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SOME PROPERTIES OF THE q-HERMITE POLYNOMIALS

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1. Introduction. Heine [**7**, p. 93] gave the following representation for the Legendre Polynomial $\{P_n(x)\}_{n=0}^{\infty}$

$$P_n(\cos\theta) = \frac{4}{\pi}\frac{2.4\ldots 2n}{3.5\ldots(2n+1)}\sum_{k=0}^{\infty} f_{k,n}\sin(n+2k+1)\theta,$$

where $f_{0,n} = 1$ and

$$f_{k,n} = \frac{1.3\ldots(2k-1)}{2.4\ldots 2k}\,\frac{(n+1)\ldots(n+k)}{(n+\frac{3}{2})(n+\frac{5}{2})\ldots(n+k+\frac{1}{2})}.$$

Szegö [**7**, p. 96] generalized this result to the Ultraspherical Polynomial set $\{C_n^{\lambda}(x)\}_{n=0}^{\infty}$ and obtained

$$(1.1)\qquad (\sin\theta)^{2\lambda-1}C_n^{\lambda}(\cos\theta) = \sum_{k=0}^{\infty} f_{k,n}^{\lambda}\sin(n+2k+1)\theta,$$

where

$$f_{k,n}^{\lambda} = \frac{2^{2-2\lambda}\Gamma(n+2\lambda)(1-\lambda)_k(n+1)_k}{\Gamma(\lambda)\Gamma(n+\lambda+1)k!(n+\lambda+1)_k},$$

$\lambda > 0$, $\Gamma(\lambda)$ is the ordinary Gamma function and $(a)_n$ is defined by

$$(1.2)\qquad (a)_n = \begin{cases} a(a+1)\ldots(a+n-1) & \text{if } n = 1, 2\ldots \\ 1 & \text{if } n = 0. \end{cases}$$

Equation (1.1) is the Fourier sine series expansion of $(\sin\theta)^{2\lambda-1}C_n^{\lambda}(\cos\theta)$. Because for each non-negative integer n, $f_{k,n}^{\lambda}$ is eventually monotonic in k and $\lim_{k\to\infty} f_{k,n}^{\lambda} = 0$, it follows from classical Fourier analysis that (1.1) converges pointwise in $(0, \pi)$ and uniformly on $[\epsilon, \pi-\epsilon]$ for $0 < \epsilon < \pi/2$.

It is well known [**5**, p. 281] that $\{C_n^{\lambda}(\cos\theta)\}_{n=0}^{\infty}$ is orthogonal on $[0, \pi]$ with weight function $(\sin\theta)^{2\lambda-1}$. In [**1**] we identified a large class of orthogonal polynomial sets that satisfy an equation of the form (1.1).

One of these polynomial sets turned out to be $\{R_n(x;q)\}_{n=0}^{\infty}$ defined by the three term recursion relation

$$(1.3)\qquad \begin{cases} R_0(x;q) = 1 \qquad R_1(x;q) = 2x \\ R_{n+1}(x;q) = 2xR_n(x;q) - (1-q^n)R_{n-1}(x;q) \qquad (n \geqq 1), \end{cases}$$

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where $|q| < 1$. From this three term recursion formula it is easy to show that

$$\lim_{q\to 1} \frac{R_n(((1-q)/2)^{1/2}x; q)}{((1-q)/2)^{n/2}} = H_n(x),$$

where $\{H_n(x)\}_{n=0}^{\infty}$ is the Hermite polynomial set [**5**, p. 188]. It is for this reason that $\{R_n(x; q)\}_{n=0}^{\infty}$ is called the q-Hermite polynomial set. This polynomial set was first introduced by Roger [**6**, p. 319] in 1894.

In this paper we study some of the properties of $\{R_n(x; q)\}$. We show that $\{R_n(x; q)\}_{n=0}^{\infty}$ is characterized by a Fourier sine series similar to (1.1) in which the coefficients satisfy a very simple recursion formula. From this fact we are able to deduce that $\{R_n(\cos\theta; q)\}_{n=0}^{\infty}$ is orthogonal on $[0, \pi]$ with respect to the weight function $\theta_1(z; q^{1/2})$, where $\theta_1(z; q)$ is one of the Theta Functions [**5**, p. 314], defined by

$$(1.4) \qquad \theta_1(z, q) = 2\sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin(2n+1)z.$$

Finally it is interesting to note that

$$(1.5) \qquad R_n(x; q) = v^n H_n(u/v; q) = u^n H_n(v/u; q),$$

where $u = x - \sqrt{x^2 - 1}$, $v = x + \sqrt{^2 - 1}$ and $H_n(x, q)$ is the polynomial set first introduced by Szegö [**8**]. $H_n(x, q)$ is defined by

$$(1.6) \qquad H_n(x; q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k,$$

where

$$(1.7) \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-k+1})}{(1-q)(1-q)\ldots(1-q^k)},$$

and

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1.$$

Carlitz [**3**] has made a detailed study of $\{H_n(x; q)\}_{n=0}^{\infty}$. By using his results, similar results can be obtained for $\{R_n(x; q)\}_{n=0}^{\infty}$. Also, Al-Salam and Chihara [**2**] have studied generalizations of $\{R_n(x; q)\}_{n=0}^{\infty}$.

