THE J-DIFFERENTIAL AND ITS INTEGRAL

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WE consider the possibility of generalising the statement

$$dy = f(x)dx \quad \longleftrightarrow \quad y(x) - y(a) = \int_a^x f dx$$

to the case

$$dy = f(x)dg(x) \quad \longleftrightarrow \quad y(x) - y(a) = \int_a^x f dg.$$

The question centres around the definition of $dg(x)$ on the one hand and of the integral $\int_a^x f dg$ on the other: as ordinarily understood, the generalisation is impossible when $g(x)$ is any arbitrary function of limited variation, or is merely not continuous. We define a differential, the vector j-differential, of a function $g(x)$ with respect to any interval I_x having x as one terminal and $x+dx$ as the other, and vector weight determined by values of a function $f(x)$ on the interval I_x, whose weighted scalar sum, for any chain of such intervals from a to x, has a limit by refinement of the chain (i.e., by subdivision of intervals), namely the σ-limit. Reciprocally, the j-differential operator, when applied to this integral, gives back the weighted j-differentials appropriate to a class of intervals with common terminal x. A reciprocal differential and integral Stieltjes calculus is thus seen to exist. Of incidental but designed advantage is the circumstance that the integration-by-parts formula and a substitution rule are smooth generalisations of the ordinary. Certain *ad hoc* limitations are of course imposed on the functions $f(x)$ and $g(x)$ but they are permitted the discontinuities, possibly simultaneous, in every subinterval of the fundamental interval $[ab]$, of functions of limited variation.

1. Definitions

We shall use the following symbols for classes:

D_1—bounded functions having discontinuities of the first kind at most;

B—functions of bounded variation;

B′—functions in the class B such that if $g(x)$ is in B' and $v_g(x)$ is the total variation of $g(x)$ over $[ax]$, then both

$$g'^{+}(x+) \equiv \lim_{\delta \to +0} [g(x+\delta) - g(x+)]/\delta$$

and $v_g'^{+}(x+)$, similarly defined, exist and are in D_1;

B*—functions in the class B′ such that, if $g(x)$ is in B*, the difference quotient $[g(x+\delta) - g(x+)]/\delta$, $\delta > 0$, converges uniformly over $[ab]$.

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The j-differential is defined by either of the equations

$$jdg = jg + g'^{+}(x+)dx, \qquad \boldsymbol{dg} = (jg, g'^{+}(x+)dx),$$

in which $jg = g(x+) - g(x)$; we shall call $\boldsymbol{dg}$ the *vector differential.* With dx non-negative and $g(x)$ continuous on the left, jdg is an approximation to Δg. The *mean* $\bar{u}(x)$ and the *vectorisation* $\boldsymbol{u}(x)$ of a function $u(x)$ are defined by the equations

$$\bar{u}(x) = \tfrac{1}{2}[u(x+)+u(x)], \qquad \boldsymbol{u}(x) = [\bar{u}(x), u(x+)].$$

Vectorisations result in economies in many places, e.g.,

$$\begin{aligned} jd(uv) &= u(x+)\,jdv + v(x)\,jdu + jv \cdot u'^{+}(x+)dx, \\ &= u(x)\,jdv + v(x+)\,jdu + ju \cdot v'^{+}(x+)dx, \\ &= u(x+)\,jdv + v(x+)\,jdu - ju \cdot jv; \end{aligned}$$

but, if each term of the right member of the following equation is understood to be the middle product (defined in § **6**), then

$$\boldsymbol{d}(\boldsymbol{uv}) = \boldsymbol{udv} + \boldsymbol{du} \cdot \boldsymbol{v}.$$

