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RESEARCH ARTICLE

Spectral Polyhedra

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Abstract

A *spectral convex set* is a collection of symmetric matrices whose range of eigenvalues forms a symmetric convex set. Spectral convex sets generalize the Schur-Horn orbitopes studied by Sanyal–Sottile–Sturmfels (2011). We study this class of convex bodies, which is closed under intersections, polarity and Minkowski sums. We describe orbits of faces and give a formula for their Steiner polynomials. We then focus on spectral polyhedra. We prove that spectral polyhedra are spectrahedra and give small representations as spectrahedral shadows. We close with observations and questions regarding hyperbolicity cones, polar convex bodies and spectral zonotopes.

Contents

1. Introduction

The symmetric group $\mathfrak{S}_d$ acts on $\mathbb{R}^d$ by permuting coordinates. We call a convex set $K \subset \mathbb{R}^d$ **symmetric** if $\sigma K = K$ for all $\sigma \in \mathfrak{S}_d$. We write $\mathrm{S}_2\mathbb{R}^d$ for the $\binom{d+1}{2}$-dimensional real vector space of symmetric d-by-d matrices. Every real symmetric matrix $A \in \mathrm{S}_2\mathbb{R}^d$ has d real eigenvalues, which we denote by $\lambda(A) \in \mathbb{R}^d$. In this note, we are concerned with **spectral convex sets**, which are sets of the form

$$\Lambda(K) \;:=\; \{A \in \mathrm{S}_2\mathbb{R}^d : \lambda(A) \in K\}, \tag{1.1}$$

where K is a symmetric convex set. The name is justified by Corollary 2.2, which asserts that $\Lambda(K)$ is indeed a convex subset of $\mathrm{S}_2\mathbb{R}^d$.

The simplest symmetric convex sets are of the form $\Pi(p) = \mathrm{conv}\{\sigma p : \sigma \in \mathfrak{S}_d\}$ for $p \in \mathbb{R}^d$. Such a symmetric polytope is called a **permutahedron** [8], and the associated spectral convex sets $\mathcal{SH}(p) := \Lambda(\Pi(p))$ were studied in [23] under the name **Schur-Horn orbitopes**. The class of spectral convex sets is strictly larger. For example, for $1 \le p \le \infty$, the unit p-norm ball in $\mathbb{R}^d$ is a symmetric convex set. The associated spectral convex set is the unit **Schatten p-norm ball** in $S_2\mathbb{R}^d$, consisting of $d \times d$ symmetric matrices with eigenvalues having p-norm at most one. It follows that the spectral convex set associated with the cube in $\mathbb{R}^d$ is the spectral norm ball in $S_2\mathbb{R}^d$, the spectral convex set associated with the octahedron in $\mathbb{R}^d$ is the nuclear norm ball in $S_2\mathbb{R}^d$, and the spectral convex set associated with the Euclidean norm ball is the Frobenius norm ball.

In Section 2, we summarize some basic, yet remarkable, geometric and algebraic properties of spectral convex sets. In particular, we observe that spectral convex sets are closed under intersections, Minkowski sums and polarity.

A **spectrahedron** is a convex set $S \subset \mathbb{R}^d$ of the form

$$S = \{x \in \mathbb{R}^d : A_0 + x_1 A_1 + \cdots + x_d A_d \succeq 0\}, \tag{1.2}$$

where $A_0, A_1, \ldots, A_d$ are symmetric matrices and $\succeq 0$ denotes positive semidefiniteness. Polyhedra are special cases of spectrahedra, since any polyhedron can be expressed in the form (1.2) with all of the A_i being diagonal matrices. Just as polyhedra arise as the feasible regions of linear programs, spectrahedra arise as the feasible regions of the more general class of semidefinite programs [28].

In Section 3, we show that **spectral polyhedra** – that is, spectral convex bodies associated to symmetric polyhedra – are spectrahedra (Theorem 3.3), generalizing the construction from [23] for Schur-Horn orbitopes. It follows that spectral polyhedra are basic semialgebraic and are examples of the very special class of *doubly spectrahedral* convex sets (i.e., spectrahedra whose polars are also spectrahedra [25]). Spectral polyhedral cones are hyperbolicity cones (see Section 5 for details). The generalized Lax conjecture asserts that every hyperbolicity cone is spectrahedral. Theorem 3.3, therefore, gives further positive evidence for the generalized Lax conjecture.

If S has a description of the form (1.2) with $n \times n$ symmetric matrices $A_0, A_1, \ldots, A_d$, then we say that S has a **spectrahedral representation of size** n. If P is a symmetric polyhedron with M orbits of defining inequalities, then the size of our spectrahedral representation of $\Lambda(P)$ is $M \cdot \prod_{i=1}^{d} \binom{d}{i}$. A lower bound on the size of a spectrahedral representation is $Md!$, obtained by considering the degree of the algebraic boundary. While spectrahedral representations give insight into the algebraic properties of spectral polyhedra, in order to solve linear optimization problems involving spectral polyhedra, it suffices to give representations as **spectrahedral shadows** (i.e., linear projections of spectrahedra). This is because one can optimize a linear functional over a spectrahedral shadow by lifting the linear functional to the spectrahedron upstairs, solving the resulting semidefinite program, and projecting the solution back into the original space [13]. In Section 4, we use a result of Ben-Tal and Nemirovski [4] to give significantly smaller representations of spectral polyhedra as spectrahedral shadows.

