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WEYL'S THEOREM, TENSOR PRODUCTS AND MULTIPLICATION OPERATORS II

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Abstract. The 'polaroid' property transfers from Banach algebra elements to their tensor product, and hence also to their induced multiplications on 'ultraprime' Banach bimodules.

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1. Introduction. Recall that an element $T \in G$ of a complex Banach algebra G, with identity I and invertible group G^{-1}, is *simply polar* ([**1**, **3**, **4**, Definition 7.3.5]) iff there is $S \in G$ for which

$$T - TST = 0 = TS - ST; \tag{1}$$

the products

$$T^\bullet = TS = ST, \;\; T^\times = STS \tag{2}$$

are uniquely determined and double commute with T. More generally $T \in G$ is polar iff T^n is simply polar for some $n \in \mathbf{N}$, and *quasi-polar* iff ([**3**, **4**, Definition 7.5.2]; cf. [**8**]) there is $E = E^2 = I - E' \in G$ for which

$$TE = ET; \; TE' \in (E'GE')^{-1}; \; TE \in QN(EGE). \tag{3}$$

Here

$$QN(G) = \{T \in G : \|T^n\|^{1/n} \to 0 \; (n \to \infty)\} \; = \; \{T \in G : I - \mathbf{C}T \subseteq G^{-1}\} \tag{4}$$

are the *quasi-nilpotent* elements of G, and necessary and sufficient for $T \in G$ to be quasi-polar is that zero is at worst an isolated point of spectrum:

$$0 \notin \operatorname{acc} \sigma(T) \subseteq \mathbf{C}. \tag{5}$$

We recall [**1**, **6**] 'isoloid' and 'polaroid' elements:

Definition 1. $T \in G$ is said to be left (resp. right) isoloid if there is implication, for arbitrary $\nu \in \mathbf{C}$,

$$T - \nu I \text{ quasi-polar } \Longrightarrow T - \nu I \text{ left (resp. right) zero divisor,} \tag{6}$$

and polaroid if

$$T - \nu I \text{ quasi-polar } \Longrightarrow T - \nu I \text{ polar,} \tag{7}$$

In this paper we show that whenever $a \in A$ and $b \in B$ are polaroid then so is $T = a \otimes b \in G = A \otimes B$, a uniformly cross-normed *tensor product* algebra, and hence also $T = L_a R_b \in G = B(M)$, induced 'elementary operators' on 'ultraprime' bimodules. We recall ([**2**, **4**, Theorems 11.7.6 and 11.6.8]) a little bit of spectral theory,

$$\sigma(a \otimes b) = \sigma(a)\sigma(b) = \sigma(L_a R_b), \tag{8}$$

with an accompanying fragment of topology: if K, H are compact subsets of $\mathbf{C}$ there is ([**6**, Theorem 6]) inclusion

$$\text{iso}(K \cdot H) \setminus \{0\} \subseteq \text{iso}(K) \cdot \text{iso}(H) \subseteq \text{iso } (K \cdot H) \cup \{0\} \tag{9}$$

and

$$\text{iso}(K \cdot H) \subseteq \text{iso}(K) \cdot H \ \cup \ K \cdot \text{iso}(H); \tag{10}$$

conversely,

$$\text{acc}(K) \cdot \text{acc}(H) \subseteq \text{acc}(K \cdot H) \subseteq \text{acc}(K) \cdot H \ \cup \ K \cdot \text{acc}(H) \subseteq \text{acc}(K \cdot H) \cup \{0\}. \tag{11}$$

As a supplement to (2.4) and (2.5),

Theorem 2. *If K, H are compact subsets of $\mathbf{C}$ there is implication*

$$0 \in (\text{iso } K \cdot H) \setminus H \Longrightarrow 0 \in \text{iso } K, \tag{12}$$

and

$$0 \in (\text{iso } K \cdot H) \cap \text{acc } H \Longrightarrow K = \{0\}. \tag{13}$$

Proof. If 0 is an isolated point of $K \cdot H$ then $0 = \lambda\mu$ with $\mu \in H$ and $\lambda \in K$, and if $0 \notin H$ then necessarily $\lambda = 0$. Now if $0 \in \text{acc } K$ then there is (λ_n) in K with $0 \neq \lambda_n \to 0$ in which case $\mu \in H \Longrightarrow 0 \neq \lambda_n\mu \to 0$, contradicting the fact that 0 is isolated in $K \cdot H$. This gives (12); towards (13) suppose that $0 \neq \lambda \in K$ and $0 \neq \mu_n \to \mu$ in H: then $0 \neq \lambda\mu_n \to 0$ in $K \cdot H$, again contradicting the status of 0 as an isolated point of $K \cdot H$ □

