THE FOURIER COEFFICIENTS OF THE MODULAR FUNCTION $\lambda(\tau)$

WILLIAM H. SIMONS

1. Introduction. In [**3**], H. Rademacher obtained a convergent series for the Fourier coefficients of the modular invariant $J(\tau)$. He found that in the expansion

$$12^3 J(\tau) = e^{-2\pi i\tau} + \sum_{m=0}^{\infty} C_m e^{2\pi i m\tau}$$

the coefficients C_m, for $m \geqslant 1$, are given by

$$C_m = \frac{2\pi}{\sqrt{m}} \sum_{k=1}^{\infty} \frac{A_k(m)}{k} I_1\left(\frac{4\pi\sqrt{m}}{k}\right), \tag{1}$$

where

$$A_k(m) = \sum_{h \bmod k}{}' e^{-\frac{2\pi i}{k}(mh+h')}, \quad hh' \equiv -1 \pmod{k},$$

and $I_1(z)$ is the Bessel function of the first order with purely imaginary argument. The $\sum'$ above indicates the sum with respect to h from 0 to $k-1$ with $(h,k) = 1$. The purpose of this paper is to discuss the Fourier coefficients of $\lambda(\tau)$, the fundamental modular function of level (Stufe) 2. It may be defined either in terms of theta-functions by

$$\lambda(\tau) = \left[\frac{\theta_2(0|\tau)}{\theta_3(0|\tau)}\right]^4 = \left[\frac{\sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}}\right]^4 \tag{2}$$

$$= 16q \prod_{n=1}^{\infty}\left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^8 = 16q[1 - 8q + 44q^2 \ldots], \quad q = e^{\pi i\tau},$$

or by the equivalent definition

$$\lambda(\tau) = \kappa^2(\tau) = \frac{e_2 - e_3}{e_1 - e_3}, \tag{3}$$

where e_1, e_2, e_3 are given in terms of the Weierstrass elliptic function $\wp(z)$ and its periods $2\omega_1$, $2\omega_2$ by

$$e_1 = \wp(\omega_1), \quad e_2 = \wp(\omega_1 + \omega_2), \quad e_3 = \wp(\omega_2).$$

The function $\lambda(\tau)$ is invariant under the substitutions of the congruence subgroup $\Gamma(2)$ of the full modular group defined by all substitutions

Received August 25, 1950. This paper was prepared for publication while the author was a member of the Summer Research Institute of the Canadian Mathematical Congress.

$$\tau' = \frac{a\tau + b}{c\tau + d},$$

where a,b,c,d are integers with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2 \quad \text{and} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1.$$

For the expansion

$$\lambda(\tau) = \sum_{m=0}^{\infty} a_m q^m, \qquad q = e^{\pi i \tau},$$

it is found that

$$a_m = \frac{\pi}{8\sqrt{m}} \sum_{\substack{k=1 \\ k \equiv 2 \pmod 4}}^{\infty} \frac{A_k(m)}{k} I_1\left(\frac{4\pi\sqrt{m}}{k}\right). \tag{4}$$

Moreover, it is found that the coefficients in the expansion of the reciprocal function

$$\mu(\tau) = \frac{1}{\lambda(\tau)} = \frac{1}{16q} + b_0 + \sum_{m=1}^{\infty} b_m e^{\pi i \tau m}$$

are given by the series

$$b_m = \frac{\pi}{8m^{\frac{1}{2}}} \sum_{\substack{k=1 \\ k \equiv 0 \pmod 4}}^{\infty} \frac{A_k(m)}{k} I_1\left(\frac{4\pi\sqrt{m}}{k}\right) \qquad (m \geqslant 1). \tag{5}$$

The method is essentially the same as that used by Rademacher. In §2 the transformation equations for $\lambda(\tau)$ are derived. The main result (4) is obtained in §§3 to 7, and equation (5) is derived in §8.

The following interesting comment was made by the referee of this paper. "The function $j(\tau)$ is determined essentially by its pole at $\tau = \infty$; it is regular everywhere else. But $1/j(\tau)$ has a pole at an interior point of the upper half-plane, and so *its* Fourier coefficients cannot be determined in as simple a manner. This situation is unavoidable with functions of the full modular group, which has but one parabolic cusp. On the other hand, the subgroup which Dr. Simons treats has 3 parabolic cusps, so it is possible to define functions which together with their reciprocals are regular in the upper half-plane by merely placing the zero and the pole at the cusps of the fundamental region. $\lambda(\tau)$ is such a function. It is of interest to note that both for $\lambda(\tau)$ and $1/\lambda(\tau)$, the Fourier coefficients are given by series which, apart from a trivial numerical factor, are composed of terms taken from the series for $j(\tau)$."