We close in Section 5 with remarks, questions and future directions regarding hyperbolic polynomials and the generalized Lax conjecture, generalizations to other Lie groups, and spectral zonotopes.

2. Spectral convex sets

Denote by $D : S_2\mathbb{R}^d \to \mathbb{R}^d$ the projection onto the diagonal and by $\delta : \mathbb{R}^d \to S_2\mathbb{R}^d$ the embedding into diagonal matrices. Many remarkable properties of spectral convex sets arise because the projection onto the diagonal, and the diagonal section, coincide.

Lemma 2.1. *If K is a symmetric convex set, then*

$$D(\Lambda(K)) = K = D(\Lambda(K) \cap \delta(\mathbb{R}^d)).$$

Before giving a proof, we introduce some notation and terminology. For a point $p \in \mathbb{R}^d$, we write $s_k(p)$ for the sum of its k largest coordinates. Recall that a point $q \in \mathbb{R}^d$ is **majorized** by p, denoted $q \trianglelefteq p$, if

$$\sum_{i=1}^{d} q_i = \sum_{i=1}^{d} p_i \qquad \text{and} \qquad s_k(q) \le s_k(p) \quad \text{for all } k = 1, \ldots, d-1. \tag{2.1}$$

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Majorization relates to permutahedra in that

$$\Pi(p) = \{q \in \mathbb{R}^d : q \trianglelefteq p\}.$$

In other words, the majorization inequalities give an inequality description of the permutahedron [8].

Proof of Lemma 2.1. Since $\Lambda(K)$ contains $\delta(K)$, the obvious inclusions are that $K \subseteq D(\Lambda(K) \cap \delta(\mathbb{R}^d)) \subseteq D(\Lambda(K))$. To show that $D(\Lambda(K)) \subseteq K$, we use Schur's insight (see, for example, [16, Theorem 4.3.45]) that for any $A \in \mathrm{S}_2\mathbb{R}^d$, we have $D(A) \trianglelefteq \lambda(A)$. Furthermore, since K is convex, $\Pi(p) \subseteq K$ for any $p \in K$. From these observations, we infer that if $A \in \Lambda(K)$, then $D(A) \in \Pi(\lambda(A)) \subseteq K$. □

Lemma 2.1 yields that spectral convex sets are, in fact, convex.

Corollary 2.2. *If K is a symmetric convex set, then* $\Lambda(K)$ *is convex.*

Proof. It is enough to show that $\mathrm{conv}(\Lambda(K)) \subseteq \Lambda(K)$. Assume that $A \in \mathrm{conv}(\Lambda(K))$. We can assume that $A = \delta(p)$ for some $p \in \mathbb{R}^d$. By definition, there are $A_1, \dots, A_m \in \Lambda(K)$ such that $\delta(p) = \sum_{i=1}^m \mu_i A_i$ with $\mu_i \geq 0$ and $\mu_1 + \cdots + \mu_m = 1$. In particular, $p = D(A) = \sum_i \mu_i D(A_i)$ and Lemma 2.1 yields $p \in K$. It follows that $A \in \Lambda(K)$. □

We identify the dual space $(\mathrm{S}_2\mathbb{R}^d)^*$ with $\mathrm{S}_2\mathbb{R}^d$ via the Frobenius inner product $\langle A, B\rangle := tr(AB)$. The **support function** of a closed convex set K is defined by

$$h_K(c) := \sup\{\langle c, p\rangle : p \in K\}.$$

Proposition 2.3. *If* $K \subset \mathbb{R}^d$ *is a symmetric closed convex set, then* $h_{\Lambda(K)}(B) = h_K(\lambda(B))$ *for all* $B \in \mathrm{S}_2\mathbb{R}^d$.

Proof. Let $B = gB'g^t$ for $g \in O(d)$ and B' diagonal. Using the fact that the trace is invariant under cyclic shifts, we see that $h_{\Lambda(K)}(B) = h_{\Lambda(K)}(B')$. Lemma 2.1 and the fact that $\langle A, B'\rangle = \langle D(A), D(B')\rangle$ finishes the proof. □

Proposition 2.3, like many of the convex analytic facts in this section, can be deduced from results of Lewis on extended real-valued spectral functions [20]. If $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is an extended real-valued symmetric function, then $f_{\mathcal{H}}(A) := f(\lambda(A))$ is the associated **spectral function**. The **Fenchel conjugate** of a function g is $g^*(c) := \sup_p \langle c, p\rangle - g(p)$. Observe that the support function $h_K(\cdot)$ of a closed convex set is the Fenchel conjugate of the indicator function $\iota_K(\cdot)$ that takes value 0 for points in K and value $+\infty$ for points not in K. For a symmetric function f, the fundamental relation $f^*_{\mathcal{H}} = (f^*)_{\mathcal{H}}$ holds [20, Theorem 2.3]. Applying this to the indicator function of a symmetric closed convex set yields Proposition 2.3.