The first integral permitting the simultaneous occurrence of discontinuities in $f(x)$ and $g(x)$ to be invented was the interior-Pollard-Moore-Stieltjes σ-integral

$$(IP)\int_a^b fdg = \lim_\sigma \Sigma_{i/\sigma[ab]} f(\xi_i)[g(x_{i+1}) - g(x_i)]$$

in which ξ_i is subject to the interior Pollard † condition $x_i < \xi_i < x_{i+1}$ which permits the simultaneity and where the index refers to points x_i, $i = 1, 2, \ldots, \nu$ of the open interval (ab) and to the end points $x_0 = a$, $x_{\nu+1} = b$, $x_i < x_{i+1}$, in which the summation is from $i = 0$ to $i = \nu$, and in which $\lim_\sigma$ is the σ-limit. The limit of a sequence of values of functions of intervals corresponding to, and given by, a sequence $\sigma_1, \sigma_2, \ldots$ in which each set σ_i (giving the ith value) is any proper subset of σ_{i+1} and such that (1) for any two sets σ_i, σ_j there is a set σ_k such that $\sigma_i \subset \sigma_k$, $\sigma_j \subset \sigma_k$, and (2) for every σ in the class of all finite subdivisions of $[ab]$ there is a set σ_i such that $\sigma \subset \sigma_i$,‡ namely the σ-limit, was first defined by Moore (**2**) and Pollard (**4**).

The left-Cauchy-Stieltjes σ-integral (**5**) and the IP-integral are the σ-limits, respectively of the functions of intervals

$$\Sigma_{i/\sigma[ab]} f(x_i)\Delta_i g, \qquad \Sigma_{i/\sigma[ab]} f(x_i+)\Delta_i g,$$

† Ref. 4, p. 123, §12; Pollard called this the "restricted" integral, Hildebrandt the "modified" integral in Ref. 1, §6, p. 273.

‡ I.e. the aggregate $\{\sigma\}$ of all finite subdivisions of $[ab]$ is (1) "directed" by the relation $\sigma_i \subset \sigma_{i+1}$ [vide E. J. McShane, *Am. Math. Monthly*, Vol. **59** (1952) p. 3] and (2) effectively exhausted by the enumeration.

where $\Delta_i g = g(x_{i+1}) - g(x_i)$, f is in D_1 and g is in B, and their integration-by-parts formulas for functions in B and continuous on the left, are

$$\int_a^b u dv + \int_a^b v du = [uv]_a^b \mp \Sigma_{a \leqq x < b} ju(x) \cdot jv(x),$$

respectively. The LM- or *left-mean-integral* is defined by the equation

$$\int_a^b f dg = \lim_\sigma \Sigma_{i/\sigma} \bar{f}(x_i) \Delta_i g; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (LM)$$

it is easily shown to be the mean of the LCSσ- and IP-integrals.

2. Existence of the Integrals

Theorem I. *When* $f(x)$ *is in* D_1 *and* $g(x)$ *is in* B, *then*

$$(IP) \int_a^b f dg \text{ exists and } (LM) \int_a^b f dg \text{ exists.}$$

The existence of the left-Cauchy-Stieltjes σ-integral (LCSσ) $\int_a^b w dg$, in which $w(x)$ is in D_1 and $g(x)$ is in B, has been established by Price [Ref. 5, Theorem 3, p. 627]. The IP- and LM-integrals are LCSσ-integrals with weight-functions $w(x)$ equal to $f(x+)$ and $\bar{f}$, respectively, when $f(x)$ is in D_1 and $g(x)$ is in B.

3. Elementary Integral Theorems

The elementary properties of linearity with respect to the integrated function, linearity with respect to the weight function, the property that

$$\int_\alpha^x f dg + \int_x^\beta f dg = \int_\alpha^\beta f dg, \; \alpha < x < \beta, \; a \leqq \alpha < \beta \leqq b,$$

and the properties

$$\int_\alpha^\alpha f dg = 0, \quad \int_a^x dg = g(x) - g(a), \quad \int_a^b \sum_{i=1}^n f_i dg_i = \sum_{i=1}^n \int_a^b f_i dg_i$$

which hold for the Riemann-Stieltjes integral also hold for the IP-, LCSσ- and LM-integrals. These elementary theorems are immediate consequences of the definition of integral as a limit or are true by definition.

