu\nu} = \begin{pmatrix} 0 & -\theta_1 & & & \\ +\theta_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\theta_{d/2} \\ & & & +\theta_{d/2} & 0 \end{pmatrix} \tag{16.45}$$

by a unitary transformation. In noncommutative gauge theory, this unitary matrix can be gauged away by a star gauge transformation of the type (16.39). Therefore, the only dependence on $\theta_{\mu\nu}$ is via $\theta_1, \ldots, \theta_{d/2}$ and the Lorentz invariance is preserved.

Remark on the $U_\theta(n)$ gauge theory

An extension of the results of this section to the group $U_\theta(n)$ is obvious. The noncommutative gauge field becomes an $n \times n$ matrix-valued field $\mathcal{A}_\mu^{ij}(x)$. The field strength is again given by Eq. (16.28) since the ordering in matrix multiplication is consistent with the ordering in the star product. The action of the noncommutative $U_\theta(n)$ gauge theory is

$$S[\mathcal{A}] = \frac{1}{4g^2} \int \mathrm{d}^d x \, \mathrm{tr}_{(n)} \, \mathcal{F}^2, \tag{16.46}$$

where $\mathrm{tr}_{(n)}$ denotes the $n \times n$ matrix trace.

The noncommutative $U_\theta(n)$ gauge theory can be obtained from the twisted Eguchi–Kawai model by choosing a more general twist with $n_i = nL^{d/2-1}$, which is described at the end of Problem 15.3 on p. 354.

16.3 One-loop renormalization

One of the main original motivations for studying quantum field theory on noncommutative spaces was the expectation that noncommutativity provides an ultraviolet regularization. We shall see in this section that this is not quite the case, while ultraviolet properties of noncommutative theories are somewhat better than those of their ordinary counterparts.

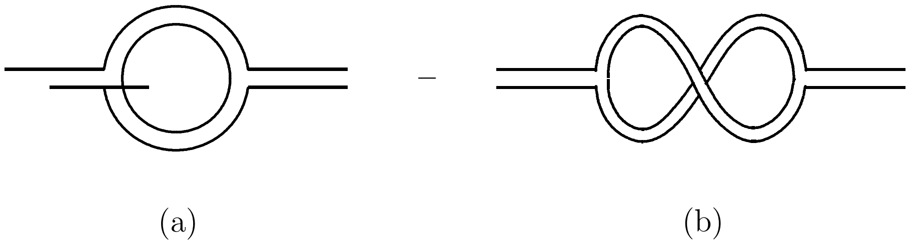

Fig. 16.1. One-loop correction to the gauge-field propagator in the noncommutative $U_\theta(1)$ gauge theory. Diagram (a) is planar and has logarithmic ultraviolet divergence. Diagram (b) is nonplanar and converges for $\theta \neq 0$. The diagrams involve momentum integrals shown in Eqs. (16.47) and (16.48). The contribution of diagram (b) is suppressed as $\theta \to \infty$ according to Eq. (16.50).

We start from the noncommutative $U_\theta(1)$ gauge theory described in the previous section, the Feynman diagrams of which have the form of ribbon graphs for a $U(N)$ Yang–Mills theory. We shall use for them the double-line notation similar to that of Sect. 11.1.

In order to construct the perturbation theory, we should first treat the gauge symmetry properly by adding the gauge-fixing and ghost terms to the action (16.29). They can be obtained easily once again by the Weyl transformation from those in the twisted Eguchi–Kawai model which are simply the reduction of the standard gauge-fixing and ghost term to zero dimensions. In the one-loop calculation of Sect. 14.5, they are given by Eq. (14.75) to quadratic order.

The one-loop corrections to the propagator are depicted in Fig. 16.1. The diagrams are the same as for a $U(N)$ Yang–Mills theory rather than $SU(N)$. The diagram in Fig. 16.1a is planar and that in Fig. 16.1b is nonplanar. The latter is not usually considered in the ordinary $SU(N)$ Yang–Mills theory, since it is associated with propagation of diagonal elements $\langle \mathcal{A}_\mu^{ii}(x)\mathcal{A}_\nu^{jj}(y)\rangle$, while the former describes an off-diagonal propagator $\langle \mathcal{A}_\mu^{ij}(x)\mathcal{A}_\nu^{ji}(y)\rangle$ with $i \neq j$.

The relative sign of the two diagrams in Fig. 16.1 is minus since the relative sign of the two terms in the commutator of $\mathcal{A}_\mu$ and $\mathcal{A}_\nu$ is minus.

For this reason, the two diagrams cancel each other in the $\theta \to 0$ limit associated with ordinary commutative Maxwell theory where there is no interaction between photons. They do not cancel however in the noncommutative case where the contributions of planar and nonplanar diagrams are different.

There is nothing special about the planar diagram in Fig. 16.1a, the contribution of which is the same as in ordinary Yang–Mills theory. Accounting for ghosts, we obtain in the Feynman gauge for the self-energy

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correction

$$\text{Fig. 16.1a} \;=\; \frac{20}{3}\lambda \int^{\Lambda} \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2(p-k)^2} \;\approx\; \frac{5}{12\pi^2}\lambda \ln \frac{\Lambda^2}{p^2} \tag{16.47}$$

in $d = 4$ with logarithmic accuracy.

Similarly, we obtain for the nonplanar diagram in Fig. 16.1b:

$$\text{Fig. 16.1b} \;=\; -\frac{20}{3}\lambda \int^{\Lambda} \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}p\theta k}}{k^2(p-k)^2} \;\approx\; -\frac{5}{12\pi^2}\lambda \ln \frac{\Lambda_{\text{eff}}^2}{p^2}\,, \tag{16.48}$$

where

$$\Lambda_{\text{eff}}^{-2} \;=\; |\,\theta p\,|^2 + \Lambda^{-2} \tag{16.49}$$

and we have assumed that $|\,p\,||\,\theta p\,| \ll 1$ for the logarithmic domain of integration to exist.

In the opposite limit of $|\,p\,||\,\theta p\,| \gg 1$, the integrand in Eq. (16.48) oscillates rapidly and the integral vanishes as

$$\text{Fig. 16.1b} \;=\; -\frac{20}{3}\lambda \int^{\Lambda} \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}p\theta k}}{k^2(p-k)^2} \;\sim\; \frac{1}{p^{2d}\det\left(\theta_{\mu\nu}\right)}\,. \tag{16.50}$$

This last formula is in accord with the general consideration of Sect. 15.2 (cf. Eq. (15.39)).

At arbitrary finite θ, the integral in Eq. (16.48) is convergent at $|\,k\,| \sim 1/|\,\theta p\,|$ so we can disregard the Λ-dependence of Λ_{eff}. Consequently, only the planar graph in Fig. 16.1a has an ultraviolet logarithmic divergence.

A very important consequence of these results is that only planar graphs contribute to the Gell-Mann–Low function of the noncommutative $U_\theta(1)$ gauge theory. For this reason it coincides [VG99, CDP00, MS99, She99, KW00] with that for ordinary Yang–Mills theory in the 't Hooft limit.

Another intriguing property of noncommutative theories is a mixing of ultraviolet and infrared scales [MRS99]. It has already been seen from Eq. (16.48) for which the RHS becomes singular at $\Lambda = \infty$ as $|\theta p| \to 0$, which plays the role of an effective ultraviolet cutoff in the coordinate space. Therefore turning on θ replaces the ultraviolet divergence by a singular infrared behavior. In other words, the infinite-cutoff limit $\Lambda \to \infty$ does not commute with the low-momentum limit $p \to 0$.