2. Orthogonality of $\{R_n(x; q)\}_{n=0}^{\infty}$. For q a real number such that $|q| < 1$,

$$\sum_{n=1}^{\infty} |q^{n(n+1)/2}|^2 < \infty.$$

Thus by the Riesz-Fischer Theorem there exists $w(\cos\theta; q) \in L^2[0, \pi]$

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such that for all non-negative integers n

$$(2.1)\qquad \int_0^\pi w(\cos\theta; q)\sin((n+1)\theta)d\theta = \begin{cases} (-1)^k q^{k(k+1)/2} & n = 2k \\ 0 & n = 2k+1. \end{cases}$$

From a well known Theorem (see [**4**], p. 196), it follows that $w(\cos\theta; q) \in L^1[0, \pi]$. We will show that $\{R_n(z; q)\}_{n=0}^\infty$ is orthogonal on $[-1, 1]$ with respect to the weight function $w(x; q)$.

In (2.1) let us make the substitution $x = \cos\theta$ to obtain

$$(2.2)\qquad \int_{-1}^1 w(x; q)U_n(x)dx = \begin{cases} (-1)^k q^{k(k+1)/2} & n = 2k \\ 0 & n = 2k+1, \end{cases}$$

where $\{U_n(x)\}_{n=0}^\infty$ is the Chebychev polynomial of the second kind (see [**5**], p. 301), defined by

$$U_n(\cos\theta) = \sin(n+1)\theta/\sin\theta \ (n \geqq 0).$$

We will extend this definition of $\{U_n(x)\}_{n=0}^\infty$ to all integers n by defining

$$(2.3)\qquad \begin{aligned} &U_{-1}(x) = 0 \\ &U_n(x) = -U_{-n-2}(x). \end{aligned}$$

It is easy to show that these extended Chebychev polynomials of the second kind satisfy a three term recursion relation of the form

$$(2.4)\qquad \begin{cases} U_0(x) = 1 \qquad U_1(x) = 2x \\ U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x). \end{cases}$$

Both $\{R_n(x)\}_{n=0}^\infty$ and $\{U_n(x)\}_{n=-\infty}^\infty$ are examples of symmetric orthogonal polynomial sets and thus for all $n \geqq 0$ and $0 \leqq n + 2k$ we have

$$R_n(x; q)U_{n+2k+1}(x) = \sum_{i=0}^{n+k} a_{n+k,i}U_{2i+1}(x).$$

By using this equation and Equation (2.2) we obtain for $n \geqq 0$ and $n + 2k \geqq 0$

$$(2.5)\qquad \int_{-1}^1 w(x; q)R_n(x; q)U_{n+2k+1}(x)dx = 0.$$

Let us define for $n \geqq -1$ and all integers k

$$(2.6)\qquad f_{k,n} = \begin{cases} \int_{-1}^1 w(x;q)R_n(x;q)U_{n+2k}(x)dx, & \text{if } n + 2k \geqq 0 \text{ and } n \neq -1 \\ 0, \quad \text{if } n + 2k < 0 \text{ or } n = -1. \end{cases}$$

It follows directly from this definition and the three term recursion formulas for $\{R_n(x; q)\}_{n=0}^\infty$ and $\{U_n(x)\}_{n=-\infty}^\infty$ that, for all integer values k

$$(2.7)\qquad f_{k,n+1} = f_{k+1,n} + f_{k,n} - (1 - q^n)f_{k+1,n-1},$$

for $n \geqq 0$.