2. The transformation equations.

LEMMA 1. *Let a, b, c, d be integers with $ad - bc = 1$, and let*

$$T = \frac{a\tau + b}{c\tau + d}.$$

https://doi.org/10.4153/CJM-1952-007-6 Published online by Cambridge University Press

Then $\lambda(T)$ *and* $\lambda(\tau)$ *are related as follows:*

	1°	2°	3°	4°	5°	6°
$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ (mod 2)	$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$
$\lambda(T)$	$\lambda(\tau)$	$\dfrac{\lambda(\tau)}{\lambda(\tau)-1}$	$\dfrac{1}{\lambda(\tau)}$	$\dfrac{1}{1-\lambda(\tau)}$	$1-\lambda(\tau)$	$1-\dfrac{1}{\lambda(\tau)}$

The lemma is an immediate consequence of the transformation equations for the theta-functions and definition (2), or of the transformation equations for e_1, e_2, e_3 and definition (3) [cf. **5**].

LEMMA 2.

$$\lambda(2\tau) = \left[\frac{\{1-\lambda(\tau)\}^{\frac{1}{2}}-1}{\{1-\lambda(\tau)\}^{\frac{1}{2}}+1}\right]^2.$$

By definition,

$$\lambda(2\tau) = \frac{\theta_2^4(0|2\tau)}{\theta_3^4(0|2\tau)}.$$

But [**5**, p. 268],

$$2\theta_2^2(0|2\tau) = \theta_3^2(0|\tau) - \theta_4^2(0|\tau),$$

and

$$2\theta_3^2(0|2\tau) = \theta_3^2(0|\tau) + \theta_4^2(0|\tau),$$

where

$$\theta_4(0|\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = 1 - 2q + 2q^4 - \dots.$$

Therefore

$$\lambda(2\tau) = \frac{\theta_3^4 - 2\theta_3^2\theta_4^2 + \theta_4^4}{\theta_3^4 + 2\theta_3^2\theta_4^2 + \theta_4^4},$$

$$\frac{\lambda(2\tau)+1}{1-\lambda(2\tau)} = \frac{\theta_3^4+\theta_4^4}{2\theta_3^2\theta_4^2},$$

$$4\left[\frac{\lambda(2\tau)+1}{1-\lambda(2\tau)}\right]^2 = \frac{\theta_3^4}{\theta_4^4} + \frac{\theta_4^4}{\theta_3^4} + 2.$$

Now

$$\frac{\theta_4^4}{\theta_3^4} = \frac{\theta_3^4-\theta_2^4}{\theta_3^4} = 1 - \frac{\theta_2^4}{\theta_3^4} = 1 - \lambda(\tau),$$

and therefore

$$4\left[\frac{\lambda(2\tau)+1}{1-\lambda(2\tau)}\right]^2 = 1 - \lambda(\tau) + \frac{1}{1-\lambda(\tau)} + 2 = \frac{[2-\lambda(\tau)]^2}{1-\lambda(\tau)},$$

so that

$$\frac{\lambda(2\tau)+1}{1-\lambda(2\tau)} = \frac{2-\lambda(\tau)}{2\{1-\lambda(\tau)\}^{\frac{1}{2}}}.$$

Solving for $\lambda(2\tau)$ gives

$$\lambda(2\tau) = \frac{\dfrac{2-\lambda(\tau)}{2\{1-\lambda(\tau)\}^{\frac{1}{2}}} - 1}{\dfrac{2-\lambda(\tau)}{2\{1-\lambda(\tau)\}^{\frac{1}{2}}} + 1}$$

$$= \frac{2-\lambda(\tau) - 2\{1-\lambda(\tau)\}^{\frac{1}{2}}}{2-\lambda(\tau) + 2\{1-\lambda(\tau)\}^{\frac{1}{2}}}$$

$$= \left[\frac{\{1-\lambda(\tau)\}^{\frac{1}{2}} - 1}{\{1-\lambda(\tau)\}^{\frac{1}{2}} + 1}\right]^2.$$

THEOREM 2. *Let k be an even integer and h and h' be integers such that $(h, k) = 1$, and $hh' \equiv -1 \pmod{k}$. Further, let*

$$\tau = 2\left(\frac{h}{k} + \frac{iz}{k}\right) \text{ and } T = 2\left(\frac{h'}{k} + \frac{i}{kz}\right).$$

Then

$$\lambda(T) = \begin{cases} \lambda(\tau) & \text{if } k \equiv 0 \pmod 4, \\ 1/\lambda(\tau) & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

Proof. Define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} h' & 2(-1-hh')/k \\ k/2 & -h \end{pmatrix}.$$

Then a, b, c, d are integers with $ad - bc = 1$, and

$$T = \frac{a\tau + b}{c\tau + d}.$$

If $k \equiv 0 \pmod 4$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2$$

and so by Lemma 1, case 1°, $\lambda(T) = \lambda(\tau)$. If $k \equiv 2 \pmod 4$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \pmod 2$$

and so by Lemma 1, case 3°, $\lambda(T) = 1/\lambda(\tau)$.