The ordinary $U(1)$ theory, where the coupling is not renormalized, is recovered at very small momenta $p \lesssim 1/\Lambda\,|\theta|$, of the order of the inverse momentum cutoff for finite θ, which are associated with very large distances $\sim \Lambda\,|\theta|$. This result is quite surprising since naively we would expect from Eq. (16.5) that the ordinary theory would be recovered at distances of the order of $\sqrt{|\theta|}$. We shall return to this aspect of the UV/IR mixing in the next section when discussing noncommutative quantum electrodynamics.

Remark on the UV/IR mixing

If we introduce an infrared cutoff by putting a noncommutative theory in a box of size ℓ, the minimal value of momentum is $p_{\min} = 2\pi/\ell$. It is related to the ultraviolet cutoff $p_{\max} = \Lambda$ by

$$p_{\min}|\theta|p_{\max} \; = \; 2\pi \tag{16.51}$$

because the position operator $\boldsymbol{x}_\mu$ and the momentum operator $\boldsymbol{P}_\nu$ are related by Eq. (16.4).

For the lattice regularization which is described in Sect. 15.2, we have $p_{\min} = 2\pi/aL$, $|\theta| = a^2L/\pi$, $p_{\max} = \pi/a$ and Eq. (16.51) is obviously satisfied.

16.4 Noncommutative quantum electrodynamics

A noncommutative extension of quantum electrodynamics (NCQED) can be constructed by the Weyl transformation of the continuum twisted Eguchi–Kawai model with fermions in the fundamental representation, the action of which is given by Eq. (15.121).

The action of noncommutative quantum electrodynamics is

$$S_{\mathrm{NCQED}} \; = \; \int \mathrm{d}^d x \left[\frac{1}{4\lambda}\mathcal{F}^2 + \bar{\psi}\gamma_\mu(\partial_\mu - \mathrm{i}\mathcal{A}_\mu \star)\psi + m\bar{\psi}\psi \right], \tag{16.52}$$

where ψ and $\bar{\psi}$ are in the fundamental representation of the star gauge group, contrary to the adjoint scalar field φ described previously in this chapter. The fields ψ and $\bar{\psi}$ are associated with "noncommutative" electrons and positrons.

Under the star gauge transformation when the gauge field is changed according to Eq. (16.30), they are transformed by

$$\psi \stackrel{\text{g.t.}}{\longrightarrow} \Omega \star \psi \,, \qquad \bar{\psi} \stackrel{\text{g.t.}}{\longrightarrow} \bar{\psi} \star \Omega^* . \tag{16.53}$$

The action (16.52) is invariant under the star gauge transformation (16.30) and (16.53).

The ordinary quantum electrodynamics with $e^2 = \lambda$ is obviously reproduced as $\theta \to 0$.

Feynman graphs of noncommutative quantum electrodynamics recall those described in Sect. 11.4 for a $U(N)$ Yang–Mills theory with quarks. It is most important that the vertex for emitting the gauge field by the fundamental matter is oriented owing to the presence of the noncommutative product. The gauge field can be emitted only to one side of the fermionic line but not to the other.

Table 16.1. Limits of noncommutative $U_\theta(1)$ gauge theory at various distances.

Distances	Theories
$r \ll \sqrt{\theta}$	Veneziano limit of QCD
$\sqrt{\theta} \lesssim r \ll \theta\Lambda$	Noncommutative $U_\theta(1)$ gauge theory
$\theta\Lambda \lesssim r$	Quantum electrodynamics

For this reason only the diagram in Fig. 11.14a on p. 230 describes the one-loop correction to the gauge-field propagator coming from fermions. This diagram is planar and there are no nonplanar diagrams with fermionic loops to this order.

This allows us to immediately conclude that the one-loop Gell-Mann–Low function of noncommutative quantum electrodynamics coincides with that (11.83) of multicolor QCD in the Veneziano limit. Given n_f species of the fermions, the one-loop Gell-Mann–Low function of noncommutative quantum electrodynamics is [Hay00]

$$\mathcal{B}(\lambda) \quad = \quad \frac{\lambda^2}{12\pi^2}\left(-11 + 2n_\mathrm{f}\right). \tag{16.54}$$

This formula shows that noncommutative quantum electrodynamics is asymptotically free at small distances for $n_\mathrm{f} \leq 5$, in contrast to ordinary QED.

Singular infrared behavior in noncommutative quantum electrodynamics is the same as in the pure noncommutative $U_\theta(1)$ gauge theory since there are no nonplanar diagrams with a fermionic loop. The usual Gell-Mann–Low function of QED is therefore reproduced at very large distances $r \gtrsim \theta\Lambda$.

To say it once again, the $\theta \to 0$ limit is not interchangeable with the $\Lambda \to \infty$ limit. Ordinary QED is reproduced for all distances when $\theta \to 0$ at fixed Λ.

In the opposite limit of $\theta \to \infty$, noncommutative quantum electrodynamics is equivalent to multicolor QCD in the Veneziano limit when the number of flavors in QCD is $N_\mathrm{f} = n_\mathrm{f} N$. This is because only the same planar diagrams survive in both cases. We have already pointed out this property in the Remark in Sect. 15.5 when describing the twisted Eguchi–Kawai model with matter in the fundamental representation. It is utilized to obtain the one-loop Gell-Mann–Low function (16.54).

These results are summarized in Table 16.1.

Remark on large but finite Λ

The value of the cutoff Λ in quantum electrodynamics (or φ^4-theory) cannot be infinite because of its "triviality" dictated by the positive sign of the Gell-Mann–Low function. The renormalized charge would vanish as $\Lambda \to \infty$. Therefore, the theory cannot be fundamental, but rather is an effective theory applicable down to the distances $\sim 1/\Lambda$ where new degrees of freedom become essential. The property of renormalizability tells us that nothing depends on this scale so the effective theory is self-consistent at large distances.

A standard way to cure the "triviality" of quantum electrodynamics is to embed it (or, strictly speaking, the $SU(2) \otimes U(1)$ electroweak theory) into an asymptotically free theory with a compact gauge group at a grand unified scale.

Remark on phenomenology in NCQED

The current experimental bound on the value of θ in our world is $\theta < (10\,\mathrm{TeV})^{-2}$. Phenomenological consequences of noncommutative quantum electrodynamics are discussed, in particular, in the recent papers [BGH01, CHK01, HPR00, Mat01, MPR00].

16.5 Wilson loops and observables

Observables in noncommutative gauge theory are to be invariant under the star gauge transformation. As has been mentioned already in Sect. 16.2, the star gauge invariance strongly restricts the allowed set of observables.

Just as in ordinary Yang–Mills theory, observables can be expressed via the Wilson loops. The standard way to derive proper formulas is to integrate over fundamental matter fields by performing the Gaussian path integral. This strategy can be repeated for noncommutative gauge theory. We describe in this section what kinds of Wilson loops then emerge.

We first define a path-dependent phase factor associated with parallel transport from the point x to the point y in an external gauge field $\mathcal{A}_\mu(x)$. The analogy with Yang-Mills theory prompts one to define

$$\mathcal{U}(C_{yx}) \stackrel{\text{def}}{=} \prod_{\xi:\, x+\xi \in C_{yx}} \star \Big[1 + \mathrm{i}\, \mathrm{d}\xi^\mu \mathcal{A}_\mu(x+\xi)\Big], \tag{16.55}$$

where the product on the RHS is the star product with respect to x, $\xi(0) = 0$, $\xi(1) \equiv \eta = y - x$ is a d-vector pointing from the initial point x of the contour to its final point y, and the ordering is along the contour C_{yx}.