An **exposed face** of a convex set K is a subset of the form $\{p \in K : \langle c, p\rangle = h_K(c)\}$ for some c. Geometrically, a (proper) exposed face is a subset of K that arises as the intersection of K and a hyperplane that supports K. Exposed faces of $\Lambda(K)$ and K come in $O(d)$- and $\mathfrak{S}_d$-orbits, respectively. The collection of exposed faces up to symmetry is a partially ordered set with respect to inclusion that we denote by $\overline{\mathcal{F}}(\Lambda(K))$ and $\overline{\mathcal{F}}(K)$, respectively. Proposition 2.3 allows us to deduce the following relationship between $\overline{\mathcal{F}}(\Lambda(K))$ and $\overline{\mathcal{F}}(K)$.

Corollary 2.4. *For any symmetric convex body* $K \subset \mathbb{R}^d$*, the posets* $\overline{\mathcal{F}}(K)$ *and* $\overline{\mathcal{F}}(\Lambda(K))$ *are canonically isomorphic.*

The **polar** of a convex set $K \subset \mathbb{R}^d$ is defined as

$$K^\circ := \{c \in \mathbb{R}^d : h_K(c) \leq 1\}.$$

It is easy to see that the polar of a symmetric convex set is symmetric. In combination with Proposition 2.3, we can deduce that the class of spectral convex sets is closed under polarity.

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Theorem 2.5. *If K is a closed symmetric convex set, then $\Lambda(K)^\circ = \Lambda(K^\circ)$.*

Proof. For $B \in S_2\mathbb{R}^d$, we have $B \in \Lambda(K)^\circ$ if and only if $1 \geq h_{\Lambda(K)}(B) = h_K(\lambda(B))$, which happens if and only if $\lambda(B) \in K^\circ$. □

Furthermore, since polyhedra are also closed under polarity, it follows that the class of spectral polyhedra is closed under polarity.

Proposition 2.3 can also be used to show that spectral convex bodies interact nicely with Minkowski sums.

Corollary 2.6. *If $K, L \subset \mathbb{R}^d$ are symmetric convex bodies, then $\Lambda(K + L) = \Lambda(K) + \Lambda(L)$.*

Proof. We compute

$$\begin{aligned} h_{\Lambda(K)+\Lambda(L)}(B) &= h_{\Lambda(K)}(B) + h_{\Lambda(L)}(B) = h_K(\lambda(B)) + h_L(\lambda(B)) \\ &= h_{K+L}(\lambda(B)) = h_{\Lambda(K+L)}(B). \end{aligned}$$ □

We can use this property to simplify the computation of basic convex-geometric invariants; cf. the book by Schneider [26]. Let $B(\mathbb{R}^d)$ denote the Euclidean unit ball in $\mathbb{R}^d$. The **Steiner polynomial** of a convex body $K \subset \mathbb{R}^d$ is

$$\mathrm{vol}(K + tB(\mathbb{R}^d)) = W_d(K) + dW_{d-1}(K)t + \cdots + \tbinom{d}{d} W_0(K)t^d.$$

The coefficients $W_i(K)$ are called **quermaßintegrals**. The following reduces the computation of Steiner polynomials of $\Lambda(K)$ to the computation of an integral over K.

Theorem 2.7. *Let $K \subset \mathbb{R}^d$ be a symmetric convex body. Then*

$$\mathrm{vol}(\Lambda(K) + tB(S_2\mathbb{R}^d)) = 2^{\frac{1}{2}d(d+3)} \prod_{r=1}^{d} \frac{\pi^{\frac{r}{2}}}{\Gamma(\frac{r}{2})} \int_{K+tB_d} \prod_{i<j} |p_j - p_i| \, dp.$$

Proof. Recall from the introduction that the unit ball in $S_2\mathbb{R}^d$ satisfies $B(S_2\mathbb{R}^d) = \Lambda(B(\mathbb{R}^d))$. In particular, using Corollary 2.6, we need to determine the volume of $\Lambda(K + tB(\mathbb{R}^d))$.

Let $\varphi : O(d) \times \mathbb{R}^d \to S_2\mathbb{R}^d$ with $\varphi(g, p) := g\delta(p)g^t$. Then by Corollary 2.2, we need to compute $\int_{\varphi(O(d)\times K')} d\mu$, where $K' := K + tB(\mathbb{R}^d)$.

The differential at $(g, p) \in O(d) \times \mathbb{R}^d$ is the linear map $D_{g,p} : T_g O(d) \times T_p\mathbb{R}^d \to T_{\varphi(g,p)} S_2\mathbb{R}^d$ with

$$D_{g,p}\varphi(Bg, u) = [g\delta(p)g^t, B] + gD(u)g^t,$$

where [,] is the Lie bracket. Now, the linear spaces $T_g O(d) \times T_p\mathbb{R}^d$ and $T_{\varphi(g,p)} S_2\mathbb{R}^d$ have the same dimension. If $g = (g_1, g_2, \ldots, g_d) \in O(d)$, then we choose as a basis for the former $g_i \wedge g_j := g_i g_j^t - g_j g_i^t \in T_g O(d)$ for $1 \leq i < j \leq d$ and the standard basis $e_1, \ldots, e_d \in T_p\mathbb{R}^d = \mathbb{R}^d$. For the latter, we choose $g_i \bullet g_j = \frac{1}{2}(g_i g_j^t + g_j g_i^t)$ for $1 \leq i < j \leq d$ and $g_i \bullet g_i$ for $i = 1, \ldots, d$. We then compute

$$D_{g,p}(g_i \wedge g_j) = (p_j - p_i)\, g_i \bullet g_j \quad \text{and} \quad D_{g,p}(e_i) = g_i \bullet g_i.$$