THEOREM 2. *Let k be an odd integer and let h and h' be integers such that $(h,k) = 1$ and $hh \equiv -1 \pmod{k}$.*
Further, let

$$\tau = \left(\frac{h}{k} + \frac{iz}{k}\right), \quad T = \left(\frac{h'}{k} + \frac{i}{kz}\right).$$

Then

$$\lambda(2\tau) = \begin{cases} [\{\lambda(T) - 1\}^{\frac{1}{2}} - \lambda(T)]^4, & \text{if } h \equiv 1 \pmod 2, \\ \left[\dfrac{\{\lambda(T)\}^{\frac{1}{2}} - 1}{\{\lambda(T)\}^{\frac{1}{2}} + 1}\right]^2, & \text{if } h \equiv 0 \pmod 2. \end{cases}$$

https://doi.org/10.4153/CJM-1952-007-6 Published online by Cambridge University Press

Proof. Define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} h' & (-1 - hh')/k \\ k & -h \end{pmatrix}.$$

Then a, b, c, d are integers with $ad - bc = 1$ and

$$T = \frac{a\tau + b}{c\tau + d}.$$

(a). Let $h' \equiv 1 \pmod 2$ and $h \equiv 1 \pmod 2$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \pmod 2,$$

and so by Lemma 1, case 3°, $\lambda(T) = 1/\lambda(\tau)$. Substituting for $\lambda(\tau)$ in Lemma 2 gives

$$\lambda(2\tau) = \left[\frac{\{1 - 1/\lambda(T)\}^{\frac{1}{2}} - 1}{\{1 - 1/\lambda(T)\}^{\frac{1}{2}} + 1}\right]^2 = \left[\frac{\{\lambda(T) - 1\}^{\frac{1}{2}} - \{\lambda(T)\}^{\frac{1}{2}}}{\{\lambda(T) - 1\}^{\frac{1}{2}} + \{\lambda(T)\}^{\frac{1}{2}}}\right]^2.$$

(b). Let $h' \equiv 1 \pmod 2$ and $h \equiv 0 \pmod 2$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \pmod 2,$$

and so by Lemma 1, case 6°, $\lambda(\tau) = 1/(1 - \lambda(T))$. Substituting for $\lambda(\tau)$ in Lemma 2 gives

$$\lambda(2\tau) = \left[\frac{\left\{\frac{\lambda(T)}{\lambda(T) - 1}\right\}^{\frac{1}{2}} - 1}{\left\{\frac{\lambda(T)}{\lambda(T) - 1}\right\}^{\frac{1}{2}} + 1}\right]^2 = \left[\frac{\{\lambda(T)\}^{\frac{1}{2}} - \{\lambda(T) - 1\}^{\frac{1}{2}}}{\{\lambda(T)\}^{\frac{1}{2}} + \{\lambda(T) - 1\}^{\frac{1}{2}}}\right]^2.$$

(c). Let $h' \equiv 0 \pmod 2$ and $h \equiv 1 \pmod 2$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \pmod 2$$

and so by Lemma 1, case 4°, $\lambda(\tau) = 1 - 1/\lambda(T)$. Substituting in Lemma 2 gives

$$\lambda(2\tau) = \left[\frac{\{\lambda(T)\}^{-\frac{1}{2}} - 1}{\{\lambda(T)\}^{-\frac{1}{2}} + 1}\right]^2 = \left[\frac{\{\lambda(T)\}^{\frac{1}{2}} - 1}{\{\lambda(T)\}^{\frac{1}{2}} + 1}\right]^2.$$

(d). Let $h' \equiv 0 \pmod 2$ and $h \equiv 0 \pmod 2$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod 2$$

and so by Lemma 1, case 5°, $\lambda(\tau) = 1 - \lambda(T)$. Substituting in Lemma 2 then gives

$$\lambda(2\tau) = \left[\frac{\{\lambda(T)\}^{\frac{1}{2}} - 1}{\{\lambda(T)\}^{\frac{1}{2}} + 1}\right]^2.$$

By combining the results of (a) with (b) and those of (c) with (d) the result of the theorem is obtained.

3. The Farey dissection. Let

$$\lambda(\tau) = f(q) = \sum_{m=1}^{\infty} a_m q^m, \qquad q = e^{\pi i \tau}.$$

Then by Cauchy's theorem,

$$a_m = \frac{1}{2\pi i}\int_C \frac{f(q)}{q^{m+1}}\, dq,$$

where the integration is in the positive sense around the circle C defined by

$$|q| = e^{-2\pi N^{-2}},$$

N being a positive integer. Using the Farey dissection of order N of the circle C, the integral may be expressed by the sum

$$a_m = \frac{1}{2\pi i}\sum_{\substack{0\leqslant h<k\leqslant N\\(h,k)=1}} \int_{\xi_{h,k}} \frac{f(q)}{q^{m+1}}\, dq,$$

where $\xi_{h,k}$ is the Farey arc corresponding to the fraction h/k in the Farey series of order N, and

$$q = \exp\left(-2\pi N^{-2} + 2\pi i\frac{h}{k} + 2\pi i\phi\right).$$

Then

$$a_m = \sum_{\substack{0\leqslant h<k\leqslant N\\(h,k)=1}} \int_{-\phi'}^{\phi''} \frac{f\left(\exp\left\{-2\pi N^{-2} + 2\pi i\frac{h}{k} + 2\pi i\phi\right\}\right)}{\exp\left(-2\pi m N^{-2} + 2\pi i m\frac{h}{k} + 2\pi i m\phi\right)}\, d\phi,$$