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Under the star gauge transformation (16.30), $\mathcal{U}(C_{yx})$ is changed as

$$\mathcal{U}(C_{yx}) \quad \stackrel{\text{g.t.}}{\longrightarrow} \quad \Omega(y) \star \mathcal{U}(C_{yx}) \star \Omega^*(x) \tag{16.56}$$

quite similarly to the phase factor in Yang–Mills theory. The property (16.56) shows that $\mathcal{U}(C_{yx})$ is indeed a parallel transporter.

This analogy with Yang–Mills theory can be made precise by returning to the continuum twisted Eguchi–Kawai model and noting that $\mathcal{U}(C_{yx})$ is the Weyl transform of the product

$$\boldsymbol{D}^\dagger(C_{\eta 0})\, \boldsymbol{U}(C_{\eta 0}) \quad \stackrel{\text{W.t.}}{\longrightarrow} \quad \mathcal{U}(C_{yx})\,, \tag{16.57}$$

where $\boldsymbol{D}(C_{\eta 0})$ is given by Eq. (15.91) and we have denoted

$$\boldsymbol{U}(C_{\eta 0}) \quad = \quad \boldsymbol{P}\, \mathrm{e}^{\mathrm{i} \int_{C_{yx}} \mathrm{d}\xi^\mu \boldsymbol{A}_\mu} \tag{16.58}$$

to emphasize that it does not depend on the position of the initial point x of the contour since $\boldsymbol{A}_\mu$ is constant.

Similarly, Eq. (16.56) is related to the Weyl transform of the operator formula

$$\boldsymbol{U}(C_{\eta 0}) \quad \stackrel{\text{g.t.}}{\longrightarrow} \quad \boldsymbol{\Omega}\, \boldsymbol{U}(C_{\eta 0})\, \boldsymbol{\Omega}^\dagger \tag{16.59}$$

which resulted from the unitary transformation

$$\boldsymbol{A}_\mu \quad \stackrel{\text{g.t.}}{\longrightarrow} \quad \boldsymbol{\Omega} \boldsymbol{A}_\mu \boldsymbol{\Omega}^\dagger\,. \tag{16.60}$$

We demonstrate this by explicit formulas for the lattice regularization in Problem 16.8.

Multiplying by $\exp\left(\mathrm{i}\eta\theta^{-1}\boldsymbol{x}\right)$ from the left, Eq. (16.57) can be represented equivalently as

$$Z(C_{\eta 0})\, \boldsymbol{U}(C_{\eta 0}) \quad \stackrel{\text{W.t.}}{\longrightarrow} \quad \mathrm{e}^{\mathrm{i}\eta\theta^{-1}x} \star \mathcal{U}(C_{yx})\,, \tag{16.61}$$

where a c-number phase factor

$$Z(C_{\eta 0}) \quad = \quad \mathrm{e}^{\mathrm{i}\eta\theta^{-1}\boldsymbol{x}} \boldsymbol{D}^\dagger(C_{\eta 0}) \tag{16.62}$$

resulted from the difference between the path ordering of operators in $\boldsymbol{D}^\dagger(C_{\eta 0})$, given by Eqs. (15.91) and (16.4), and the symmetric ordering in $\exp\left(\mathrm{i}\eta\theta^{-1}\boldsymbol{x}\right)$. For a straight line, the difference disappears and we have $Z(C_{\eta 0}) = 1$.

In Yang–Mills theory, the trace of a phase factor for a closed loop is gauge invariant. Since the trace over the Hilbert space is substituted by the integral according to Eq. (15.106), we define

$$\mathcal{W}_{\text{clos}}(C) \quad = \quad \int \mathrm{d}^d x\, \mathcal{U}(C_{xx}) \tag{16.63}$$

which is star gauge invariant as can be easily seen using Eq. (16.16). This determines closed Wilson loops in noncommutative gauge theory.

The role of the integration over the initial point x of the contour C_{xx} is to parallel transport the contour over the space. Since an average over quantum fluctuations of the field $\mathcal{A}_\mu$ is invariant under translation, the (normalized) average of the closed Wilson loop is given simply by

$$W_{\mathrm{NC}}(C) \;=\; \frac{1}{V}\left\langle \mathcal{W}_{\mathrm{clos}}(C)\right\rangle \;=\; \left\langle \mathcal{U}(C_{xx})\right\rangle . \tag{16.64}$$

It recovers the average of the closed Wilson loop in Maxwell's theory as $\theta \to 0$.

Quite surprisingly there exists yet another kind of star gauge-invariant object in noncommutative gauge theory – open Wilson loops. They are given by [IIK00]

$$\mathcal{W}_{\mathrm{open}}(C_{\eta 0}) \;=\; \int \mathrm{d}^d x \, \mathcal{U}(C_{yx})\, \mathrm{e}^{\mathrm{i}\eta_\mu \theta^{-1}_{\mu\nu} x_\nu}, \tag{16.65}$$

where the integration over x translates the contour as a whole so $\eta = y - x$ does not change under the translation. The closed Wilson loop (16.63) corresponds to $y = x$ (or $\eta = 0$) in Eq. (16.65).

Problem 16.5 Show that the open Wilson loop (16.65) is star gauge invariant.

Solution The star gauge invariance of the open Wilson loop in noncommutative gauge theory can be shown as follows:

$$\begin{aligned}
&\int \mathrm{d}^d x \, \mathcal{U}(C_{yx})\; \mathrm{e}^{\mathrm{i}\eta\theta^{-1}x} \\
&\quad \xrightarrow{\text{g.t.}} \quad \int \mathrm{d}^d x \; \Omega(x+\eta) \star \mathcal{U}(C_{yx}) \star \Omega^*(x)\, \mathrm{e}^{\mathrm{i}\eta\theta^{-1}x} \\
&\quad \overset{(16.16)}{=} \quad \int \mathrm{d}^d x \, \mathcal{U}(C_{yx})\, \Omega^*(x) \star \, \mathrm{e}^{\mathrm{i}\eta\theta^{-1}x} \star \Omega(x+\eta) \\
&\quad \overset{(16.38)}{=} \quad \int \mathrm{d}^d x \, \mathcal{U}(C_{yx})\; \mathrm{e}^{\mathrm{i}\eta\theta^{-1}x} \star \Omega^*(x+\eta) \star \Omega(x+\eta) \\
&\quad \overset{(16.32)}{=} \quad \int \mathrm{d}^d x \, \mathcal{U}(C_{yx})\; \mathrm{e}^{\mathrm{i}\eta\theta^{-1}x}.
\end{aligned} \tag{16.66}$$

The noncommutative Wilson loop is simply related to the integral over space of the Weyl transform (16.61) with $\boldsymbol{U}(C_{\eta 0})$ given by the operator expression (16.58) for the (open or closed) contour $C_{\eta 0}$. Since the trace over the Hilbert space and the integral are related by Eq. (15.106), the open Wilson loop (16.65) is simply proportional to the trace of $\boldsymbol{U}(C_{\eta 0})$ for an open contour:

$$\mathcal{W}_{\mathrm{open}}(C_{\eta 0}) \;=\; Z(C_{\eta 0})\,(2\pi)^{d/2}\, \mathrm{Pf}\,\theta \;\; \mathrm{tr}_{\mathcal{H}}\, \boldsymbol{U}(C_{\eta 0})\, , \tag{16.67}$$

where the c-number phase factor $Z(C_{\eta 0})$ is given by Eq. (16.62).

