2
Lattices

The best evidence we have for confinement in a non-Abelian gauge theory of the strong interactions comes by way of Wilson's (1974) formulation on a space-time lattice. At first this prescription seems a little peculiar because the vacuum is not a crystal. Indeed, experimentalists work daily with relativistic particles showing no deviations from the continuous symmetries of the Lorentz group. Why, then, have theorists in recent years spent so much time describing field theory on the scaffolding of a space-time lattice?

The lattice represents a mathematical trick. It provides a cutoff removing the ultraviolet infinities so rampant in quantum field theory. As with any regulator, it must be removed after renormalization. Physics can only be extracted in the continuum limit, where the lattice spacing is taken to zero.

But infinities and the resulting need for renormalization have been with us since the beginnings of relativistic quantum mechanics. The program for electrodynamics has had immense success without recourse to discrete space. Why reject the time-honored perturbative renormalization procedures in favor of a new cutoff scheme?

We are driven to the lattice by the rather unique feature of confinement in the strong interactions. This phenomenon is inherently non-perturbative. The free theory with vanishing coupling constant has no resemblance to the observed physical world. Renormalization group arguments, to be presented in detail in later chapters, indicate severe essential singularities when hadronic properties are regarded as functions of the gauge coupling. This contrasts sharply with the great successes of quantum electrodynamics, where perturbation theory was central. Most conventional regularization schemes are based on the Feynman expansion; some process is calculated until a divergence is met in a particular diagram, and this divergence is then removed. To go beyond the diagrammatic approach, one needs a non-perturbative cutoff. Herein lies the main virtue of the lattice, which directly eliminates all wavelengths less than twice the lattice spacing. This occurs before any expansions or approximations are begun.

On a lattice, a field theory becomes mathematically well-defined and can

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be studied in various ways. Lattice perturbation theory, although somewhat awkward, recovers all the conventional results of other regularization schemes. Discrete space-time, however, is particularly well-suited for a strong coupling expansion. Remarkably, confinement is automatic in this limit where the theory reduces to one of quarks on the ends of strings with a finite energy per unit length. Most recent research has concentrated on showing that this phenomenon survives the continuum limit.

A lattice formulation emphasizes the close connections between field theory and statistical mechanics. Indeed, the strong coupling treatment is equivalent to a high temperature expansion. The deep ties between these disciplines are manifest in the Feynman path integral formulation of quantum mechanics (Feynman, 1948; Dirac, 1933, 1945). In Euclidian space, a path integral is equivalent to a partition function for an analogous statistical system. The square of the field theoretical coupling constant corresponds directly to the temperature. Thus, the particle physicist has available the full technology of the condensed matter theorist.

Confinement is natural in the strong coupling limit of the lattice theory; however, this is not the region of direct physical interest, for which a continuum limit is necessary. The coupling constant on the lattice represents a bare coupling at a length scale of the lattice spacing. Non-Abelian gauge theories possess the property of asymptotic freedom, which means that in the short distance limit the effective coupling goes to zero. This remarkable phenomenon allows predictions for the observed scaling behavior in deeply inelastic collisions. Indeed, this was one of the original motivations for a non-Abelian gauge theory of the strong interactions. The consequence for the lattice theory, however, is that the bare coupling must be taken to zero as the lattice spacing decreases towards the continuum limit. Thus we are inevitably led out of the high temperature regime and into a low temperature domain. Along the way in a general statistical system one might expect to encounter phase transitions. Such qualitative shifts in the physical characteristics of a system can only hamper the task of showing confinement in the non-Abelian theory. In later chapters we will present evidence that such troublesome transitions can be avoided in the four-dimensional $SU(3)$ gauge theory of the nuclear force.

Although our ultimate goal with lattice gauge theory is an understanding of hadronic physics, many interesting phenomena arise which are peculiar to the lattice. We will see non-trivial phase structure occurring in a variety of models, some of which do not correspond to any continuum field theory. The lattice formulation is highly non-unique and thereby spurious transitions can be alternately introduced and removed. We will also see that the statistical mechanics of gauge models displays curious analogies with magnetic systems in half the number of space-time dimensions. Even quantum electrodynamics shows interesting structure in certain lattice formulations. This rich spectrum of phenomena has led to the recent popularity of lattice field theories and motivates this book.