where

$$(6)\qquad \begin{aligned} \phi' &= \frac{h}{k} - \frac{h+h_1}{k+k_1} = \frac{1}{k(k+k_1)},\\ \phi'' &= \frac{h+h_2}{k+k_2} - \frac{h}{k} = \frac{1}{k(k+k_2)}, \end{aligned}$$

h_1/k_1, h/k, h_2/k_2 being three consecutive terms of the Farey series of order N. For convenience the double sum over $0 \leqslant h < k \leqslant N$ with $(h,k) = 1$ will be denoted by

$$\sum_{h,k}^{N}.$$

Then

$$a_m = \exp(2\pi m N^{-2})\sum_{h,k}^{N} \exp\left(-2\pi i m\frac{h}{k}\right) \cdot \int_{-\phi'}^{\phi''} f\left(\exp\left\{-2\pi N^{-2} + 2\pi i\frac{h}{k} + 2\pi i\phi\right\}\right)\exp(-2\pi i m\phi)\, d\phi$$

$$= \exp(2\pi m N^{-2}) \sum_{h,k}^{N} \exp\left(-2\pi i m \frac{h}{k}\right)$$

$$\cdot \int_{-\phi'}^{\phi''} f\left(\exp\left\{2\pi i\left(\frac{h}{k} + \frac{iz}{k}\right)\right\}\right) \exp(-2\pi i m\phi) d\phi,$$

where $z = k(N^{-2} - i\phi)$.

Now let the above summation be broken up into three sums Σ_1, Σ_2, Σ_3, the first consisting of those terms for which $k \equiv 1 \pmod 2$, the second those for which $k \equiv 2 \pmod 4$, and the third those for which $k \equiv 0 \pmod 4$, and let I_1, I_2, and I_3, be the parts of a_m corresponding to Σ_1, Σ_2, Σ_3 respectively. Thus

$$a_m = I_1 + I_2 + I_3.$$

4. Evaluation of the integral I_3.

$$I_3 = \exp(2\pi m N^{-2}) \sum_{\substack{k=1 \\ k\equiv 0 \pmod 4}}^{N} \sum_{\substack{h=0 \\ (h,k)=1}}^{k-1} \exp\left(-2\pi i m \frac{h}{k}\right)$$

$$\cdot \int_{-\phi'}^{\phi''} f\left(\exp\left\{2\pi i\left(\frac{h}{k} + \frac{iz}{k}\right)\right\}\right) \exp(-2\pi i m\phi) d\phi.$$

Applying the transformation equation of Theorem 1, for $k \equiv 0 \pmod 4$ gives

$$I_3 = \exp(-2\pi m N^{-2}) \sum_{\substack{k=1 \\ k\equiv 0 \pmod 4}}^{N} \sum_{\substack{h=0 \\ (h,k)=1}}^{k-1} \exp\left(-2\pi i m \frac{h}{k}\right)$$

$$\cdot \int_{-\phi'}^{\phi''} f\left(\exp\left\{2\pi i\left(\frac{h'}{k} + \frac{i}{kz}\right)\right\}\right) \exp(-2\pi i m\phi) d\phi.$$

But

$$f(q) = \lambda(\tau) = \sum_{n=1}^{\infty} a_n q^n, \qquad q = \exp \pi i\tau,$$

and so, substituting for $f(q)$, rearranging terms, and putting $\omega = N^{-2} - i\phi$, $z = k\omega$, gives

$$I_3 = \sum_{\substack{k=1 \\ k\equiv 0 \pmod 4}}^{N} \sum_{\substack{h=0 \\ (h,k)=1}}^{k-1} \int_{-\phi'}^{\phi''} \sum_{n=1}^{\infty} a_n \exp\left\{\frac{2\pi i}{k}(nh' - mh)\right\} \exp\left(2\pi m\omega - \frac{2\pi n}{k^2\omega}\right) d\phi.$$

Use is now made of a result due to Estermann [2]. Let ϕ' and ϕ'' be defined by (6), and let

$$g(N, \phi, h, k) = \begin{cases} 1, & \text{for } -\phi' \leqslant \phi \leqslant \phi'', \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$g = \sum_{r=1}^{k} b_r \exp\{2\pi i r h'/k\},$$

where h' is an integer satisfying $hh' \equiv -1 \pmod k$, and b_r is independent of h and

$$\sum_{r=1}^{k} |b_r| < \log 4k.$$

Introducing the function $g(N,\phi,h,k)$ into the integral I_3 gives

$$I_3 = \sum_{\substack{k=1 \\ k\equiv 0 \pmod 4}}^{N} \sum_{n=1}^{\infty} a_n \int_{-1/k(N+1)}^{1/k(N+1)} \sum_{r=1}^{k} b_r \exp\left(2\pi i r \frac{h'}{k}\right)$$

$$\cdot \exp\left(2\pi m\omega - \frac{2\pi n}{k^2\omega}\right) \sum_{h \bmod k}{}' \exp\left\{\frac{2\pi i}{k}(nh' - mh)\right\} d\phi.$$