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The RHS of Eq. (16.67) is obviously invariant under the unitary transformation (16.60) both for closed and open contours. It is of the same type as the expression under the average in Eq. (15.113), while the first trace is replaced by $Z(C_{\eta 0})$. These are the same for the closed Wilson loops.

As was shown in [AMN00a], the closed Wilson loops (16.63) naturally appear in the sum-over-path representation of the matter correlator $\langle\bar{\psi}(x)\star\psi(x)\rangle_{\psi}$, while the open Wilson loops (16.65) do so for the correlator $\langle\bar{\psi}(x+\eta)\star\exp\left(\mathrm{i}\eta\theta^{-1}x\right)\star\psi(x)\rangle_{\psi}$. Both correlators are invariant under the star gauge transformation (16.53). The second one is invariant owing to Eq. (16.38).

Noncommutative extensions of local operators, such as $\operatorname{tr}\mathcal{F}^2$, are constructed using the open Wilson loops in [GHI00].

Remark on the definition of open Wilson loops

The RHS of Eq. (16.67) looks slightly different from the expression under the sign of averaging in Eq. (15.113) since the first trace is replaced by $Z(C_{\eta 0})$, which only coincide for the closed Wilson loops. This difference between the two factors becomes inessential for the averages of open Wilson loops since they vanish in the noncommutative gauge theory owing to translational invariance

$$
\begin{aligned}
\Big\langle \mathcal{W}_{\mathrm{open}}(C_{yx})\Big\rangle &= W_{\mathrm{NC}}(C)\int \mathrm{d}^d x\,\mathrm{e}^{\mathrm{i}\eta\theta^{-1}x} \\
&= (2\pi)^d \det\theta\,\delta^{(d)}\left(x-y\right)W_{\mathrm{NC}}(C)\,. \qquad (16.68)
\end{aligned}
$$

Note that the vanishing of the open Wilson loops in the twisted Eguchi–Kawai model was guaranteed by the first trace in Eq. (15.113).

This difference of the two definitions of the open Wilson loops is a result of historical reasons. Once again they are essentially the same in the large-N limit where the averages factorize.

Problem 16.6 Obtain an explicit expression for the noncommutative phase factor expanding in $\mathcal{A}_\mu$.

Solution The calculation is similar to that in Problem 5.2 on p. 89. Expanding in $\mathcal{A}_\mu$, we obtain finally

$$
\begin{aligned}
\mathcal{U}(C_{yx}) &= \sum_{k=0}^{\infty}\mathrm{i}^k\int_0^\eta \mathrm{d}\xi_1^{\mu_1}\cdots\int_0^\eta \mathrm{d}\xi_{k-1}^{\mu_{k-1}}\int_0^\eta \mathrm{d}\xi_k^{\mu_k}\,\theta(k,k-1,\ldots,1) \\
&\quad\times\mathcal{A}_{\mu_k}(x+\xi_k)\star\mathcal{A}_{\mu_{k-1}}(x+\xi_{k-1})\star\cdots\star\mathcal{A}_{\mu_1}(x+\xi_1)\,,
\end{aligned}
\qquad (16.69)
$$

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where the star product is with respect to x and the theta function orders the points ξ_i along the contour. This formula is simply the Weyl transform of an operator version of Eq. (5.27).

Problem 16.7 Derive the loop equation in noncommutative $U_\theta(1)$ gauge theory as $\theta \to \infty$.

Solution The derivation is similar to that in Yang–Mills theory. Applying the operator $\partial_\mu \delta/\delta\sigma_{\mu\nu}$ to $\mathcal{U}(C_{xx})$ at the point $z \in C_{xx}$, we have

$$\begin{aligned}\partial_\mu \frac{\delta}{\delta\sigma_{\mu\nu}(z)}\mathcal{U}(C_{xx}) & \\ = \;& \mathcal{U}(C_{xz}) \star (\mathrm{i}\,\partial_\mu \mathcal{F}_{\mu\nu} + \mathcal{A}_\mu \star \mathcal{F}_{\mu\nu} - \mathcal{F}_{\mu\nu} \star \mathcal{A}_\mu)\,(z) \star \mathcal{U}(C_{zx})\,.\end{aligned} \tag{16.70}$$

This calculation is purely geometrical and results in the insertion of the noncommutative Maxwell equation at the point z. Replacing it by $-\mathrm{i}\delta/\delta\mathcal{A}_\nu(z)$ in the average, we obtain [AMN99]

$$\partial_\mu \frac{\delta}{\delta\sigma_{\mu\nu}(z)} W_{\mathrm{NC}}(C) \;=\; \lambda \int_0^\eta \mathrm{d}\xi^\nu \left\langle \mathcal{U}(C_{xz}) \star \delta^{(d)}(x+\xi-z) \star \mathcal{U}(C_{zx}) \right\rangle. \tag{16.71}$$

Using translational invariance of the average and the identity

$$\begin{aligned}&\int \mathrm{d}^d x\, f_1(x) \star \delta\,(x+\xi-z) \star f_2(x)\Big|_{z=x} \\ &= \frac{1}{(2\pi)^d\,|\det\theta_{\mu\nu}|} \int \mathrm{d}^d x\, f_1(x)\,\mathrm{e}^{-\mathrm{i}\xi\theta^{-1}x} \int \mathrm{d}^d y\, f_2(y)\,\mathrm{e}^{\mathrm{i}\xi\theta^{-1}y}\end{aligned} \tag{16.72}$$

which can be easily derived from Eq. (16.44), Eq. (16.71) finally takes the form [AD01]

$$\partial_\mu \frac{\delta}{\delta\sigma_{\mu\nu}(x)} \left\langle \mathcal{W}_{\mathrm{clos}}(C) \right\rangle \;=\; \frac{\lambda}{(2\pi)^d\,|\det\theta_{\mu\nu}|} \oint_C \mathrm{d}z^\nu \left\langle \mathcal{W}_{\mathrm{open}}(C_{xz})\,\mathcal{W}_{\mathrm{open}}(C_{zx}) \right\rangle. \tag{16.73}$$

Note that Eq. (16.73) relates the average of closed Wilson loops to the correlator of two open Wilson loops. The latter has a factorized part and a connected part:

$$\begin{aligned}&\left\langle \mathcal{W}_{\mathrm{open}}(C_{xz})\,\mathcal{W}_{\mathrm{open}}(C_{zx}) \right\rangle \\ &= \left\langle \mathcal{W}_{\mathrm{open}}(C_{xz}) \right\rangle \left\langle \mathcal{W}_{\mathrm{open}}(C_{zx}) \right\rangle + \left\langle \mathcal{W}_{\mathrm{open}}(C_{xz})\,\mathcal{W}_{\mathrm{open}}(C_{zx}) \right\rangle_{\mathrm{conn}}.\end{aligned} \tag{16.74}$$

The resulting equation for the factorized parts is the Weyl transform of the loop equation in the continuum twisted Eguchi–Kawai model owing to Eq. (16.67) relating the Wilson loops in both cases. Remember, that the volume $V = N(2\pi)^{d/2}\,\mathrm{Pf}\,\theta$ provides the correct normalization.