If $a \in A$ and $b \in B$ are left, or right, isoloid then so is $a \otimes b \in A \otimes B$: this follows from Theorem 7 of [**6**], cf. [**10**], applied to the operators L_a and R_b. The polaroid property also transfers:

Theorem 3. *If $a \in A$ and $b \in B$ are polaroid then so is $T = a \otimes b \in G = A \otimes B$.*

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Proof. If $0 \neq \nu \in \text{iso}\,\sigma(a \otimes b)$ then by (9) there is $(\lambda, \mu) \in \mathbf{C}^2$ for which

$$\lambda \in \text{iso}\,\sigma(a), \quad \mu \in \text{iso}\,\sigma(b), \quad \lambda\mu = \nu: \tag{14}$$

then with $p = p^2 \in A$ obtained from the analogue of (3) with $a - \lambda \in A$ in place of $T \in G$, and $q = q^2 \in B$ doing the same job for $b - \mu \in B$ we have, with $p' = 1 - p$, $q' = 1 - q$ and $\nu = \lambda\mu$,

$$\begin{aligned} T &= a \otimes b - \nu(1 \otimes 1) = (a \otimes b - \nu(1 \otimes 1))(p' \otimes q') \\ &\quad + ((a - \lambda) \otimes (b - \mu) + \lambda \otimes (b - \mu) + (a - \lambda) \otimes \mu)(p \otimes q + p' \otimes q + p \otimes q') \\ &= ((a - \lambda) \otimes (b - \mu) + \lambda \otimes (b - \mu) + (a - \lambda) \otimes \mu)(p \otimes q) \\ &\quad + ((a - \lambda) \otimes b + \lambda \otimes (b - \mu))(p \otimes q') + (a \otimes (b - \mu) + (a - \lambda) \otimes \mu)(p' \otimes q) \\ &\quad + (a \otimes b - \nu(1 \otimes 1))(p' \otimes q'). \end{aligned}$$

Now $T(p \otimes q)$ is the sum of three commuting nilpotents in $(p \otimes q)G(p \otimes q)$, each of $T(p \otimes q')$ and $T(p' \otimes q)$ is the commuting sum of an invertible and a nilpotent, while finally the invertibility of $T(p' \otimes q')$ in $(p' \otimes q')G(p' \otimes q')$ is (8), and $T \in G$ is therefore polar.

It remains to consider the case

$$\nu = 0 \in \text{iso}\,\sigma(a \otimes b) \subseteq (\text{iso}\,\sigma(a))\sigma(b) \cup \sigma(a)(\text{iso}\,\sigma(b)): \tag{15}$$

necessarily $0 \in \sigma(a) \cup \sigma(b)$ and there are several possibilities. Note that there is implication

$$a \in A \textit{ polar}, \quad b \in B \textit{ polar} \implies a \otimes b \in A \otimes B \textit{ polar}. \tag{16}$$

If $0 \in (\text{iso}\,\sigma(a \otimes b)) \setminus \sigma(b)$ then $b \in B^{-1}$ and hence, by (12), $0 \in \text{iso}\,\sigma(a)$. Thus 0 is a pole for $a \in A$ and an (honorary!) pole of $b \in B$. If $0 \in (\text{iso}\,\sigma(a)) \cap (\text{acc}\,\sigma(b))$ then necessarily, by (13), $\sigma(a) = \{0\}$ and hence also $\sigma(a \otimes b) = \{0\}$. Since $a \in A$ is polar and quasi-nilpotent it is also nilpotent, and hence also $a \otimes b$. If $0 \in (\text{iso}\,\sigma(a \otimes b)) \cap (\text{iso}\,\sigma(b))$ then we again consider cases: either $0 \notin \sigma(a)$ in which case a is invertible and b is polar, or $0 \in \text{acc}\,\sigma(a)$, in which case b and hence also $a \otimes b$ are nilpotent, or finally $0 \in \text{iso}\,\sigma(a)$, in which case both a and b are polar □

The extension to multiplication operators is almost automatic:

Corollary 4. *If $a \in A$ and $b \in B$ are left isoloid, or polaroid, then so is $T = L_a R_b \in G = B(M)$, for an ultraprime Banach (A, B) bimodule M.*

Proof. The prime condition **[5]**,

$$L_a R_b = 0 \in B(M) \implies 0 \in \{a, b\} \subseteq A \cup B, \tag{17}$$

says that the 'elementary operators' induced by A and B on M are just the tensor product of the algebras $L_A \subseteq B(M)$ and $R_B \subseteq B(M)$, and hence of the algebras A and B^{op}, obtained by reversing the multiplication in B, while the ultraprime condition,

$$\|L_a R_b\| = \|a\|\,\|b\|, \tag{18}$$

ensures that the operator norm of $B(M)$ induces a uniform cross-norm on the tensor product □

The Browder spectrum is given, with a little help from the punctured neighbourhood theorem, by

$$\beta_{ess}(T) = \sigma_{ess}(T) \cup \text{acc}\, \sigma(T) = \omega_{ess}(T) \cup \text{acc}\, \sigma(T): \tag{19}$$

in [**6**] this was written as $\omega_{ess}^{comm}(T)$. It is clear from (6) and the inclusion ([**6**, Theorem 3])

$$\sigma_{ess}(a \otimes b) \subseteq \sigma_{ess}(a)\sigma(b) \cup \sigma(a)\sigma_{ess}(b) \tag{20}$$

that

$$\beta_{ess}(a \otimes b) \subseteq \beta_{ess}(a)\sigma(b) \cup \sigma(a)\beta_{ess}(b) \subseteq \beta_{ess}(a \otimes b) \cup \{0\}. \tag{21}$$

The obstacle to the transfer of Browder's theorem lies in the slightly complicated form ([**6**, equation (6.6)]) of the Weyl spectrum of a tensor product. We begin by simplifying (21):

THEOREM 5. *If $a \in A = B(X)$ and $b \in B = B(Y)$ then*

$$\beta_{ess}(a \otimes b) = \beta_{ess}(a)\sigma(b) \cup \sigma(a)\beta_{ess}(b). \tag{22}$$

Proof. We recall ([**6**, Theorem 4]) the inclusion

$$(a^{-1}(0) \otimes Y) \cup (X \otimes b^{-1}(0)) \subseteq (a \otimes b)^{-1}(0 \otimes 0), \tag{23}$$

which ensures that, if both X and Y are infinite-dimensional, the operator $a \otimes b$ cannot have a non-trivial finite-dimensional null space: hence

$$0 \in \sigma(a \otimes b) \Longrightarrow 0 \in \omega_{ess}(a \otimes b) \subseteq \beta_{ess}(a \otimes b); \tag{24}$$

this with (20) gives (22) □

If we also look at dual operators we can improve (6.3) to

$$0 \in \sigma(a \otimes b) \Longrightarrow 0 \in \sigma_{ess}(a \otimes b). \tag{25}$$

Our observation now is that for any operators $a \in A$ and $b \in B$ for which 'Browder's theorem holds' simultaneously for a, b and $a \otimes b$, the Weyl spectrum of $a \otimes b$ is comparatively simple:

THEOREM 6. *If Browder's theorem holds for $a \in A = B(X)$ and $b \in B = B(Y)$ then the following are equivalent:*

$$\omega_{ess}(a \otimes b) = \omega_{ess}(a)\sigma(b) \cup \sigma(a)\omega_{ess}(b). \tag{26}$$

$$\beta_{ess}(a \otimes b) = \omega_{ess}(a \otimes b). \tag{27}$$

Proof. If (26) holds then (cf. [**10**]) (27) follows from (24) and Browder's theorem for a and b; conversely (27) and (22) give (26) □

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Theorem 6 has been obtained for Hilbert spaces by Kubrusly and Duggal ([**9**, Proposition 7]). Kitson *et al.* [**7**] has a specific example in which the equivalent conditions of Theorem 6 both fail: with the forward and backward shifts u and v on $Y = \ell_2$, for which

$$vu = 1 \neq uv \in 1 + \{c \in B(Y) : \dim c(Y) < \infty\}, \tag{28}$$

take

$$A = B(X), \quad X = Y \oplus Y, \quad a = (1 - uv) \oplus \left(\tfrac{1}{2}u - 1\right), \quad b = -(1 - uv) \oplus \left(\tfrac{1}{2}v + 1\right). \tag{29}$$

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