Hence, under the identification $g_i \wedge g_j \mapsto g_i \bullet g_j$ and $e_i \mapsto g_i \bullet g_i$, $D_{g,p}\varphi$ has eigenvalues $p_j - p_i$ for $i < j$ as well as 1 with multiplicity d. This yields

$$\int_{\varphi(O(d)\times K')} d\mu = \int_{O(d)\times K'} |\det D_{g,p}\varphi| \, dg dp = \int_{O(d)} dg \int_{K'} \prod_{i<j} |p_j - p_i| \, dp.$$

Together with Hurwitz formula for the volume of $O(d)$, this yields the claim. □

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The **algebraic boundary** $\partial_{\mathrm{alg}}K$ of a full-dimensional closed convex set $K \subset \mathbb{R}^d$ is, up to scaling, the unique polynomial $f_K \in \mathbb{R}[x_1, \dots, x_d]$ of minimal degree that vanishes on all points $q \in \partial K$; see [27] for more information. Throughout, we assume K is semialgebraic. If K is symmetric, then f_K is a symmetric polynomial – that is, $f_K(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(d)}) = f_K(x_1, \dots, x_d)$ for all $\sigma \in \mathfrak{S}_d$. By the fundamental theorem of symmetric polynomials, there is a polynomial $F_K(y_1, \dots, y_d) \in \mathbb{R}[y_1, \dots, y_d]$ such that $f_K(x_1, \dots, x_d) = F_K(e_1, \dots, e_d)$, where e_i is the i-th elementary symmetric polynomial.

For $A \in \mathrm{S}_2\mathbb{R}^d$, let $\det(A + tI) = t^d + \eta_1(A)t^{d-1} + \cdots + \eta_d(A)$ be its characteristic polynomial. The coefficients $\eta_i(A)$ are polynomials in the entries of A, and it is easy to see that $\eta_i(gAg^t) = \eta_i(A)$. In fact, every polynomial h such that $h(gAg^t) = h(A)$ for all $g \in O(d)$ and $A \in \mathrm{S}_2\mathbb{R}^d$ can be written as a polynomial in $\eta_1, \dots, \eta_d$; see [14, Ch. 12.5.3].

Proposition 2.8. *Let $K \subset \mathbb{R}^d$ be a full-dimensional symmetric closed convex set. Then the algebraic boundary of $\Lambda(K)$ is given by $F_K(\eta_1, \dots, \eta_d)$. In particular, $\partial_{\mathrm{alg}}K$ and $\partial_{\mathrm{alg}}\Lambda(K)$ have the same degree.*

Proof. A point $A \in \Lambda(K)$ is in the boundary if and only if $\lambda(A) \in \partial K$. Thus, $\partial_{\mathrm{alg}}\Lambda(K)$ is invariant under the action of $O(d)$ by conjugation and hence can be written as a polynomial $F(\eta_1, \dots, \eta_d)$. For any (symmetric) matrix A, $\eta_i(A) = e_i(\lambda(A))$ for $i = 1, \dots, d$. Thus, $F_K(\eta_1, \dots, \eta_d)$ is a polynomial that vanishes on the boundary of $\Lambda(K)$. To see that it is of minimal degree, we note $\partial_{\mathrm{alg}}\Lambda(K)$ vanishes on $\partial\Lambda(K) \cap \delta(\mathbb{R}^d) \cong \partial K$. Since the collection of polynomials e_i and η_i are algebraically independent with corresponding degrees, this implies that $\partial_{\mathrm{alg}}\Lambda(K) = F(\eta_1, \dots, \eta_d)$ has degree as least as large as $\partial_{\mathrm{alg}}K = F_K(e_1, \dots, e_d)$. □

3. Spectrahedra

In this section, we show that spectral polyhedra are spectrahedra. For $P = \Pi(p)$ a permutahedron and $\mathcal{SH}(p) = \Lambda(P)$, a Schur-Horn orbitope, this was shown in [23]. We briefly recall the construction, which will then be suitably generalized.

A point $q \in \mathbb{R}^d$ is contained in $\Pi(p)$ if and only if $q \trianglelefteq p$. This condition can be rewritten in terms of linear inequalities. For $I \subseteq [d]$, we write $q(I) = \sum_{i \in I} q_i$. Recall that for a point $p \in \mathbb{R}^d$, we write $s_k(p)$ for the sum of its k largest coordinates. Then $q \trianglelefteq p$ if and only if

$$s_d(p) \;=\; q([d]) \qquad \text{and} \qquad s_{|I|}(p) \;\geq\; q(I) \qquad \text{for all } \varnothing \neq I \subsetneq [d].$$

If p is generic – that is, $p_i \neq p_j$ for $i \neq j$ – then it is easy to show that the system of $2^d - 2$ linear inequalities is irredundant.