Each average in the factorized part is proportional to a delta-function as a result of Eq. (16.68), which should be treated by introducing a regularization as is discussed in Sect. 14.4. Since the connected correlator is suppressed at large θ as $1/\det\theta$, we arrive at Eq. (12.59) as $\theta \to \infty$.

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16.6 Compactification to tori

To describe compactification of noncommutative theories to tori, we start again from the twisted reduced models. We consider a lattice regularization of noncommutative gauge theories in order to make the results of this and the next sections rigorous.

A compactification of reduced models to a d-torus $\mathbb{T}^d$ can be described [CDS98] by imposing the quotient condition

$$A_\mu + 2\pi R_\mu \delta_{\mu\nu} \;=\; \Omega_\nu A_\mu \Omega_\nu^\dagger \tag{16.75}$$

on A_μ. Here Ω_ν are unitary transition matrices like those in Eq. (15.76).

In a moment we shall see that the twisted Eguchi–Kawai model with imposed quotient condition (16.75) and a certain choice of Ω_ν describes, at $N = \infty$, the noncommutative $U_\theta(1)$ gauge theory on a torus. This explains the terminology used in this section.

Taking the trace of Eq. (16.75), we see that a solution only exists for infinite matrices (= Hermitian operators).

Motivated by the discretization (15.2) of the Heisenberg commutation relation (15.87) at finite N by the unitary matrices, we exponentiate A_μ according to Eq. (15.108) with a dimensional parameter a to obtain

$$\mathrm{e}^{2\pi \mathrm{i} a \delta_{\mu\nu} R_\mu} U_\mu \;=\; \Omega_\nu U_\mu \Omega_\nu^\dagger \,. \tag{16.76}$$

This U_μ is unitary and Eq. (16.76) is an $N \times N$ matrix discretization of Eq. (16.75) which has solutions (described below) for finite N.

Taking the trace of Eq. (16.76), we conclude that U_μ should be traceless, which is the case for the twist eaters Γ_μ. Taking the determinant of Eq. (16.76), we conclude that $aR_\mu N$ should be integral. The consistency of Eq. (16.76) also requires

$$\Omega_\mu \Omega_\nu \;=\; Z_{\mu\nu}\, \Omega_\nu \Omega_\mu \tag{16.77}$$

with $Z_{\mu\nu} \in Z(N)$. The quotient condition (16.76) is compatible with the gauge symmetry (14.39) if Ω commutes with the transition matrices Ω_ν.

Let us choose

$$\Omega_\mu \;=\; \prod_\nu \Gamma_\nu^{m\varepsilon_{\mu\nu}}, \tag{16.78}$$

where m is an integer and

$$\varepsilon_{\mu\nu} \;=\; \begin{pmatrix} 0 & +1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & +1 \\ & & & -1 & 0 \end{pmatrix}. \tag{16.79}$$

These Ω_μ obviously obey Eq. (16.77).

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Then a particular solution to Eq. (16.76) with $aR_\mu = m/L$ is given by $U_\mu = \Gamma_\mu$, while a general solution is

$$U_\mu \;=\; \Gamma_\mu \widetilde{U}_\mu \,, \tag{16.80}$$

with $\widetilde{U}_\mu$ obeying

$$\widetilde{U}_\mu \;=\; \Omega_\nu \widetilde{U}_\mu \Omega_\nu^\dagger \,. \tag{16.81}$$

We are interested in a very special solution [AMN99] to Eq. (16.81) at finite N when m is a divisor of L so that $n = L/m$ is an integer. Then a solution to Eq. (16.81) can be written as

$$\widetilde{U}_\mu^{ij} \;=\; \frac{1}{m^d} \sum_{k \in \mathbb{Z}_m^d} (J_k^n)^{ij}\, U_\mu(k) \,, \tag{16.82}$$

where J_k are defined in Eq. (15.22). Here k_μ runs from 1 to m since $\Gamma_\mu^L = 1$. This $\widetilde{U}_\mu$ obviously commutes with Ω_ν given by Eq. (16.78).

Given the c-number coefficients $U_\mu(k)$ which describe the dynamical degrees of freedom, we can use a Fourier transformation to obtain the field

$$\mathcal{U}_\mu(x) \;=\; \frac{1}{m^d} \sum_{k \in \mathbb{Z}_m^d} \mathrm{e}^{2\pi \mathrm{i}\, x \varepsilon k / am}\, U_\mu(k) \tag{16.83}$$

which is periodic on an m^d lattice (or equivalently on a discrete torus $\mathbb{T}_m^d$). The spatial extent of the lattice is therefore $\ell = am$.

The field $\mathcal{U}_\mu(x)$ describes the same degrees of freedom as the (constraint) $N \times N$ matrix U_μ^{ij}, while the unitarity condition $U_\mu U_\mu^\dagger = 1$ is rewritten as

$$\mathcal{U}_\mu(x) \star \mathcal{U}_\mu^*(x) \;=\; 1 \tag{16.84}$$

similarly to Eq. (16.32) in the continuum.

The lattice star product in Eq. (16.84) is given by

$$f_1(x) \star f_2(x) \;=\; \frac{1}{m^d} \sum_{y,z} \mathrm{e}^{-2\mathrm{i} y_\mu \theta^{-1}_{\mu\nu} z_\nu} f_1(x+y) f_2(x+z) \tag{16.85}$$

with

$$\theta_{\mu\nu} \;=\; -\frac{a^2 mn}{\pi} \varepsilon_{\mu\nu} \;=\; -\frac{\ell^2}{\pi} \frac{n}{m} \varepsilon_{\mu\nu} \,. \tag{16.86}$$

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This expression for $\theta_{\mu\nu}$ is of the same type as Eq. (15.38) for the given simplest twist with*

$$n_{\mu\nu} = 2L^{d/2-1}\varepsilon_{\mu\nu} . \tag{16.87}$$

These formulas follow from comparing the expansions (16.82) with (16.83) and using Eq. (15.26). As $a \to 0$, Eq. (16.85) recovers the integral representation (16.24) of the star-product (16.8) in the continuum.

The twisted Eguchi–Kawai model (15.65) (in general, with the quotient condition (16.76)) can be rewritten identically as a noncommutative $U_\theta(1)$ lattice gauge theory. Given the relations (16.82) and (16.83) between matrices and fields, we rewrite the action (15.64) of the twisted Eguchi–Kawai model as

$$\frac{N}{g^2}S_{\mathrm{TEK}} = \frac{1}{2\lambda}\sum_{x\in\mathbb{T}^d_m}\sum_{\mu\neq\nu}\Big(1-\mathcal{U}^*_\nu(x)\star\mathcal{U}^*_\mu(x+a\hat{\nu})\star\mathcal{U}_\nu(x+a\hat{\mu})\star\mathcal{U}_\mu(x)\Big), \tag{16.88}$$

where the coupling constant $\lambda = g^2N$.

Analogously, the (constraint) measure $\mathrm{d}U_\mu$ turns into the Haar measure

$$\prod_{x,\mu}\mathrm{d}\,\mathcal{U}_\mu(x) = \prod_{k,\mu}\mathrm{d}U_\mu(k) . \tag{16.89}$$

In fact, both (constraint) $\mathrm{d}U_\mu$ and $\prod_{x,\mu}\mathrm{d}\,\mathcal{U}_\mu(x)$ are simply given by the RHS of Eq. (16.89) since the degrees of freedom are the same.