The latter sum is a Kloosterman sum [4;1] and has the estimate $O(k^{2/3+\epsilon}m^{1/3})$. Also, the real part of $2\pi n/k^2\omega$ is

$$\Re\left(\frac{2\pi n}{k^2(N^{-2} - i\phi)}\right) = \frac{2\pi n N^{-2}}{k^2(N^{-4} + \phi^2)} \geqslant \frac{2\pi n}{k^2 N^{-2} + k^2 N^2 \phi''^2}$$

$$\geqslant \frac{2\pi n}{1 + 1} = \pi n,$$

and

$$\Re(2\pi m\omega) = 2\pi m N^{-2}.$$

Therefore

$$|I_3| = O\left(\sum_{\substack{k=1 \\ k\equiv 0 \pmod 4}}^{N} \sum_{n=1}^{\infty} a_n e^{-\pi n} \int_{-1/k(N+1)}^{1/k(N+1)} \sum_{r=1}^{k} |b_r| \exp\,(2\pi m N^{-2}) k^{2/3+\epsilon} m^{1/3} d\phi\right)$$

$$= O\left(\sum_{\substack{k=1 \\ k\equiv 0 \pmod 4}}^{N} m^{1/3} k^{2/3+\epsilon} \log 4k \int_{-1/k(N+1)}^{1/k(N+1)} d\phi\right)$$

$$= O\left(\sum_{\substack{k=1 \\ k\equiv 0 \pmod 4}}^{N} m^{1/3} k^{2/3+\epsilon} \frac{1}{kN}\right)$$

and so

$$|I_3| = O\left(N^{-1} \sum_{k=1}^{N} k^{-1/3+\epsilon} m^{1/3}\right)$$

$$= O\left(N^{-1/3+\epsilon} m^{1/3}\right).$$

5. Evaluation of the integral I_2.

$$I_2 = \exp\,(2\pi m N^{-2}) \sum_{\substack{k=1 \\ k\equiv 2 \pmod 4}}^{N} \sum_{\substack{h=0 \\ (h,k)=1}}^{k-1} \exp\left(-2\pi i m \frac{h}{k}\right)$$

$$\cdot \int_{-\phi'}^{\phi''} f\left(\exp\left\{2\pi i\left(\frac{h}{k} + \frac{iz}{k}\right)\right\}\right) \exp\,(-2\pi i m\phi) d\phi.$$

Now, by Theorem 1, with $k \equiv 2 \pmod 4$, and putting $q = e^{\pi i\tau}$ and $q' = e^{\pi i T}$,

https://doi.org/10.4153/CJM-1952-007-6 Published online by Cambridge University Press

$$f(q) = \lambda(\tau) = \frac{1}{\lambda(\mathrm{T})} = f_1(q') = \left[\frac{\theta_3(0|\mathrm{T})}{\theta_2(0|\mathrm{T})}\right]^4$$

$$= \frac{1}{16q'} + \sum_{n=0}^{\infty} b_n q'^n$$

$$= \frac{1}{16q'} + f_2(q').$$

Therefore

$$I_2 = \exp\,(2\pi m N^{-2}) \sum_{\substack{k=1\\k\equiv 2(\mathrm{mod}\ 4)}}^{N} \sum_{\substack{h=0\\(h,k)=1}}^{k-1} \exp\left(-2\pi i m\frac{h}{k}\right)$$

$$\cdot \int_{-\phi'}^{\phi''} f_1\left(\exp\left\{2\pi i\left(\frac{h'}{k} + \frac{i}{kz}\right)\right\}\right) \exp\,(-2\pi i m\phi)d\phi$$

$$= I_{2,1} + I_{2,2},$$

where $f_1(q')$ is replaced by $1/16q'$ in $I_{2,1}$ and by $f_2(q')$ in $I_{2,2}$. Introducing the function $g(N,\phi,h,k)$ into the integral $I_{2,2}$ and proceeding as in §4 gives

$$|I_{2,2}| = O\left(\sum_{\substack{k=1\\k\equiv 2(\mathrm{mod}\ 4)}}^{N} \sum_{n=0}^{\infty} b_n e^{-\pi n} \int_{-1/k(N+1)}^{1/k(N+1)} \sum_{r=1}^{k} |b_r| \exp\,(2\pi m N^{-2}) k^{2/3+\epsilon} m^{1/3} d\phi\right)$$

$$= O\left(\frac{1}{N}\sum_{k=1}^{N} k^{2/3+\epsilon} m^{1/3}\right)$$

$$= O(N^{-1/3+\epsilon} m^{1/3}).$$

Next,

$$I_{2,1} = \exp\,(2\pi m N^{-2}) \sum_{\substack{k=1\\k\equiv 2\ (\mathrm{mod}\ 4)}}^{N} \sum_{\substack{h=0\\(h,k)=1}}^{k-1} \exp\left(-2\pi i m\frac{h}{k}\right)$$

$$\cdot \int_{-\phi'}^{\phi''} \frac{1}{16} \exp\left(-2\pi i\left\{\frac{h'}{k} + \frac{i}{kz}\right\}\right) \exp\,(-2\pi i m\phi)d\phi$$

$$= \frac{1}{16} \sum_{\substack{k=1\\k\equiv 2\ (\mathrm{mod}\ 4)}}^{N} \sum_{\substack{h=0\\(h,k)=1}}^{k-1} \exp\left(\frac{-2\pi i}{k}\left\{mh + h'\right\}\right)$$

$$\cdot \int_{-\phi'}^{\phi''} \exp\left(2\pi m\omega + \frac{2\pi}{k^2\omega}\right) d\phi$$

$$= \frac{i}{16} \sum_{\substack{k=1\\k\equiv 2\ (\mathrm{mod}\ 4)}}^{N} \sum_{\substack{h=0\\(h,k)=1}}^{k-1} \exp\left(-\frac{2\pi i}{k}\left\{mh + h'\right\}\right)$$

$$\cdot \int_{N^{-2}-i\phi''}^{N^{-2}+i\phi'} \exp\left(2\pi m\omega + \frac{2\pi}{k^2\omega}\right) d\omega.$$