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We will now prove by mathematical induction on n that for all non-negative integers k

(2.8) $$f_{k,n} = \frac{(-1)^k q^{k(k+1)/2}[q]_{n+k}}{[q]_k},$$

where

(2.9) $$[a]_k = \begin{cases} (1-a)(1-aq)\ldots(1-aq^{k-1}) & k = 1, 2, 3\ldots \\ 1 & k = 0. \end{cases}$$

For $n = 0$ we obtain from Definition (2.6) and Equation (2.2)

$$f_{k,0} = \int_{-1}^{1} w(x)U_{2k}(x)dx = (-1)^k q^{k(k+1)/2},$$

and for $n = 1$ we obtain in the same manner

$$\begin{aligned} f_{k,1} &= \int_{-1}^{1} w(x;q)R_1(x;q)U_{1+2k}(x)dx \\ &= \int_{-1}^{1} w(x)(U_{2+2k}(x) + U_{2k}(x))dx \\ &= ((-1)^{k+1}q^{(k+1)(k+2)/2} + (-1)^k q^{k(k+1)/2}) \\ &= (-1)^k q^{k(k+1)/2}(1 - q^{k+1}). \end{aligned}$$

Thus Equation (2.8) is true for all non-negative integers k, and $n = 0$ or 1. Now let us make the induction hypothesis that Equation (2.8) is true for all non-negative integers k, and $n = 0, 1, 2\ldots m$. By Equation (2.7) and the induction hypothesis we obtain for all non-negative integers k

$$\begin{aligned} f_{k,m+1} &= \frac{(-1)^{k+1}q^{(k+1)(k+2)/2}[q]_{m+k+1}}{[q]_{k+1}} + \frac{(-1)^k q^{k(k+1)/2}[q]_{m+k}}{[q]_k} \\ &\quad - \frac{(1-q^m)(-1)^{k+1}q^{(k+1)(k+2)/2}[q]_{m+k}}{[q]_{k+1}} = \frac{(-1)^k q^{k(k+1)/2}[q]_{m+1+k}}{[q]_k}. \end{aligned}$$

Now we have developed everything that is required in order to show that $\{R_n(x;q)\}_{n=0}^{\infty}$ is orthogonal on $[-1, 1]$ with respect to the weight function $w(x, q)$ which is defined by Equation (2.1). We will show that for m and n non-negative integers

(2.10) $$\int_{-1}^{1} R_n(x;q)R_m(x;q)w(x;q)dx = \delta_{n,m}[q]_n,$$

where $\delta_{n,m}$ is the Kronecher delta. It is an easy exercise to show that Equation (2.8) is equivalent to

(2.11) $$\int_{-1}^{1} R_n(x;q)U_m(x)w(x;q)dx = \begin{cases} 0 & 0 \leqq m < n \\ [q]_n & m = n. \end{cases}$$

From Equation (2.5) we know that for $n \geqq 0$ and $n + 2k \geqq 0$,

$$\int_{-1}^{1} R_n(x;q)U_{n+2k+1}(x)w(x;q)dx = 0,$$

and from Equations (2.8) and (2.6) we have that for $n \geqq 0$,

$$\int_{-1}^{1} R_n(x; q) U_n(x) w(x; q) dx = [q]_n.$$

Thus in order to show that (2.10) is true we need only show that for all non-negative integers n, $f_{k,n} = 0$, for $-n \leqq 2k < 0$. We use mathematical induction on n to show that for all negative integers k

$$f_{k,n} = 0.$$

By the definition of $f_{k,n}$ as given by Equation (2.6)

$$(2.12) \quad f_{k,0} = 0,$$

for k a negative integer. Also for the case $n = 1$ we have from the definition of $f_{k,n}$ that $f_{-1,1} = 0$, and from Equations (2.7) and (2.12) that

$$f_{k,1} = 0,$$

for k a negative integer. Now let us make the induction hypothesis that for all negative integers k

$$f_{k,n} = 0,$$

for $n = 0, 1, 2 \ldots m$. By Equation (2.7) we have

$$(2.13) \quad f_{k,m+1} = f_{k+1,m} + f_{k,m} - (1 - q^m) f_{k+1,m-1}.$$

Thus from the induction hypothesis $f_{k,m+1} = 0$ for $k = -2, -3 \ldots$. For $k = -1$ we obtain from Equations (2.13) and (2.8), and the induction hypothesis

$$\begin{aligned} f_{-1,m+1} &= f_{0,m} - (1 - q^m) f_{0,m-1} \\ &= [q]_m - (1 - q^m)[q]_{m-1} \\ &= 0. \end{aligned}$$

Therefore for all negative integers k and non-negative integers n, $n, f_{k,n} = 0$. Therefore $\{R_n(x; q)\}_{n=0}^{\infty}$ is orthogonal on $[-1, 1]$ with respect to the weight function

$$(2.14) \quad w(x; q) = \frac{2}{\pi} \sqrt{1 - x^2} \sum_{k=0}^{\infty} (-1)^k q^{k(k+1)/2} U_{2k}(x).$$

3. A characterization of $\{R_n(x; q)\}_{n=0}^{\infty}$. In this section we wish to find a characterization of $\{R_n(x; q)\}_{n=0}^{\infty}$.