The action (16.88) is invariant under the lattice star gauge transformations

$$\mathcal{U}_\mu(x) \stackrel{\text{g.t.}}{\longrightarrow} \Omega(x+a\hat{\mu})\star\mathcal{U}_\mu(x)\star\Omega^*(x) , \tag{16.90}$$

where $\Omega(x)$ is star unitary.

The usual twisted Eguchi–Kawai model (without the quotient condition) is associated with $n=1$. Then $\Omega_\mu = 1$ (remember that $\Gamma^L_\mu = 1$) and Eq. (16.76) becomes trivial. The large-N limit of the usual twisted reduced models can be associated [AII00] with noncommutative theories on $\mathbb{R}^d$ as has already been discussed in the Remark on p. 382.

For $n > 1$ (that is the twisted Eguchi–Kawai model with the quotient condition), the noncommutativity parameter (16.86) can be kept finite as $N \to \infty$, even for a finite ℓ if the dimensionless noncommutativity parameter

$$\Theta_{\mu\nu} \stackrel{\text{def}}{=} \frac{2\pi}{\ell^2}\theta_{\mu\nu} \tag{16.91}$$

* The discrepancy in a factor of 4 is because we have changed the definition of the lattice momentum: $p_\mu = 2\pi\varepsilon_{\mu\nu}k_\nu/aL = \pi n_{\mu\nu}k_\nu/aN$, which is more natural for even $n_{\mu\nu}$ and odd L (see the footnote on p. 356).

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is kept finite (and becomes an irrational number as $N \to \infty$). This means that the resulting continuum noncommutative theory lives on a torus [CDS98]. The case of finite N corresponds [AMN99] to the noncommutative lattice gauge theory (16.88) which is a lattice regularization of this continuum theory. The noncommutative theory on $\mathbb{R}^d$ is reached as $\ell \to \infty$.

Remark on finite Heisenberg–Weyl group

The noncommutative lattice gauge theory can be constructed [BM99, AMN00b] on the basis of finite-dimensional representations of the Heisenberg–Weyl group without the use of the matrix approximation. Equation (16.86) relating θ and the lattice size then emerges as a consistency condition.

Remark on Wilson loops on the lattice

A lattice contour C consisting of J links is given by the set of unit vectors $\hat{\mu}_j$ associated with the direction of each link j ($j = 1, \ldots, J$) forming the contour (cf. Eq. (6.40)). The parallel transporter from the point x to the point $y = x + \eta$ ($\eta = a \sum_j \hat{\mu}_j$) along C_{yx} is given by

$$\mathcal{U}(C_{yx}) \; = \; \mathcal{U}_{\mu_J}\Big(x + a\sum_{j=1}^{J-1} \hat{\mu}_j\Big) \star \cdots \star \mathcal{U}_{\mu_2}(x + a\hat{\mu}_1) \star \mathcal{U}_{\mu_1}(x)\,. \tag{16.92}$$

It is star gauge covariant,

$$\mathcal{U}(C_{yx}) \;\xrightarrow{\text{g.t.}}\; \Omega(x+\eta) \star \mathcal{U}(C) \star \Omega^*(x)\,, \tag{16.93}$$

under the star gauge transformation (16.90) of the link variable.

The lattice analog of the open Wilson loop (16.65) is

$$\mathcal{W}_{\text{open}}(C_{\eta 0}) \;=\; \sum_x \mathrm{e}^{\mathrm{i}\eta_\mu \theta^{-1}_{\mu\nu} x_\nu} \star \mathcal{U}(C_{yx})\,, \tag{16.94}$$

where $\eta_\mu = a n j_\mu$ with integer-valued j_μ (modulo possible windings). It is star gauge invariant.

The continuum limit of Eq. (16.94) determines star gauge-invariant Wilson loops in noncommutative gauge theory. The open loops (16.65), which exist in addition to closed Wilson loops, can have an arbitrary value of η on $\mathbb{R}^d$ [IIK00]. On $\mathbb{T}^d$ the open Wilson loops are star gauge invariant only for discrete values of η measured in the units of $2\pi\theta/\ell$ [AMN99].

We shall see in the next section that the open Wilson loops in noncommutative $U_\theta(1)$ gauge theory for integral $m/n = \tilde{p}$ are Morita equivalent to the Polyakov loops in a $U(\tilde{p}^{d/2})$ Yang–Mills theory on a smaller torus of period $\ell/\tilde{p}$ with twisted boundary conditions.

Problem 16.8 Find a map between the Wilson loops in the twisted Eguchi–Kawai model and the noncommutative lattice gauge theory.

Solution The map between the Wilson loops in the twisted Eguchi–Kawai model and the noncommutative $U_\theta(1)$ gauge theory on the lattice can be written down explicitly using the relation (15.47) and (15.48) between matrices and fields. We consider for simplicity the twisted Eguchi–Kawai model without the quotient condition.

We mention first that $\Delta^{ij}(x)$ defined in Eq. (15.40) obeys the property

$$\Delta(y) \;=\; D(C_{yx})\Delta(x)D^\dagger(C_{yx})\,, \tag{16.95}$$

where $D(C_{yx})$ is given by Eq. (15.7) but the RHS is path-independent as is shown in Problem 15.1 on p. 352.

It also satisfies

$$\Delta^{ij}(x)\star\Delta^{kl}(x) \;=\; \frac{1}{N}\delta^{il}\Delta^{kj}(x) \tag{16.96}$$

as a consequence of the formula

$$\sum_{m\in\mathbb{Z}^d_L} J^{ij}_m J^{kl}_{n-m}\,\mathrm{e}^{\pi\mathrm{i}\sum_{\mu,\nu} m_\mu n_{\mu\nu} n_\nu/N} \;=\; N\delta^{il}J^{kj}_n \tag{16.97}$$

which recovers the completeness condition (15.25) for $n=0$.

Given Eqs. (16.95) and (16.96), we have

$$N\operatorname{tr}\Big[A\Delta(x+\eta)\Big]\star N\operatorname{tr}\Big[B\Delta(x)\Big] \;=\; N\operatorname{tr}\Big[D^\dagger(C_{yx})AD(C_{yx})B\Delta(x)\Big], \tag{16.98}$$

where $y=x+\eta$. Equation (16.98) is simply a matrix analog of the extension of Eq. (16.23) for the case when y does not coincide with x.

Noting that

$$\mathcal{U}_\mu(x) \;=\; N\operatorname{tr}\Big[\widetilde{U}_\mu\Delta(x)\Big] \tag{16.99}$$

and applying Eq. (16.98) several times, we obtain

$$\mathcal{U}(C_{yx}) \;=\; N\operatorname{tr}\Big[D^\dagger(C_{yx})\,U(C_{yx})\,\Delta(x)\Big] \tag{16.100}$$

with $U(C_{yx})=\prod_{C_{yx}} U_{\mu_i}$. This is a lattice analog of Eq. (16.57).