Now,

https://doi.org/10.4153/CJM-1952-007-6 Published online by Cambridge University Press

$$\phi' = \frac{1}{k(k_1 + k)} \leqslant \frac{1}{k(N+1)}$$

and

$$\phi'' = \frac{1}{k(k_2 + k)} \leqslant \frac{1}{k(N+1)},$$

and so

$$I_{2,1} = -\frac{1}{16} \sum_{\substack{k=1 \\ k\equiv 2 \pmod 4}}^{N} \sum_{\substack{h=0 \\ (h,k)=1}}^{k-1} \exp\left(-\frac{2\pi i}{k}\left\{mh + h'\right\}\right)$$

$$\cdot \left[\int^{(0+)} \exp\left(2\pi m\omega + \frac{2\pi}{k^2\omega}\right)d\omega - \left\{\int_{N^{-2}+i\phi'}^{N^{-2}+i/k(N+1)}\right.\right.$$

$$+ \int_{N^{-2}+i/k(N+1)}^{-N^{-2}+i/k(N+1)} + \int_{-N^{-2}+i/k(N+1)}^{-N^{-2}-i/k(N+1)} + \int_{-N^{-2}-i/k(N+1)}^{N^{-2}-i/k(N+1)}$$

$$\left.\left. + \int_{N^{-2}-i/k(N+1)}^{N^{-2}-i\phi''}\right\} \exp\left(2\pi m\omega + \frac{2\pi}{k^2\omega}\right)d\omega\right]$$

(7) $$= \frac{\pi}{8} \sum_{\substack{k=1 \\ k\equiv 2 \pmod 4}}^{N} A_k(m)\frac{1}{2\pi i}\int^{(0+)} \exp\left(2\pi m\omega + \frac{2\pi}{k^2\omega}\right)d\omega$$

$$+ K_1 + K_2 + K_3 + K_4 + K_5,$$

where

$$A_k(m) = \sum_{h \bmod k} \exp\left(-\frac{2\pi i}{k}\{mh + h'\}\right).$$

Now

$$K_1 = \frac{i}{16} \sum_{\substack{k=1 \\ k\equiv 2 \pmod 4}}^{N} A_k(m)\int_{N^{-2}+i\phi}^{N^{-2}+i/k(N+1)} \exp\left(2\pi m\omega + \frac{2\pi}{k^2\omega}\right)d\omega.$$

Introducing the function $g(N,\phi,h,k)$, and integrating from $N^{-2}+ i/k(N + k)$ to $N^{-2}+ i/k(N + 1)$ gives

$$|K_1| = O\left(\sum_{k=1}^{N} k^{2/3+\epsilon}m^{1/3} \log 4k \frac{1}{kN}\right) = O(N^{-1/3+\epsilon}m^{1/3}).$$

Similarly

$$|K_5| = O(N^{-1/3+\epsilon}m^{1/3}).$$

In K_2,

$$\omega = u + i/k(N+1), \quad -N^{-2} \leqslant u \leqslant N^{-2},$$

$$\Re\left(\frac{1}{\omega}\right) = \frac{u}{u^2 + 1/k^2(N+1)^2} < N^{-2}k^2(N+1)^2 < k^2,$$

so that

$$\left|\exp\left(2\pi m\omega + \frac{2\pi}{k^2\omega}\right)\right| \leqslant \exp\left(2\pi m N^{-2} + 2\pi\right).$$

Therefore

$$|K_2| = O\left(\sum_{k=1}^{N} k^{2/3+\epsilon} m^{1/3} N^{-2}\right) = O(N^{-1/3+\epsilon} m^{1/3}).$$

Similarly

$$|K_4| = O(N^{-1/3+\epsilon} m^{1/3}).$$

Again, in K_3,

$$\omega = -N^{-2} + iv, \quad -\frac{1}{k(N+1)} \leqslant v \leqslant \frac{1}{k(N+1)}.$$

Also

$$\Re(\omega) = -N^{-2} < 0$$

$$\Re\left(\frac{1}{\omega}\right) = \Re\left(\frac{1}{-N^{-2} + iv}\right) = \frac{-N^{-2}}{N^{-4} + v^2} < 0,$$

and hence

$$\left| \exp\left(2\pi m\omega + \frac{2\pi}{k^2\omega}\right)\right| < 1.$$

Therefore

$$|K_3| = O\left(\sum_{k=1}^{N} k^{2/3+\epsilon} m^{1/3} \frac{1}{kN}\right) = O(N^{-1/3+\epsilon} m^{1/3}).$$

