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TOPOLOGICAL LOCALIZATION, CATEGORY AND COCATEGORY

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It is easy to see that a localization (in the sense of [**9**]) of a simply connected co H-space (equivalently a simply connected space of Liusternik-Schnirelman category one) is again a co H-space. (All spaces in this paper will be pointed and have the based homotopy type of a connected CW complex; and all maps will preserve base-points.) We show that the category of a simply connected space does not increase on localizing. We give an example to show that the hypothesis simple-connectivity is crucial. In strong contrast, the dual result only requires connectivity.

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1. We first recall some properties of topological localization. Let l be a (possibly empty) set of primes contained in $\mathbf{Z}$, the integers; and let $\mathbf{Z}_l$ denote the integers localized at the primes in l. (Thus $\mathbf{Z}_{\{0\}}$ is the rational numbers.)

Definition 1 [**9**]. A space Y is *local* if $\pi_*(Y)$ is local, i.e., $\pi_*(Y)$ is a $\mathbf{Z}_l$-module. A map $l{:}X \to X_l$ into a local space X_l is a *localization* of X if given any local space Y and a map

$$X \xrightarrow{f} Y,$$

there is a *unique* map $f_l{:}X_l \to Y$ making the diagram

$$\begin{tikzcd} X \arrow[rr, "l"] \arrow[dr, "f"'] & & X_l \arrow[dl, dashed, "f_l"] \\ & Y & \end{tikzcd}$$

commute. We refer the reader to [**9**] for the construction and properties of $l{:}X \to X_l$. In particular, localizations always exist for simply connected spaces.

There are three well-known equivalent definitions of category for connected CW complexes: one in terms of open coverings, one in terms of closed coverings and one in terms of the "*fat* wedge". Since it is not clear how localization behaves with respect to coverings or fat wedges we adopt another approach.

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LEMMA 2 [**3**]. *There is a sequence of fibrations*

$$F_n(X) \hookrightarrow E_n(X) \xrightarrow{\gamma_n(X)} X,$$

defined for any space X, such that

(i) *$\gamma_0(X)$ is the standard path fibration over X;*

(ii) *if X is a path-connected CW complex, $\gamma_n(X)$ has a cross section if, and only if* cat $X \leqq n$; (We take cat $* = 0$.)

(iii) *for any space X*, Cat $E_n(X) \leqq n$ *where* Cat *denotes the strong Liusternik-Schnirelmann category of a space* [**3**];

(iv) $F_n(X) \simeq \Omega X * \Omega X * \ldots * \Omega X$ ($n + 1$ fold join), *and hence is* $(k(n+1)-2)$-*connected if X is $(k-1)$-connected, $k \geqq 2$.* (In fact $\gamma_n(X)$ is obtained from $\gamma_{n-1}(X)$ as follows: Let $E_n(X)$ result from $E_{n-1}(X)$ by erecting a reduced cone over $F_{n-1}(X)$, and $\gamma_{n-1}(X)$ is extended by mapping the cone to a point. Converting this map into a homotopy equivalent fibration, we get $\gamma_n(X)$.)

We will exhibit a map f_l so that

$$\begin{tikzcd} E_n(X)_l \arrow[rr, "f_l"] \arrow[dr, "\gamma_n(X)_l"'] & & E_n(X_l) \arrow[dl, "\gamma_n(X_l)"] \\ & X_l & \end{tikzcd}$$

homotopy commutes. Then if cat $X \leqq n$, $\gamma_n(X)$ and hence $\gamma_n(X_l)$ has a section. Thus $E_n(X_l)$ dominates X_l and so by [**1**, 2.4], cat $X_l \leqq$ cat $E_n(X_l)$. Ganea [**3**] has shown that for any space Y, cat $Y \leqq$ Cat Y, and hence by (iii) above

$$\text{cat } X_l \leqq \text{Cat } E_n(X_l) \leqq n.$$

PROPOSITION 3. *Suppose X is a simply connected space.*

(i) *l: $X \to X_l$ induces a map $E_n(l)$: $E_n(X) \to E_n(X_l)$ such that*

$$\begin{tikzcd} E_n(X) \arrow[r, "E_n(l)"] \arrow[d, "\gamma_n(X)"'] & E_n(X_l) \arrow[d, "\gamma_n(X_l)"] \\ X \arrow[r, "l"] & X_l \end{tikzcd}$$

commutes.

(ii) *$E_n(X)$ is simply connected for each $n \geqq 0$; and $E_n(X_l)$ is a local space.*

(iii) *There is a map f_l: $E_n(X)_l \to E_n(X_l)$ such that diagram* (A) *above commutes.*

Proof. (i) This is a routine check.

(ii) The first part is true since we have a fibration $F_n(X) \hookrightarrow E_n(X) \to X$ with $F_n(X)$ at least 2-connected by Lemma 2 (iv).