The Mean-value Lemma. *When* $w(x)$ *is in* D_1 *and* $g(x)$ *is in* B, *numbers* w_1, w_2 *between the l.u.b. and g.l.b., inclusively, of the weight function* $w(x)$ *on any closed interval* $[x, x+\delta]$, *always exist such that*

$$\int_x^{x+\delta} w dg = \frac{w_1 + w_2}{2} \int_x^{x+\delta} dg + \frac{w_1 - w_2}{2} \int_x^{x+\delta} dv_g,$$

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and, when $f(x)$ is in D_1 and $g(x)$ is in B',

$$\lim_{\delta\to+0}\frac{1}{\delta}\int_{x+}^{x+\delta} fdg = f(x+)g'^{+}(x+);$$

these equations hold for each of the four† kinds of integral just mentioned; supplementary thereto are the equations

$$(LCS\sigma)\int_x^{x+} fdg = f(x)jg,\quad (IP)\int_x^{x+} fdg = f(x+)jg,\quad (LM)\int_x^{x+} fdg = \bar{f}jg.$$

To prove this lemma, let $g(x)-g(a)$ be represented as the difference between two non-decreasing functions $p(x)$ and $q(x)$ such that $p(x)+q(x) = v_g(x)$. Then, for any σ restricted to the subinterval $[x, x+\delta]$ of $[ab]$ and, with $x_i \leqq \xi_i \leqq x_{i+1}$, $x_i \varepsilon \sigma$, $x_{i+1}\varepsilon\sigma$, there are the inequalities

$$\inf w\,.\,\Sigma_{i/\sigma}\Delta_i p \quad \leqq \quad \Sigma_{i/\sigma} w(\xi_i)\Delta_i p \quad \leqq \quad \sup w\,.\,\Sigma_{i/\sigma}\Delta_i p$$

for $p(x)$, and similarly for $q(x)$, which hold up to, and including, their norm and σ-limits. The equations

$$\int_x^{x+\delta} wdp = w_1\int_x^{x+\delta} dp, \qquad \int_x^{x+\delta} wdq = w_2\int_x^{x+\delta} dq,$$

are seen to be implied. The first statement in the lemma follows by virtue of the equations $\Delta g = \Delta p - \Delta q$, $v = p+q$, and the linearity property of the integrated function. Next, the limit $\dfrac{1}{\delta}\displaystyle\int_{x+}^{x+\delta} fdg$ is seen, by a Cauchy test, to exist and, by a slight adjustment of first statement and proof,

$$\lim_{\delta\to+0}\frac{1}{\delta}\int_{x+}^{x+\delta} fdg = \lim_{\delta\to+0}\left\{\frac{w_1^*+w_2^*}{2}\frac{1}{\delta}\int_{x+}^{x+\delta} dg + \frac{w_1^*-w_2^*}{2}\frac{1}{\delta}\int_{x+}^{x+\delta} dv_g\right\},$$

where w_1^* and w_2^* are numbers between the l.u.b. and g.l.b. of $f(x)$ on the semi-open interval $(x, x+\delta]$ and both having the limit $f(x+)$ as $\delta\to+0$; the second statement is implied.

To obtain the supplementary results, let F_0 be a number such that $F_0 > |f(x)|$, $a \leqq x \leqq b$, let the point x now be called x_0 and let σ consist of ν points in $(x_0, x_0+\delta)$ together with the end points, $\nu+2$ points in all. Then, in the case of the LCSσ-integral,

$$\left|\textstyle\sum_{i=0}^{\nu} f(x_i)\Delta_i g - f(x_0)[g(x_1)-g(x_0)]\right| < F_0\textstyle\sum_{i=1}^{\nu}|\Delta_i g| = F_0[v_g(x_0+\delta)-v_g(x_1)].$$

Since the variation v_g of $g(x)$ over the semi-open interval $(x_0, x_0+\delta]$ tends to zero as $\delta\to 0$, it follows that to each $\varepsilon>0$ there corresponds a $\delta_0(\varepsilon)>0$ such that, for all $0<\delta<\delta_0$,

$$\left|\textstyle\sum_{i/\sigma} f(x_i)\Delta_i g - f(x_0)\Delta_0 g\right| < \varepsilon;$$

† In the case of the Riemann-Stieltjes integral, it is first necessary that the left members of the equations exist.