Collecting these results together and substituting back into (7) gives

$$I_{2,1} = \frac{\pi}{8} \sum_{\substack{k=1 \\ k\equiv 2 \pmod 4}}^{N} A_k(m) L_k(m) + O(N^{-1/3+\epsilon} m^{1/3}),$$

where [6; 3]

$$L_k(m) = \frac{1}{2\pi i}\int^{(0+)} \exp\left(2\pi m\omega + \frac{2\pi}{k^2\omega}\right) d\omega = \frac{1}{k\sqrt{m}} I_1\left(\frac{4\pi\sqrt{m}}{k}\right),$$

$I_1(z)$ being the Bessel function of the first order with purely imaginary argument. Therefore

$$I_2 = \frac{\pi}{8\sqrt{m}} \sum_{\substack{k=1 \\ k\equiv 2 \pmod 4}}^{N} \frac{A_k(m)}{k} I_1\left(\frac{4\pi\sqrt{m}}{k}\right) + O(N^{-1/3+\epsilon} m^{1/3}).$$

6. Evaluation of the integral I_1. In I_1, consider

$$\tau = \frac{h}{k} + \frac{iz}{k}, \qquad T = \frac{h'}{k} + \frac{i}{kz}$$

so that

$$f\left(\exp\left\{2\pi i\left(\frac{h}{k} + \frac{iz}{k}\right)\right\}\right) = f(\exp\{2\pi i\tau\}) = \lambda(2\tau).$$

Then, by Theorem 2, with $t = \exp \pi i T$,

$$\lambda(2\tau) = 1 + 16it^{\frac{1}{2}} - 128t + \ldots$$

when $h' \equiv 1 \pmod 2$, and

$$\lambda(2\tau) = 1 - 16t^{\frac{1}{2}} + 128t - \ldots$$

https://doi.org/10.4153/CJM-1952-007-6 Published online by Cambridge University Press

when $h' \equiv 0 \pmod 2$. These may be combined by replacing t by $t' = \exp \pi i(T + h') = t \exp(\pi i h')$, giving

$$\lambda(2\tau) = 1 + 16t'^{\frac{1}{2}} + 128t' + \dots$$
$$= \sum_{n=0}^{\infty} u_n t'^{\frac{1}{2}n}.$$

Applying the transformations of Theorem 2 to the integrand of I_1 gives

$$I_1 = \exp(2\pi m N^{-2}) \sum_{\substack{k=1 \\ k\equiv 1 \pmod 2}}^{N} \sum_{\substack{h=0 \\ (h,k)=1}}^{k-1} \exp\left(-2\pi i m \frac{h}{k}\right)$$
$$\cdot \int_{-\phi'}^{\phi''} \sum_{n=0}^{\infty} u_n \exp\left\{\frac{\pi i n}{2}\left(\frac{h'}{k} + \frac{i}{kz}\right)\right\} \exp\left(\frac{\pi i n h'}{2}\right) \exp(-2\pi i m\phi)\, d\phi$$
$$= \sum_{\substack{k=1 \\ k\equiv 1 \pmod 2}}^{N} \sum_{n=0}^{\infty} (-1)^{nh'/2} \int_{-1/k(N+1)}^{1/k(N+1)} \sum_{r=1}^{k} b_r \exp\left(2\pi i r \frac{h'}{k}\right)$$
$$\cdot \exp\left(2\pi m\omega - \frac{\pi n}{2k^2\omega}\right) \sum_{\substack{h=0 \\ (h,k)=1}}^{k-1} \exp\left\{\frac{\pi i}{2k}(nh' - 4mh)\right\} d\phi.$$

Now the latter sum in the integrand is an incomplete Kloosterman sum for which we have [**2**; **4**] the estimate

$$O(k^{2/3+\epsilon}(4m, k)^{1/3}) = O(k^{2/3+\epsilon} m^{1/3}).$$

Also

$$\Re\left(\frac{\pi n}{2k^2\omega}\right) = \frac{\pi n N^{-2}}{2(k^2N^{-2} + k^2N^2\phi''^2)} \geqslant \frac{\pi n}{4}.$$

Therefore

$$|I_1| = O\left(\sum_{k=1}^{N} k^{2/3+\epsilon} m^{1/3} \sum_{h=0}^{\infty} |u_n| e^{-\pi n/4} \exp 2\pi m N^{-2} \int_{-1/k(N+1)}^{1/k(N+1)} d\phi\right)$$
$$= O\left(\frac{1}{N} \sum_{k=1}^{N} k^{-1/3+\epsilon} m^{1/3}\right)$$
$$= O(N^{-1/3+\epsilon} m^{1/3}).$$

7. The convergent series for a_m. Collecting together the results of §§4, 5, and 6, we have

$$a_m = I_1 + I_2 + I_3$$
$$= \frac{\pi}{8\sqrt{m}} \sum_{\substack{k=1 \\ k\equiv 2 \pmod 4}}^{N} \frac{A_k(m)}{k} I_1\left(\frac{4\pi\sqrt{m}}{k}\right) + O(N^{-1/3+\epsilon} m^{1/3}).$$