Under the gauge transformation (14.39) the RHS of Eq. (16.100) transforms as

$$\begin{aligned}
N\operatorname{tr}\Big[D^\dagger(C_{yx})\,U(C_{yx})\,\Delta(x)\Big] &\;\stackrel{\text{g.t.}}{\longrightarrow}\; N\operatorname{tr}\Big[D^\dagger(C_{yx})\,\Omega\,U(C_{yx})\,\Omega^\dagger\Delta(x)\Big]\\
&=\; N\operatorname{tr}\Big[D^\dagger(C_{yx})\,\Omega\,D(C_{yx})\,D^\dagger(C_{yx})\,U(C_{yx})\,\Omega^\dagger\Delta(x)\Big]\\
&\stackrel{(16.98)}{\longrightarrow}\; \Omega(y)\star\mathcal{U}(C_{yx})\star\Omega^*(x)
\end{aligned} \tag{16.101}$$

which is a lattice version of Eq. (16.56).

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For a closed contour, $D(C_{xx})$ in Eq. (16.100) is a c-number and summing over x by $\sum_x \Delta(x) = 1$, we arrive at

$$\sum_x \mathcal{U}(C_{xx}) \;=\; \operatorname{tr} D^\dagger(C)\, \operatorname{tr} U(C)\,. \tag{16.102}$$

This reproduces Eq. (15.68) for the Wilson loops in the twisted Eguchi–Kawai model after averaging and dividing by the volume factor of $L^d = N^2$ which appears on the LHS owing to Eq. (16.64).

For an open contour, we obtain analogously

$$\sum_x \mathcal{U}(C_{yx})\, \mathrm{e}^{\mathrm{i}\eta\theta^{-1}x} \;=\; Z(C_{\eta 0})\, N \operatorname{tr} U(C_{\eta 0})\,, \tag{16.103}$$

where $Z(C_{\eta 0}) = J_{\eta/a} D^\dagger(C_{\eta 0})$ is a lattice analog of the c-number phase factor (16.62) (cf. Eq. (15.60)). Equation (16.103) is a lattice analog of Eq. (16.67).

16.7 Morita equivalence

The continuum noncommutative gauge theory with rational values of the dimensionless noncommutativity parameter Θ defined in Eq. (16.91) has an interesting property known as Morita equivalence [Sch98].* We shall describe it for the lattice regularization associated with the simplest twist (16.87), assuming that the ratio $m/n = \tilde{p}$ is an integer.

Then the noncommutative $U_\theta(1)$ gauge theory on an m^d periodic lattice is equivalent to ordinary $U(p)$ Yang–Mills theory with $p = \tilde{p}^{d/2}$ on a smaller $n^d = (m/\tilde{p})^d$ lattice with twisted boundary conditions and the coupling $g^2 = \lambda/p$ (where λ is the coupling of the $U_\theta(1)$ gauge theory).

In the previous section we have discussed the equivalence of the twisted Eguchi–Kawai model (with the quotient condition in general) with $N = (mn)^{d/2}$ and the noncommutative $U_\theta(1)$ gauge theory on $\mathbb{T}^d_m$. Both theories have the same m^d degrees of freedom, which are described either by the (constraint) $N \times N$ matrix (16.82) or the lattice field (16.83).

In the matrix language, the noncommutativity emerges since

$$J^n_k J^n_q \;=\; J^n_{k+q}\, \mathrm{e}^{2\pi \mathrm{i} k\varepsilon q\, n/m} \tag{16.104}$$

as it follows from the general Eq. (15.26) for the given simplest twist.

In the noncommutative language, this noncommutativity resides in the star product

$$\mathrm{e}^{2\pi \mathrm{i}\, k\varepsilon x/\ell} \star \mathrm{e}^{2\pi \mathrm{i}\, q\varepsilon x/\ell} \;=\; \mathrm{e}^{2\pi \mathrm{i}\,(k+q)\varepsilon x/\ell}\, \mathrm{e}^{2\pi \mathrm{i}\, k\varepsilon q\, n/m} \tag{16.105}$$

as follows from the definition (16.85).

* It is often defined in a broader sense relating the values of $\Theta_{\mu\nu}$ in two equivalent noncommutative theories (see the review [KS00] and references therein).

When $m = \tilde{p}n$, a third equivalent model exists where the same dynamical degrees of freedom are described by a $p \times p$ matrix-valued field $\widetilde{U}_\mu^{ab}(\tilde{x})$ on an n^d lattice, the sites of which are denoted by $\tilde{x}$.

Let us introduce $p \times p$ twist eaters $\widetilde{\Gamma}_\nu$ obeying the Weyl–'t Hooft commutation relation

$$\widetilde{\Gamma}_\mu \widetilde{\Gamma}_\nu = \widetilde{Z}_{\mu\nu} \widetilde{\Gamma}_\nu \widetilde{\Gamma}_\mu \,, \qquad \widetilde{Z}_{\mu\nu} = \mathrm{e}^{4\pi\mathrm{i}\varepsilon_{\mu\nu}/\tilde{p}} \tag{16.106}$$

and $\tilde{p}$ is also assumed to be odd. The integers $\tilde{p}$ and p play, respectively, the same role as L and N above.

A solution to Eq. (16.81) can then be represented as

$$\widetilde{U}_\mu^{ab}(\tilde{x}) = \frac{1}{m^d} \sum_{k \in \mathbb{Z}_m^d} \tilde{J}_k^{ab}\, \mathrm{e}^{2\pi\mathrm{i}\, \tilde{x}\varepsilon k / a\tilde{p}n}\, U_\mu(k) \,. \tag{16.107}$$

Here we have introduced a basis on $gl(p;\mathbb{C})$ given by the formulas similar to Eq. (15.22):

$$\tilde{J}_k = \prod_\mu \widetilde{\Gamma}_\mu^{k_\mu}\, \mathrm{e}^{-2\pi\mathrm{i} \sum_{\mu<\nu} k_\mu \varepsilon_{\mu\nu} k_\nu / \tilde{p}}, \tag{16.108}$$

where $k \in \mathbb{Z}_{\tilde{p}}^d$. They obey, in particular,

$$\tilde{J}_k \tilde{J}_q = \tilde{J}_{k+q}\, \mathrm{e}^{2\pi\mathrm{i}k\varepsilon q/\tilde{p}} \qquad (\mathrm{mod}\ \tilde{p}) \,. \tag{16.109}$$

The action of the third model is just the ordinary Wilson lattice action

$$S = \frac{1}{2} \sum_{\tilde{x} \in \widetilde{\mathbb{T}}_n^d} \sum_{\mu \neq \nu} \left(1 - \frac{1}{p} \operatorname{tr}_{(p)} \widetilde{U}_\nu^\dagger(\tilde{x}) \widetilde{U}_\mu^\dagger(\tilde{x} + a\hat{\nu}) \widetilde{U}_\nu(\tilde{x} + a\hat{\mu}) \widetilde{U}_\mu(\tilde{x}) \right), \tag{16.110}$$

while the coupling constant $g^2 = \lambda/p$. The field $\widetilde{U}_\mu(\tilde{x})$ is quasi-periodic on $\widetilde{\mathbb{T}}_n^d$ and obeys the twisted boundary conditions

$$\widetilde{U}_\mu(\tilde{x} + an\hat{\nu}) = \widetilde{\Gamma}_\nu^\dagger \widetilde{U}_\mu(\tilde{x}) \widetilde{\Gamma}_\nu \tag{16.111}$$

since

$$\widetilde{\Gamma}_\mu \tilde{J}_k \widetilde{\Gamma}_\mu^\dagger = \tilde{J}_k\, \mathrm{e}^{4\pi\mathrm{i}\varepsilon_{\mu\nu} k_\nu / \tilde{p}}. \tag{16.112}$$

It is of the type of Eq. (15.78) with $\Omega_\nu = \widetilde{\Gamma}_\nu^\dagger$. Therefore, $\widetilde{Z}_{\mu\nu} \in Z(p)$ in Eq. (16.106) represents the non-Abelian 't Hooft flux.