For $1 \leq k \leq d$, the **k-th linearized Schur functor** $\mathcal{L}_k$ is a linear map from $\mathrm{S}_2\mathbb{R}^d$ to $\mathrm{S}_2 \bigwedge^k \mathbb{R}^d$ such that the eigenvalues of $\mathcal{L}_k(A)$ are precisely $\lambda(A)(I) = \sum_{i \in I} \lambda(A)_i$ for $I \subseteq [d]$ and $|I| = k$. Therefore, $\mathcal{SH}(p)$ is precisely the set of points $A \in \mathrm{S}_2\mathbb{R}^d$ such that

$$s_d(p) \;=\; tr(A) \qquad \text{and} \qquad s_k(p)\, I_{\binom{d}{k}} \;\succeq\; \mathcal{L}_k(A) \qquad \text{for all } 1 \leq k < d. \tag{3.1}$$

The simplest symmetric polyhedron has the form

$$P_{a,b} \;=\; \{x \in \mathbb{R}^d : \langle \sigma a, x \rangle \leq b \text{ for } \sigma \in \mathfrak{S}_d\},$$

where $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$. In general, a symmetric polyhedron has the form

$$P \;=\; \{x \in \mathbb{R}^d : \langle \sigma a_i, x \rangle \leq b_i \text{ for } \sigma \in \mathfrak{S}_d \text{ and } i = 1, \dots, M\} \;=\; \bigcap_{i=1}^{M} P_{a_i, b_i}.$$

Since $\Lambda(K \cap L) \;=\; \Lambda(K) \cap \Lambda(L)$, it suffices to focus on the case $P_{a,b}$.

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To extend the representation (3.1) directly, for each general $a \in \mathbb{R}^d$, we would need a linear map $\mathcal{L}_a$ from $\mathrm{S}_2\mathbb{R}^d$ to S_2V with $\dim V = d!$ such that the eigenvalues of $\mathcal{L}_a(A)$ are precisely $\langle \sigma a, \lambda(A)\rangle$ for all $\sigma \in \mathfrak{S}_d$. For $a = (1, \ldots, 1, 0, \ldots, 0)$ with k ones, this is realized by the linearized Schur functors.

Proposition 3.1. *For $d = 2$, set*

$$\mathcal{L}_a(A) \;:=\; a_1 A + a_2\, adj(A)\,,$$

where $adj(A)$ is the adjugate (or cofactor) matrix. Then $A \mapsto \mathcal{L}_a(A)$ is a linear map satisfying the above requirements.

Proof. Since $d = 2$, the map $A \mapsto adj(A)$ is linear. The matrices A and $adj(A)$ can be simultaneously diagonalized, and hence, it suffices to assume that $A = \delta(\lambda_1, \lambda_2)$. In that case, $adj(A) = \delta(\lambda_2, \lambda_1)$, which proves the claim. □

The construction above only works for $d = 2$, and we have not been able to construct such a map for $d \geq 3$.

Question 1. Does $\mathcal{L}_a$ exist for $d \geq 3$?

We pursue a different approach toward a spectrahedral representation by considering a redundant set of linear inequalities for $P_{a,b}$. An ordered collection $\mathcal{I} = (I_1, \ldots, I_d)$ of subsets $I_j \subseteq [d]$ is called a **numerical chain** if $|I_j| = j$ for all j. A numerical chain is a **chain** if additionally $I_1 \subset I_2 \subset \cdots \subset I_d$. Chains are in bijection to permutations $\sigma \in \mathfrak{S}_d$ via $I_j = \{\sigma(1), \ldots, \sigma(j)\}$. For $I \subseteq [d]$, we write $\mathbf{1}_I \in \{0,1\}^d$ for its characteristic vector.

Let us assume that $a = (a_1 \geq a_2 \geq \cdots \geq a_d)$, and set $a_{d+1} := 0$. For a numerical chain $\mathcal{I}$, we define

$$a^{\mathcal{I}} \;:=\; (a_1 - a_2)\mathbf{1}_{I_1} + (a_2 - a_3)\mathbf{1}_{I_2} + \cdots + (a_{d-1} - a_d)\mathbf{1}_{I_{d-1}} + a_d \mathbf{1}_{I_d}. \tag{3.2}$$

Proposition 3.2. *Let $a = (a_1 \geq a_2 \geq \cdots \geq a_d)$ and $b \in \mathbb{R}$. Then*

$$P_{a,b} \;=\; \{x \in \mathbb{R}^d : \langle a^{\mathcal{I}}, x\rangle \leq b \text{ for all numerical chains } \mathcal{I}\}.$$

Proof. Let Q denote the right-hand side. To see that $Q \subseteq P_{a,b}$, we note that if $\mathcal{I}$ is a chain corresponding to a permutation σ, then $a^{\mathcal{I}} = \sigma a$.