Finally, letting $N \to \infty$, we get

$$(4) \qquad a_m = \frac{\pi}{8\sqrt{m}} \sum_{\substack{k=1 \\ k\equiv 2 \pmod 4}}^{\infty} \frac{A_k(m)}{k} I_1\left(\frac{4\pi\sqrt{m}}{k}\right).$$

As a numerical example we may compare the actual value of a_{16} with the value obtained from the series (4). Thus $a_{16} = -316342272$. Using the series for a_{16}, we have

$$a_{16} = \frac{\pi}{32} \sum_{\substack{k=1 \\ k\equiv 2 \pmod 4}}^{\infty} \frac{A_k(16)}{k} I_1\left(\frac{16\pi}{k}\right),$$

$$\frac{\pi}{64} A_2\,(16) I_1\left(\frac{16\pi}{2}\right) = -316342253.1678$$

$$\frac{\pi}{192} A_6\,(16) I_1\left(\frac{16\pi}{6}\right) = -\qquad 18.6991$$

$$\frac{\pi}{320} A_{10}(16) I_1\left(\frac{16\pi}{10}\right) = -\qquad 0.0935.$$

8. The reciprocal function $\mu(\tau)$.

THEOREM 3. *Let*

$$\mu(\tau) = g(q) = \frac{1}{\lambda(\tau)} = \left[\frac{\theta_3(0|\tau)}{\theta_2(0|\tau)}\right]^4$$

$$= \frac{1}{16q}(1 + 8q + 20q^2 + \ldots)$$

$$= \frac{1}{16q} + \sum_{m=0}^{\infty} b_m q^m.$$

Then, for $m > 0$,

(5) $$b_m = \frac{\pi}{8\sqrt{m}} \sum_{\substack{k=1 \\ k\equiv 0 \pmod 4}}^{\infty} \frac{A_k(m)}{k} I_1\left(\frac{4\pi\sqrt{m}}{k}\right).$$

Proof. Since the analysis in this case is essentially the same as for $\lambda(\tau)$, we will only outline the proof. The transformation equations for $\mu(\tau)$ may be obtained directly from those for $\lambda(\tau)$. Now, by Cauchy's theorem, for $m > 0$,

$$b_m = \frac{1}{2\pi i}\int_C \frac{g(q)}{q^{m+1}} dq,$$

where, as before, C is the circle of radius $|q| = \exp(-2\pi N^{-2})$. Therefore

$$b_m = \exp\,(2\pi m N^{-2}) \sum_{k=1}^{N} \sum_{\substack{h=0 \\ (h,k)=1}}^{k-1} \exp\left(-2\pi i m\frac{h}{k}\right)$$

$$\cdot \int_{-\phi'}^{\phi''} g\left(\exp\left\{2\pi i\left(\frac{h}{k} + \frac{iz}{k}\right)\right\}\right) \exp\,(-2\pi i m\phi) d\phi.$$

Let $b_m = b_{m,1} + b_{m,2} + b_{m,3}$, where $b_{m,1}$ consists of the terms of b_m for which $k \equiv 1 \pmod 2$, $b_{m,2}$ those for which $k \equiv 2 \pmod 4$, and $b_{m,3}$ those for which $k \equiv 0 \pmod 4$. Then it may be shown that

https://doi.org/10.4153/CJM-1952-007-6 Published online by Cambridge University Press

$$b_{m,1} = O(N^{-1/3+\epsilon} m^{1/3}), \quad b_{m,2} = O(N^{-1/3+\epsilon} m^{1/3})$$

and

$$b_{m,3} = \frac{\pi}{8\sqrt{m}} \sum_{\substack{k=1 \\ k \equiv 0 \pmod 4}}^{N} \frac{A_k(m)}{k} I_1\left(\frac{4\pi\sqrt{m}}{k}\right) + O(N^{-1/3+\epsilon} m^{1/3}).$$

Then letting $N \to \infty$ we get equation (5).

Similar results may be obtained for the Fourier coefficients of powers of $\lambda(\tau)$ and $\mu(\tau)$. However, these are omitted here since the method used in obtaining them is merely a repetition of that given for $\lambda(\tau)$.

References

1. H. Davenport, *On certain exponential sums*, J. Reine Angew. Math., Bd. 169 (1933), 158-176.
2. T. Estermann, *Vereinfachter Beweis eines Satz von Kloosterman*, Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität, Bd. 7 (1939), 82-98, especially 94.
3. H. Rademacher, *The Fourier coefficients of the modular invariant* $J(\tau)$, Amer. J. Math., vol. 60 (1938), 501-512.
4. H. Salié, *Zur Abschatzung der Fourierkoeffizienten ganzer Modulformen*, Math. Z., Bd. 36 (1933), 263-278.
5. J. Tannery and J. Molk, *Théorie des fonctions elliptiques*, Tome II (Paris, 1896), 290.
6. G. N. Watson, *Theory of Bessel functions* (Cambridge, 1922), 181.

The University of British Columbia

https://doi.org/10.4153/CJM-1952-007-6 Published online by Cambridge University Press