The number of degrees of freedom of the third model is $n^d p^2 = m^d$ for $p = \tilde{p}^{d/2}$ and coincides with those in the other two equivalent models.

For $n = 1$ when $\tilde{p} = m$ and $p = N$, the third model lives on a unit hypercube with twisted boundary conditions and coincides with the twisted

Eguchi–Kawai model as is shown in Problem 15.6 on p. 366. Therefore, the derivation of noncommutative gauge theories from the twisted Eguchi–Kawai model is the simplest example of Morita equivalence.

In the continuum limit ($N \to \infty$) when the twisted Eguchi–Kawai model is formulated via operators, the noncommutative $U_\theta(1)$ gauge theory lives on $\mathbb{T}^d$ with period ℓ and is Morita equivalent at the rational value of $\Theta_{\mu\nu}$ to the ordinary $U(p)$ gauge theory on a smaller torus $\widetilde{\mathbb{T}}^d$ with twisted boundary conditions and period $\tilde{\ell} = \ell/\tilde{p}$. Its coupling constant is given by $g^2 = \lambda/p$. This twisted torus is precisely the one which first appeared in Eq. (16.75) since $\tilde{\ell}_\mu = 1/R_\mu$.

The lattice regularization makes these results rigorous. An arbitrary rational value of $\Theta_{\mu\nu}$ can be obtained for the most general twist described in Problem 15.3 on p. 354. And vice versa, a continuum noncommutative theory with an arbitrary irrational value of the $\Theta_{\mu\nu}$ can be obtained starting from the ordinary Yang–Mills theory on a twisted torus as $p \to \infty$ or, equivalently, the (constraint) twisted Eguchi–Kawai model as $N \to \infty$.

Remark on constraint TEK

The results of this section show that ordinary Yang–Mills theory on a twisted torus (i.e. with the 't Hooft flux) can be represented as the twisted Eguchi–Kawai model with constraint matrices. This fact was not known in the 1980s.

Remark on fundamental matter

The results of this and the previous sections can be extended [AMN00a] to the presence of matter. Let $\varphi(x)$ be a scalar matter field in the fundamental representation of $U_\theta(1)$. The matter part of the action is

$$S_{\rm mat} \;=\; -\sum_{x,\mu} \varphi^*(x+a\hat{\mu}) \star \mathcal{U}_\mu(x) \star \varphi(x) + M^2 \sum_x \varphi^*(x)\varphi(x) \tag{16.113}$$

and is invariant under the star gauge transformation

$$\varphi(x) \;\xrightarrow{\text{g.t.}}\; \Omega(x) \star \varphi(x)\,, \qquad \varphi^*(x) \;\xrightarrow{\text{g.t.}}\; \varphi^*(x) \star \Omega^*(x) \tag{16.114}$$

simultaneously with that (16.90) for $\mathcal{U}_\mu(x)$. Equation (16.114) is similar to Eq. (16.53) in the continuum.

At a rational value of Θ, the action (16.113) on a torus is Morita equivalent to

$$S_{\rm mat} \;=\; -\sum_{\tilde{x},\mu} \mathrm{tr}_{(p)}\, \phi^\dagger(\tilde{x}+a\hat{\mu}) \widetilde{U}_\mu(\tilde{x}) \phi(\tilde{x}) + M^2 \sum_{\tilde{x}} \mathrm{tr}_{(p)}\, \phi^\dagger(\tilde{x})\phi(\tilde{x})\,, \tag{16.115}$$

where $\phi^{ab}(\tilde{x})$ is a $p \times p$ matrix-valued field on a n^d lattice which obeys the twisted boundary conditions

$$\phi(\tilde{x} + \tilde{\ell}\hat{\nu}) = \widetilde{\Gamma}_\nu^\dagger \phi(\tilde{x}) \widetilde{\Gamma}_\nu \tag{16.116}$$

similar to Eq. (16.111) for the gauge field.

The index a of ϕ^{ab} plays the role of color, while b plays the role of flavor (labeling species). The color symmetry is local, while the flavor symmetry is global. In particular, the model (16.115) reduces for $n = 1$ to the twisted Eguchi–Kawai model with fundamental matter [Das83], the action of which is given by Eq. (15.117).

The continuum limit of the above formulas is obvious. The continuum $U_\theta(1)$ gauge theory with fundamental matter (noncommutative QED) is reproduced as $N \to \infty$. For $\theta \to \infty$ it is equivalent to large-N QCD on $\mathbb{R}^d$ in the Veneziano limit when the number of flavors of fundamental matter is proportional to the number of colors so the matter survives in the large-N limit. This makes the results of Sect. 16.4 rigorous since they are now obtained with regularization.

Remark on classical solutions

Noncommutative theories admit a whole zoo of classical solutions: instantons [NS98], solitons [GMS00], monopoles [GN00]. Some of them, such as instantons in the noncommutative $U_\theta(1)$ gauge theory or solitons in a four-dimensional noncommutative scalar theory, are new in the sense that they do not exist in ordinary cases. Some of them, such as monopoles in the noncommutative $U_\theta(1)$ gauge theory, are counterparts of the known solutions, which are usually associated with an infinite action. Now they become essential since turning on θ regularizes tension of the Dirac string. More on these classical solutions can be found in the reviews [Nek00, Har01].

It is intriguing whether or not these classical solutions in noncommutative gauge theories can give us a new insight into the problems of large-N QCD.

Bibliography to Part 4

Reference guide

There are no books on the reduced models. The existing reviews include those by Migdal [Mig83] and Das [Das87]. I would recommend to read the well-written original papers [GK82, GO83b] as well as the others cited in the text. A modern survey of the twist-eating solutions is given by González-Arroyo [Gon98].

The reduced models were discovered by Eguchi and Kawai [EK82]. The quenched Eguchi–Kawai model was introduced by Bhanot, Heller and Neuberger [BHN82] and elaborated in [Par82, GK82, DW82, Mig82]. The twisted Eguchi–Kawai model is constructed by González-Arroyo and Okawa [GO83a, GO83b] for the Yang–Mills theory and by Eguchi and Nakayama [EN83] for scalar fields.

The literature on noncommutative theories is vast. A mathematical background is described in the books by Connes [Con94], Landi [Lan97], and Madore [Mad99a]. The Weyl transformation is presented in the books by Weyl [Wey31] and Wong [Won98]. The properties of the star product are considered in [BFF78]. Its relation to the large-N limit is discussed in the review [Ran92] and the original papers cited therein. Some recent reviews on noncommutative theories and their applications are [Mad99b, KS00, DN01], which contain a comprehensive list of references. The classical solutions in noncommutative theories are discussed in the lectures by Nekrasov [Nek00] and Harvey [Har01].

The recent interest in noncommutative gauge theories came from string theory [SW99]. Their relation to the twisted reduced models was pointed out in [AII00]. Its extension to the original toroidal construction of [CDS98] was given in [AMN99]. The Wilson loops in noncommutative gauge theories are constructed in [IIK00]. The lattice regularization of noncommutative gauge theories is described in [AMN00b]. The issue of the UV/IR mixing in noncommutative quantum field theories is discussed in the paper [MRS99]. Subtleties with renormalization of noncommutative field theories are discussed in [CR00].

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