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To see that $E_n(X_l)$ is a local space, we need only show that the fibre $F_n(X_l)$ is local [**9**, Chapter 2]; and this follows inductively from the following short exact sequence of Švarc [**10**]:

$$0 \to \sum_{i+j=r-1} \tilde{H}_i(A) \otimes \tilde{H}_j(B) \to \tilde{H}_r(A * B) \to \sum_{i+j=r-2} \tilde{H}_i(A) \otimes \tilde{H}_j(B) \to 0$$

(for if any two terms are local, so is the third by [**9**, Chapter 1].) The induction begins at $n = 0$ since X_l is local and ΩX_l is the fibre of $P(X_l, *) \to X_l$ and hence is local by [**9**, Chapter 2].

(iii) This is an easy consequence of the universality of Sullivan's construction stated in Definition 1, and (i) and (ii) above. (We note that an easy induction shows that f_l induces an isomorphism of $\mathbf{Z}_l$-homology and since $E_n(X_l)$ is local and simply connected, f_l will be a homotopy equivalence.)

We can thus state

THEOREM 4. *If X_l is a localization of a simply connected space X then* $\operatorname{cat} X_l \leqq \operatorname{cat} X$.

It follows from [9] that the rationalization, $X_{\{0\}}$, of X is a localization of X_l for any l, and so $\operatorname{cat} X_{\{0\}} \leqq \operatorname{cat} X_l \leqq \operatorname{cat} X$. Peter Hilton has raised (in private communication) the question of when $\operatorname{cat} X$ is the supremum of $\operatorname{cat} X_p$ over all primes p. We remark that X has to be simply connected. For, Ganea has pointed out in [**6**] that the Eilenberg MacLane space $K(\mathbf{Q}, 1)$ has category two, and $K(\mathbf{Q}, 1)$ is clearly the rationalization of $K(\mathbf{Z}, 1) = S^1$. Thus Theorem 4 is false for non-simply connected spaces.

2. In [**2**], Bousfield and Kan have extended the theory of localization to apply to spaces X for which the action of $\pi_1(X)$ on each $\pi_n(X)$ is nilpotent—see [**2**, p. 58] for the definition; and have characterized nilpotent spaces as those spaces for which each stage of its Postnikov system can be refined to a finite composition of principal fibrations with abelian fibre [**2**, III. 5.3].

We refer the reader to [**4**] for the definition of cocategory. James [**7**] and Ganea [**4**; **5**] have shown that H-spaces are precisely the spaces of cocategory one. Since the action of $\pi_1(X)$ on $\pi_*(X)$ is trivial for an H-space, it follows that $\operatorname{cocat} X \leqq 1$ implies that the action of $\pi_1(X)$ on $\pi_*(X)$ is nilpotent. More generally,

THEOREM 5 (Berstein). *If* $\operatorname{cocat} X < \infty$, *then for each n the action of $\pi_1(X)$ on $\pi_n(X)$ is nilpotent.*

Proof. Ganea [**5**, 3.14] showed that if X_m denotes the mth stage of a Postnikov system for X, then $\operatorname{cocat} X_m \leqq \operatorname{cocat} X$. The result therefore follows from

PROPOSITION 6 (Berstein). *The action of $\pi_1(X)$ on $\pi_*(X)$ is nilpotent if, and only if for each m* $\operatorname{cocat} X_m < \infty$, *where X_m denotes the m^{th} stage of a Postnikov system for X.*

Proof. Since the action may be written in terms of Whitehead products, and the Whitehead product length is always bounded above by the cocategory [**4**, 3.17], the sufficiency is clear. Conversely, if the action is nilpotent, the action of $\pi_1(X_m) = \pi_1(X)$ on $\pi_*(X_m)$ is nilpotent and thus the mth stage Postnikov system may be refined into a finite tower of principal fibrations. Hence by [**4**, 6.3], cocat $X_m < \infty$.

We now outline the proof of the dual of Theorem 4, viz.

THEOREM 7. *If* cocat $X \leqq k$, *and* X_l *denotes a localization of* X, *then* cocat $X_l \leqq k$.

Proof. By Theorem 5, X_l makes sense. Ganea [**4**] has defined a sequence of cofibrations $X \hookrightarrow F_k(X) \to B_k(X)$ satisfying the duals of Lemma 2 (i), (ii) and (iii). (Warning: the dual of (iv) is false—see [**4**, Remark 3.5].) Nevertheless, $F_k(X_l)$ is a local space, and we can construct a map f_l: $F_k(X_l) \to F_k(X_l)$ as before; and it is easy to see that f_l induces isomorphisms of homology groups. Now $\pi_1(F_k(X)) = 0$ by [**4**, 6.9], and so by [**9**, Chapter 2], f_l is a homotopy equivalence of local spaces.

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