and, corresponding to the same ε, there also exists a $\sigma_0(\varepsilon)$ on $[x_0, x_0+\delta]$ such that for all $\sigma \supset \sigma_0$ and $0<\delta<\delta_0$,

$$\left| \int_{x_0}^{x_0+\delta} fdg - f(x_0)j_0 g \right| \leqq \left| \int_{x_0}^{x_0+\delta} fdg - \sum_{i/\sigma} f(x_i)\Delta_i g \right|$$
$$+ \left| \sum_{i/\sigma} f(x_i)\Delta_i g - f(x_0)\Delta_0 g \right| + \left| f(x_0)\Delta_0 g - f(x_0)j_0 g \right| \leqq 3\varepsilon;$$

for we can always provide that the point x_1 of σ_0 be so close to x_0 that $| g(x)-g(x_0)-j_0 g | < \varepsilon/F_0$ whenever $x_0 \leqq x \leqq x_1$, and therefore such that $| f(x_0)\Delta_0 g - f(x_0)j_0 g | < \varepsilon$. This 3ε-inequality implies the first supplementary result and since it holds when for $f(x)$ we write $f(x+)$ or $\bar{f}(x)$ it implies the second or third supplementary result, respectively, as the case may be.

4. Elementary Properties of $\phi(x) \equiv \int_a^x fdg$.

For all four integrals (in the case of the Riemann-Stieltjes integral, it is first necessary that the integral exist), we have the properties, f in D_1,

P_1 When g is in B, then $\phi(x)$ is in B.

P_2 When g is in B', then $\phi'^{+}(x+)$ exists and is bounded.

P_3 When g is in B^*, then $(\phi(x+\delta)-\phi(x+))/\delta$, $\delta>0$, converges uniformly over $[ab]$, the convergant being written $\phi'^{+}(x+)$.

For the integral indicated in each case and for g in B', we have

P_4	$jd\phi = f(x+)\,jg + f(x+)g'^{+}(x+)dx,$	(IP)
P_5	$jd\phi = f(x)\,jg + f(x+)g'^{+}(x+)dx,$	(LCSσ)
P_6	$jd\phi = \bar{f}jg + f(x+)g'^{+}(x+)dx = \boldsymbol{fdg}$	(LM)

To prove P_1, let $W_0 > | w(x) |$, $a \leqq x \leqq b$ and let i be the index of any point of $\sigma[ax]$ and I_i the corresponding interval $[x_i, x_{i+1}]$. Then, invoking the mean-value lemma for each of the $\nu+1$ intervals I_i, we have a boundedness relation

$$\Sigma_i \mid \Delta_i \phi_i \mid = \Sigma_i \left| \left(\frac{w_1+w_2}{2}\right)_i \Delta_i g + \left(\frac{w_1-w_2}{2}\right)_i \Delta_i v_g \right|$$
$$< W_0\{\Sigma \mid \Delta_i g \mid + \Sigma \Delta_i v_g\} = 2W_0 v_g(b),$$

which is uniform with respect to the class of subdivisions $\sigma[ax]$ and terminal x. The proofs of P_2 and P_3 are each essentially the same as that of the second statement of the mean-value lemma. To prove P_4, let now i be the index of any point of any set σ on $[x_0, x_0+\delta]$; then

$$\mid \Sigma_{i/\sigma} f(x_i+)\Delta_i g - f(x_0+)\Delta_0 g \mid \leqq F_0 \sum_{i=1}^{\nu} \mid \Delta_i g \mid = F_0[v_g(x_0+\delta) - v_g(x_1)],$$
$$\therefore \lim_{\delta\to 0} \Sigma_{i/\sigma} f(x_i+)\Delta_i g = f(x_0+)\,.\,jg(x_0).$$

To each $\varepsilon>0$ there corresponds a $\delta_0(\varepsilon)$ such that, for all $0<\delta<\delta_0$,

$$\mid \Sigma_{i/\sigma} f(x_i+)\Delta_i g - f(x_0+)\,.\,j_0 g \mid < \varepsilon,$$

and for any such δ a set $\sigma[x_0, x_0+\delta]$ exists such that

$$\left|\int_{x_0}^{x_0+\delta} fdg - f(x_0+)j_0g\right| \leqq \left|\int_{x_0}^{x_0+\delta} fdg - \Sigma_{i/\sigma} f(x_i+)\Delta_i g\right| + \left|\Sigma_{i/\sigma} f(x_i+)\Delta_i g - f(x_0+)j_0 g\right| < 2\varepsilon.$$

Hence

$$j\int_a^x fdg = f(x+)jg.$$

This jump result and the derivative result in the lemma, together, give P_4. This proof of P_4 has an obvious adaption to the case of the LCSσ-integral to give P_5; P_6 follows immediately from the fact that the LM-integral is the mean of the IP- and LCSσ-integrals.