Let the polynomial set $\{E_n^{\lambda}(x)\}_{n=0}^{\infty}$ be defined by

$$E_n^{\lambda}(x) = \frac{n!}{(1 + \lambda)_n} C_n^{\lambda}(x) \qquad (n \geqq 0),$$

where $\{C_n^{\lambda}(x)\}_{n=0}^{\infty}$ is the Ultraspherical Polynomial set and $(1 + \lambda)_n$ is

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defined by (1.2). It follows directly from Equation (1.1) that

$$(3.1)\qquad (1-x^2)^{\lambda-1/2}E_n^{\lambda}(x) = \frac{2^{2-2\lambda}\Gamma(n+2\lambda)}{\Gamma(\lambda)\Gamma(n+\lambda+1)}\sqrt{1-x^2}\sum_{k=0}^{\infty} g_{k,n}^{\lambda}U_{n+2k}(x),$$

where

$$(3.2)\qquad g_{k,n}^{\lambda} = \frac{k-\lambda}{k}g_{k-1,n+1}^{\lambda}.$$

Equations (3.1) and (3.2) suggest studying the polynomial sets $\{A_n(x)\}_{n=0}^{\infty}$ such that there exists a function $w(x)$ and a sequence of real numbers $\{\alpha_k\}_{n=0}^{\infty}$ having the property that the Fourier Chebychev expansion of $w(x)A_n(x)$ is

$$(3.3)\qquad w(x)A_n(x) \sim \frac{2}{\pi}\sum_{k=0}^{\infty} h_{k,n}U_{n+2k}(x)$$

where

$$h_{0,n} \neq 0$$

and

$$(3.4)\qquad h_{k,n} = \alpha_k h_{k-1,n+1} \qquad (k \geqq 1,\, n \geqq 0).$$

In [**1**] we find the three term recursion relation of all these polynomial sets and study some of their properties.

It is easy to show (see [**1**]) that all polynomial sets $\{A_n(x)\}_{n=0}^{\infty}$ that satisfy Equation (3.3) are symmetric and orthogonal on $[-1, 1]$ with respect to the weight function $w(x)$. It is well known (see [**1**]) that such symmetric orthogonal polynomial sets satisfy a three term recursion formula of the form

$$(3.5)\qquad \begin{cases} A_0(x) = 1 \qquad A_1(x) = 2b_1x \\ A_n(x) = 2b_nxA_{n-1}(x) - \lambda_nA_{n-2}(x) \qquad (n \geqq 2), \end{cases}$$

where $\{b_n\}_{n=0}^{\infty}$ and $\{\lambda_n\}_{n=0}^{\infty}$ are real non-zero sequences.

We note from Equation (1.3) that in order for Equation (3.5) to be the three term recursion relation for $\{R_n(x)\}_{n=0}^{\infty}$ we require $b_1 = b_2$ and $2b_1b_2 > \lambda_2$. Now we wish to prove the following theorem.

THEOREM 3.1. *Let* $\{A_n(x)\}_{n=0}^{\infty}$ *be any polynomial set satisfying Equations* (3.3), (3.4) *and* (3.5). *Also let* $\{R_n(x)\}_{n=0}^{\infty}$ *be defined by Equation* (1.3).

$$R_n(x; q) = A_n(x)/b_1^{\,n}$$

if and only if $b_1 = b_2$ *and* $2b_1b_2 > \lambda_2 > 0$.

Proof. Because $\{A_n(x)\}_{n=0}^{\infty}$ satisfies (3.3) and (3.5) we have for $n \geqq 2$

$$0 = \int_{-1}^{1} w(x)A_n(x)U_{n-2}(x)dx = \int_{-1}^{1} w(x)[b_nA_{n-1}(x)(U_{n-1}(x) + U_{n-3}(x)) - \lambda_nA_{n-2}(x)U_{n-2}(x)]dx = b_nh_{0,n-1} - \lambda_nh_{0,n-2}.$$

Thus if we let $\gamma_n = \lambda_n/b_n$ we obtain

$$(3.6) \qquad h_{0,n} = \gamma_{n+1}f_{0,n-1} = \left(\prod_{i=2}^{n+1} \gamma_i\right)h_{0,0}.$$

By combining this equation with Equation (3.4) we obtain

$$(3.7) \qquad h_{k,n} = \prod_{i=1}^{k} \alpha_i \prod_{j=2}^{n+k+1} \gamma_j h_{0,0}.$$