For the reverse inclusion, it suffices to show that $a^{\mathcal{I}} \trianglelefteq a$, which implies that $\langle a^{\mathcal{I}}, x\rangle \leq b$ is a valid inequality for $P_{a,b}$. Using the fact that $s_k(p + q) \leq s_k(p) + s_k(q)$, we compute

$$s_k(a^{\mathcal{I}}) \leq \sum_{j=1}^{d} (a_j - a_{j+1}) s_k(\mathbf{1}_{I_j}) = \sum_{j=1}^{k-1} j(a_j - a_{j+1}) + k \sum_{j=k}^{d} (a_j - a_{j+1}) = a_1 + \cdots + a_k = s_k(a).$$

Similarly, $s_d(a^{\mathcal{I}}) = a_1 + \cdots + a_d$, which completes the proof. □

Recall that for matrices $A \in \mathrm{S}_2\mathbb{R}^d$ and $B \in \mathrm{S}_2\mathbb{R}^e$, the tensor product $A \otimes B$ is a symmetric matrix of order de with eigenvalues $\lambda_i(A) \cdot \lambda_j(B)$ for $i = 1, \ldots, d$ and $j = 1, \ldots, e$. For $a = (a_1 \geq \cdots \geq a_d)$, let

$$\widehat{\mathcal{L}}_a : \textstyle\bigwedge^1 \mathbb{R}^d \otimes \bigwedge^2 \mathbb{R}^d \otimes \cdots \otimes \bigwedge^d \mathbb{R}^d \;\to\; \bigwedge^1 \mathbb{R}^d \otimes \bigwedge^2 \mathbb{R}^d \otimes \cdots \otimes \bigwedge^d \mathbb{R}^d$$

be the linear map given by

$$\widehat{\mathcal{L}}_a(A) \;:=\; \sum_{j=1}^{d} (a_j - a_{j+1}) I_{\binom{d}{1}} \otimes \cdots \otimes I_{\binom{d}{j-1}} \otimes \mathcal{L}_j(A) \otimes I_{\binom{d}{j+1}} \otimes \cdots \otimes I_{\binom{d}{d}}.$$

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Theorem 3.3. *Let $P = P_{a_1,b_1} \cap \cdots \cap P_{a_M,b_M}$ be a symmetric polyhedron. Then $A \in \Lambda(P)$ if and only if*

$$b_i\, I \succeq \widehat{\mathcal{L}}_{a_i}(A) \text{ for } i = 1, 2, \ldots, M.$$

Proof. Since $\Lambda(P) = \bigcap_{i=1}^{M} \Lambda(P_{a_i,b_i})$, it is enough to show that $A \in \Lambda(P_{a,b})$ if and only if $bI \succeq \widehat{\mathcal{L}}_a(A)$.

Let $a = (a_1 \geq a_2 \geq \cdots \geq a_d)$ and $A \in \mathrm{S}_2\mathbb{R}^d$ with $v_1, \ldots, v_d$ an orthonormal basis of eigenvectors. For $I = \{i_1 < i_2 < \cdots < i_k\}$ a subset of $[d]$, we write $v_I := v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k} \in \bigwedge^k \mathbb{R}^d$. Then a basis of eigenvectors for $\widehat{\mathcal{L}}_a(A)$ is given by

$$v_{\mathcal{I}} \; := \; v_{I_1} \otimes v_{I_2} \otimes \cdots \otimes v_{I_d}\,,$$

where $\mathcal{I}$ ranges of all numerical chains. The eigenvalue of $\widehat{\mathcal{L}}_a(A)$ corresponding to $v_{\mathcal{I}}$ is precisely $\langle a^{\mathcal{I}}, \lambda(A)\rangle$. Hence, A satisfies the given linear matrix inequalities for a if and only if $\sum_i \lambda_i(A) = \sum_i a_i$ and $\langle a^{\mathcal{I}}, \lambda(A)\rangle \leq b$ for all $\mathcal{I}$. By Proposition 3.2, this is the case if and only if $\lambda(A) \in P_{a,b}$ or, equivalently, $A \in \Lambda(P_{a,b})$. □

The spectrahedral representation given in Theorem 3.3 for $\Lambda(P)$, where P is a symmetric polyhedron in $\mathbb{R}^d$ with M orbits of facets, is of size

$$M \cdot \prod_{i=1}^{d} \binom{d}{i}.$$

So the spectrahedral representation is of order $M2^{d^2}$; see [19].

If

$$K \;=\; \{x \in \mathbb{R}^d : A_0 + x_1 A_1 + \cdots + x_d A_d \succeq 0\}$$

is a spectrahedral representation of a convex set K with $A_0, \ldots, A_d \in \mathrm{S}_2\mathbb{R}^m$ and A_0 positive definite, then $h(x) = \det(A_0 + x_1 A_1 + \cdots + x_d A_d)$ vanishes on ∂K. Hence, the size of a spectrahedral representation is bounded from below by the degree of $\partial_{\mathrm{alg}} K$. If P is a symmetric polytope with M full orbits of facets, then its algebraic boundary has degree $M \cdot d!$. From the discussion following Proposition 2.8, we can deduce that the degree of $\partial_{\mathrm{alg}}\Lambda(P)$ is also $M \cdot d!$, and so that any spectrahedral representation of $\Lambda(P)$ has size at least $M \cdot d!$. While interesting from an algebraic point of view, spectrahedral representations of symmetric polytopes are clearly impractical for computational use. In the next section, we discuss substantially smaller representations as projections of spectrahedra.