Theorem II. *When f is in D_1 and g is in B^* and $dx = \Delta x$ is relevant to σ, then*

$$\int_a^b fdg = \lim_\sigma \Sigma_\sigma \boldsymbol{fdg}.$$

Proof. To each $\varepsilon>0$, there corresponds a $\sigma_1(\varepsilon)$ such that for all $\sigma \supset \sigma_1$

$$\left|\int_a^b fdg - \Sigma_{i/\sigma}\bar{f}(x_i)\Delta_i g\right| < \varepsilon,$$

and a $\delta_0(\varepsilon)$, independently of x, such that for all $0<\delta<\delta_0$

$$\left|\frac{g(x+\delta)-g(x+)}{\delta} - g'^{+}(x+)\right| < \varepsilon.$$

Let σ_2 be of norm less than δ_0. Then, for all $\sigma \supset \sigma_1+\sigma_2$,

$$\begin{aligned}\left|\int_a^b fdg - \Sigma_{i/\sigma}\bar{f}dg\right| &\leqq \left|\int_a^b fdg - \Sigma\bar{f}\Delta g\right| + \left|\Sigma\bar{f}(\Delta g - dg)\right| \\ &\leqq \left|\int_a^b fdg - \Sigma\bar{f}\Delta g\right| + \left|\Sigma_{i/\sigma}\bar{f}\left(\frac{g(x_{i+1})-g(x_i+)}{\Delta_i x} - g'^{+}(x_i+)\right)\Delta_i x\right| \\ &< \varepsilon + F_0(b-a)\varepsilon,\end{aligned}$$

where $F_0 > |\bar{f}(x)|$ on $[ab]$. Thus $\int_a^b fdg = \lim_\sigma \Sigma_\sigma \bar{f}dg$. The theorem follows now from the inequality

$$\left|\int_a^b fdg - \Sigma_\sigma \boldsymbol{fdg}\right| \leqq \left|\int_a^b fdg - \Sigma_\sigma \bar{f}dg\right| + \Sigma_{i/\sigma}\left|\bar{f}dg - \boldsymbol{fdg}\right|,$$

from the equation $\boldsymbol{fdg} = \bar{f}dg + \tfrac{1}{2}jf\cdot g'^{+}(x+)dx$, from the boundedness of $g'^{+}(x+)$ and since the kth jump-abscissa of $f(x)$ can be covered by an interval of length $\frac{\varepsilon}{2^k}$.

We may use the symbol $\int_a^b \boldsymbol{f}d\boldsymbol{g}$ for the right-hand member of the just established theorem and have the *corollary*: $\int_a^b dg = \int_a^b jdg$. But it is to be noted that $\int_a^b d\boldsymbol{g} = \boldsymbol{g}(b)-\boldsymbol{g}(a)$ is false.

5. Integration-by-parts

In the case of the IP-integral, if both functions are in B and one or other is continuous on the left at each common point of discontinuity, then the integration-by-parts formula mentioned in § **1** holds [Ref. 1, § 6.2, p. 274]. The same is true in the case of the LCSσ-integral. But, for functions continuous on the left, the LM-integral is the same as the *mean* integral

$$(SM)\int_a^b fdg \equiv \lim_\sigma \Sigma_{i/\sigma}\tfrac{1}{2}(f(x_i)+f(x_{i+1}))\Delta_i g$$

of H. L. Smith and therefore has the ordinary integration-by-parts formula [Ref. 1, § 7]. Since the definition of the jump-differential, as given, requires continuity on the left in order that $jdg-\Delta g$ be an infinitesimal in Δx at all points, a desideratum, we assume from now on that all functions are continuous on the left and that all integrals are SM-integrals or, what is then the same thing, LM-integrals.