By definition

$$h_{k,n} = \int_{-1}^{1} w(x)A_n(x)U_{n+2k}(x)dx.$$

Thus by using this fact and the three term recursion formula for $\{A_n(x)\}_{n=0}^{\infty}$ and $\{U_{n+2k}(x)\}_{n=0}^{\infty}$ we obtain

$$(3.8) \qquad h_{k,n} = b_n(h_{k,n-1} + h_{k+1,n-1}) - \lambda_nh_{k+1,n-2}.$$

Now by combining Equation (3.7) and (3.8) we obtain

$$(3.9) \qquad \gamma_{n+k}(b_n^{-1} - \alpha_k) + \alpha_k\gamma_n - 1 = 0$$

for $n = 1, 2, 3 \ldots$ and $1 \leqq k \leqq m$, where $\gamma_1 = 0$ and m is defined to be the smallest integer such that $\alpha_m = 0$. If all the α_i's are not equal to zero then $m = \infty$.

In Equation (3.9) let $n = 2$ and 1, to obtain

$$(3.10) \quad \gamma_{k+2}(b_2^{-1} - \alpha_k) + \gamma_2\alpha_k - 1 = 0 \qquad (1 \leqq k \leqq m)$$

and

$$(3.11) \quad \alpha_k = b_1^{-1} - \gamma_{k+1}^{-1} \qquad (1 \leqq k \leqq m)$$

respectively. By using Equation (3.11) to eliminate α_k from (3.10) and by using the fact that $b_1 = b_2$ and $\gamma_1 = 0$ we obtain

$$(3.12) \quad \gamma_{k+2} + \gamma_{k+1}(\gamma_2b_1^{-1} - 1) = \gamma_2 \qquad (0 \leqq k \leqq m).$$

This is a first order non-homogeneous finite difference equation with constant coefficients. By using standard methods it is easy to show that its solution is

$$(3.13) \quad \gamma_k = b_1[1 - (1 - \gamma_2b_1^{-1})^{k-1}] \qquad (1 \leqq k \leqq m).$$

By using Equation (3.13) to substitute for γ_k in Equation (3.11) we obtain

$$(3.14) \quad \alpha_n = (1 - \gamma_2 b_1^{-1})^n \{b_1[1 = \gamma_2 b_1^{-1})^n - 1]\}^{-1}.$$

Thus we see that $m = 0$ or $m = \infty$. By letting $k = 1$ in Equation (3.9) and then using Equation (3.11) and (3.13) we obtain

$$(3.15) \quad b_n = b_1.$$

Finally we note that if we let $q = 1 - \gamma_2 b_1^{-1}$ and use the fact that $2b_1 b_2 > \lambda_2 > 0$ we obtain

$$|q| < 1.$$

From Equations (1.3), (3.5), (3.13) and (3.15) we obtain

$$R_n(x; q) = A_n(x)/b_1^n$$

and from Equation (3.14)

$$(3.16) \quad \alpha_n = \frac{q^n}{q^n - 1}.$$

The converse follows directly from the three term recursion relation (1.3) and the fact that $|q| < 1$.

4. The weight function $w(x; q)$. We now wish to study the weight function $w(x; q)$ of $\{R_n(x; q)\}_{n=0}^{\infty}$.

From Equation (2.13) we see that the Fourier sine series expansion of $w(\cos\theta; q)$ is given by

$$w(\cos\theta; q) = \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k q^{k(k+1)/2} \sin(2k+1)\theta$$
$$= \frac{2}{\pi} q^{-1/8} \sum_{k=0}^{\infty} (-1)^k (q^{1/2})^{k^2+k+1/4} \sin(2k+1)\theta.$$

By comparing this with the Theta Function $\theta_1(z, q)$ as defined in [**5**, p. 314] by

$$(4.1) \quad \theta_1(z, q) = 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin(2n+1)z,$$

we obtain

$$(4.2) \quad w(\cos z; q) = \frac{q^{-1/8}}{\pi} \theta_1(z; q^{1/2}).$$

$\theta_1(z, q)$ has an infinite product representation (see [**5**], p. 334);

$$\theta_1(z, q) = 2q^{1/4} \sin z \prod_{n=1}^{\infty} (1 - q^{2n})(1 - 2q^{2n} \cos 2z + q^{4n}).$$

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Therefore,

$$(4.3) \qquad w(\cos z; q) = \frac{2}{\pi} \sin z \prod_{n=1}^{\infty} (1 - q^n)(1 - 2q^n \cos 2z + q^{2n}).$$

Equation (4.3) agrees with results obtained by Al-Salam and Chihara [**2**, p. 28].

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