4. Spectrahedral shadows

In this section, we give a representation of $\Lambda(K)$ as a spectrahedral shadow (i.e., a linear projection of a spectrahedron) when K is, itself, a symmetric spectrahedral shadow, by a direct application of results from [4]. The aim of this section is to illustrate the significant reductions in size possible by using projected spectrahedral representations.

It is convenient to use slightly different notation in this section, to emphasize that we do not need to construct an explicit representation of the symmetric convex set K, to get a representation of $\Lambda(K)$. To this end, let $\mathbb{R}^d_{\downarrow} = \{p \in \mathbb{R}^d \;:\; p_1 \geq p_2 \geq \cdots \geq p_d\}$. For $L \subseteq \mathbb{R}^d_{\downarrow}$, define

$$\Pi(L) = \operatorname{conv}(\mathfrak{S}_d \cdot L),$$

the convex hull of the orbit of L under $\mathfrak{S}_d$. This is the inclusion-wise minimal symmetric convex set containing L. We recover the usual permutahedron of a point $p \in \mathbb{R}^d_{\downarrow}$ by $\Pi(p)$.

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In Theorem 4.2, we give a representation of $\Lambda(\Pi(L))$ as a spectrahedral shadow whenever $L \subseteq \mathbb{R}^d_{\downarrow}$ is a spectrahedral shadow. We use the following result of Ben-Tal and Nemirovski [4, Section 4.2, 18c].

Lemma 4.1. *Let $1 < k < d$ and $t \in \mathbb{R}$. Then a matrix $A \in \mathrm{S}_2\mathbb{R}^d$ satisfies $s_k(\lambda(A)) \le t$ if and only if there are $Z \in \mathrm{S}_2\mathbb{R}^d$ and $s \in \mathbb{R}$ such that*

$$Z \succeq 0, \quad Z - A + sI_d \succeq 0, \quad \textit{and} \quad t - ks - tr(Z) \ge 0.$$

For the case $k = 1$, we obtain the simpler representation $s_1(\lambda(A)) = \max \lambda(A) \le t$ if and only if $tI - A \succeq 0$.

Theorem 4.2. *If $L \subseteq \mathbb{R}^d_{\downarrow}$ is convex, then*

$$\Lambda(\Pi(L)) = \{A \in \mathrm{S}_2\mathbb{R}^d \ : \ \exists p \in L \text{ such that } \lambda(A) \trianglelefteq p\}. \tag{4.1}$$

If $L \subseteq \mathbb{R}^d_{\downarrow}$ is the projection of a spectrahedron of size r, then $\Lambda(\Pi(L))$ is the projection of a spectrahedron of size $r + 2d^2 - 2d - 2$.

Proof. Let C denote the right-hand side of (4.1). We first show that C is convex and is the projection of a spectrahedron of size $r + 2d^2 - 2d - 2$. Since $p \in L \subseteq \mathbb{R}^d_{\downarrow}$, we can write $s_k(p) = \sum_{i=1}^k p_i$, which is linear in p. Then, using Lemma 4.1, the conditions $tr(A) = \sum_i p_i$ and $s_k(\lambda(A)) \le \sum_{i=1}^k p_i$ for $1 \le k \le d-1$ define a convex set in A and p. Moreover, this set can be encoded by linear matrix inequalities involving matrices of size $(d-2)(2d+1) + d$, for a total size of $r + (d-2)(2d+1) + d = r + 2d^2 - 2d - 2$.

To check that $\Lambda(\Pi(L)) = C$, since both sides are spectral convex sets, it is enough to check that their diagonal projections are equal. Since $\Pi(L)$ is symmetric, $D(\Pi(L)) = \Pi(L)$. The diagonal projection $D(C)$ is a symmetric convex set containing L, so $D(C) \supseteq \Pi(L)$. For the reverse inclusion, if $A \in C$, then there exists $p \in L$ such that $\lambda(A) \trianglelefteq p$, but then $A \in \Lambda(\Pi(p)) \subseteq \Lambda(\Pi(L))$. □

We now specialize to the case of $\Lambda(P)$ where P is a symmetric polyhedron with the origin in its interior.

Proposition 4.3. *Suppose that $P \subseteq \mathbb{R}^d$ is a symmetric polyhedron with M orbits of facets that contains the origin in its interior. Then $\Lambda(P)$ is the projection of a spectrahedron of size $M + 2d^2 - 2d - 2$.*

Proof. We will argue that $\Lambda(P^\circ) = \Lambda(P)^\circ$ is the projection of a spectrahedron of size $M + 2d^2 - 2d - 2$ and then appeal to the fact that if C has a projected spectrahedral representation, then C° has a representation as a projection of a spectrahedron of the same size [15, Proposition 1]. By our assumptions on P, we have that $(\Lambda(P)^\circ)^\circ = \Lambda(P)$.