6. The Substitution Theorem

The substitution theorem, in the case where $g(x) = \int_a^x udv$ is known [$\lim_\sigma \Delta f \Delta u \Delta v = 0$ is assumed, vide Ref. 1, § 7]. A generalisation which permits the functions to have simultaneous points of discontinuity, is given when we vectorise the functions, the vector multiplications being properly understood. Ordinary products become vector products. By the *lower product* of any two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is to be understood the scalar function $\Sigma u_i v_i$, *middle product* the vector whose ith element is $u_i v_i$, and *upper product* the square matrix with element $u_i v_j$ in the ith row and jth column. Three vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ have a lower product $\Sigma u_i v_i w_i$, a middle product with ith element $u_i v_i w_i$, but no (defined) upper product. No distinguishing symbols are necessary for our purpose as the kind of product intended can be inferred from the context in all cases. Thus, in the case of the triple-product $\boldsymbol{fudv}$ of Theorem III, the context implies the lower product. But, to avoid ambiguity in a term, we may *diagonalise* a vector $\boldsymbol{u}$, i.e. call $[\boldsymbol{u}]$ the square matrix with the elements of $\boldsymbol{u}$ along the principal diagonal and zeros in the remaining places, and have $\boldsymbol{f}[\boldsymbol{u}]\boldsymbol{dv}$, unambiguously, the lower product.

Theorem III. *When f and u are in D_1, g and v are in B^*, and* $\boldsymbol{dg} = \boldsymbol{udv}$, *then*

$$\int_a^b fdg = \int_a^b \boldsymbol{fudv}.$$

Proof. The result follows from the equation $\int_a^b fdg = \int_a^b \boldsymbol{fdg}$, both formally and in fact, if by $\int_a^b \boldsymbol{fudv}$ we understand $\lim_\sigma \Sigma \boldsymbol{f}[\boldsymbol{u})\boldsymbol{dv}$. For the left member of the integral equation of the theorem has the approximation $\Sigma_\sigma \boldsymbol{fdg}$ and this is equal to $\Sigma_\sigma \boldsymbol{f}[\boldsymbol{u})\boldsymbol{dv}$ for all $\sigma \supset \sigma_0(\varepsilon)$, with dx in both appropriate to σ; the limit of the first exists and so the limit of the second exists. Since $\bar{f} \cdot \bar{u} = \overline{fu} - \frac{1}{4} jf.ju$, it is evident that $\int_a^x fdg = \int_a^x fudv$ if, and only if, at each point on $[ab]$ at least one of the three functions is continuous.

A *corollary*, by P_6, is that $jd\int_a^x fdg = \boldsymbol{f}[\boldsymbol{u})\boldsymbol{dv}$. Another *corollary* is that when $\sum_{k=1}^n f_k \boldsymbol{dg}_k = \boldsymbol{udv}$ holds at all points on $[ab]$ and the integrals exist, then

$$\sum_{k=1}^n \int_a^b f_k \boldsymbol{dg}_k = \int_a^b \boldsymbol{udv}.$$

7. The Fundamental Theorem

By P_6, $\int_a^x fdg$ is a function such that its j-differential $\boldsymbol{fdg}$ exists for all x, $a \leqq x < b$. By Theorem II, the aggregate of all sums $\Sigma \boldsymbol{fdg}$ appropriate to all sets σ on $[ax]$ has the σ-limit $\int_a^x \boldsymbol{fdg}$. With $dx = x_{i+1} - x_i$ appropriate to σ, P_6 and II together imply the essential theorem:

Theorem IV. When f is in D_1 and g is in B^*, then $jd\int_a^x \boldsymbol{fdg} = \boldsymbol{fdg}$.

Taking the j-differential of the integral of the vectorially weighted vector-differential regains the vectorially weighted vector-differential appropriate to the variable terminal x of integration.

The reciprocity, by virtue of the corollary to Theorem III;

$$jdy = \boldsymbol{f}(x)\boldsymbol{dg}(x) \qquad \longleftrightarrow \qquad y(x) - y(a) = \int_a^x \boldsymbol{fdg}$$

is now seen to hold.

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