Since the origin is in the interior of P, we know that P° is a symmetric polytope with M orbits of vertices. Each orbit of vertices meets $\mathbb{R}^d_{\downarrow}$, and thus, $\Lambda(P) = \Lambda(\Pi(\{v_1, \dots, v_M\}))$ for some $v_1, \dots, v_M \in \mathbb{R}^d_{\downarrow}$. Let $L = \operatorname{conv}\{v_1, \dots, v_M\} \subseteq \mathbb{R}^d_{\downarrow}$, and note that

$$L = \{\mu_1 v_1 + \cdots + \mu_M v_M \ : \ \mu_1, \dots, \mu_M \ge 0, \ \mu_1 + \cdots + \mu_M = 1\}$$

gives a representation of L as the projection of a polyhedron with M facets, and so a representation as the projection of a spectrahedron of size M. Finally, since $\Pi(L) = \Pi(\{v_1, \dots, v_M\})$, it follows from Theorem 4.2 applied to $\Lambda(\Pi(L))$ that $\Lambda(P)^\circ = \Lambda(P^\circ)$ is the projection of a spectrahedron of size $M + 2d^2 - 2d - 2$. □

5. Remarks, questions and future directions

Hyperbolicity cones and the generalized Lax conjecture

A multivariate polynomial $f \in \mathbb{R}[x_1, \dots, x_d]$, homogeneous of degree m, is **hyperbolic** with respect to $e \in \mathbb{R}^d$ if $f(e) \neq 0$ and for each $x \in \mathbb{R}^d$, the univariate polynomial $t \mapsto f_x(t) := f(x - te)$ has only real roots. Associated with (f, e) is a closed convex cone $C_{f,e} \subseteq \mathbb{R}^d$, defined as the set of points $x \in \mathbb{R}^d$ for which all roots of f_x are nonnegative. A major question in convex algebraic geometry, known as the *generalized (set-theoretic) Lax conjecture* (see [29]), asks whether every hyperbolicity cone is a spectrahedron.

If $C = \{x \in \mathbb{R}^d : \langle \sigma a_i, x\rangle \geq 0, \text{ for all } \sigma \in \mathfrak{S}_d \text{ and } i = 1, 2, \dots, M\}$ is a symmetric polyhedral cone containing $e = (1, 1, \dots, 1)$ in its interior, then it is the hyperbolicity cone associated with the degree $M \cdot d!$ symmetric polynomial

$$f(x) = \prod_{i=1}^{M} \prod_{\sigma \in \mathfrak{S}_d} \langle \sigma a_i, x\rangle.$$

The spectral polyhedral cone $\Lambda(C)$ is the hyperbolicity cone associated with the polynomial $F(X) = f(\lambda(X))$ and $e = I \in \mathrm{S}_2\mathbb{R}^d$. This follows from Proposition 2.8 and is a special case of an observation of Bauschke, Güler, Lewis and Sendov [2, Theorem 3.1]. One can view Theorem 3.3 as providing further evidence for the generalized Lax conjecture since it shows that every member of this family of hyperbolicity cones is, in fact, a spectrahedron.

Given a symmetric hyperbolic polynomial f, one natural way to produce a new symmetric hyperbolic polynomial, and an associated symmetric hyperbolicity cone, is to take the directional derivative $D_e f$ in the direction $e = (1, 1, \dots, 1)$, an example of a *Renegar derivative*. This operation commutes with passing to the associated spectral objects. Indeed, taking the Renegar derivative $D_e f$ and then constructing the spectral convex cone $\Lambda(C_{D_e f, e})$ gives the same result as constructing the spectral hyperbolic polynomial $F(X) = f(\lambda(X))$ and then taking the hyperbolicity cone of $D_I F$, the Renegar derivative in the direction $I \in \mathrm{S}_2\mathbb{R}^d$. For example, the hyperbolicity cones associated with the elementary symmetric polynomials are symmetric convex cones that arise by repeatedly taking Renegar derivatives starting with $f(x) = x_1 x_2 \cdots x_d$ in the direction $e = (1, 1, \dots, 1)$. Brändén [10] established that these cones are all spectrahedral; see also [22, 24]. Building on this result, Kummer [18] has shown that the associated spectral hyperbolicity cones are also spectrahedral.

Categories and Adjointness

For a group G acting on a real vector space V, let us write $\mathcal{K}(V)^G$ for the class of G-invariant convex bodies $K \subset V$. We can interpret the construction of spectral bodies as a map

$$\Lambda : \mathcal{K}^{\mathfrak{S}_d}(\mathbb{R}^d) \ \to \ \mathcal{K}^{O(d)}(\mathrm{S}_2\mathbb{R}^d).$$

It follows from Lemma 2.1 that the map that takes $A \in \mathrm{S}_2\mathbb{R}^d$ to $\{\sigma\lambda(A) : \sigma \in \mathfrak{S}_d\}$ extends to a map

$$\lambda : \mathcal{K}^{O(d)}(\mathrm{S}_2\mathbb{R}^d) \ \to \ \mathcal{K}^{\mathfrak{S}_d}(\mathbb{R}^d) \tag{5.1}$$

such that $\lambda \circ \Lambda$ and $\Lambda \circ \lambda$ are the identity maps. It would be very interesting to see if this can be phrased in categorical terms that would explain the reminiscence of adjointness of functors in Proposition 2.3.

Polar convex bodies

In [5, 6], Biliotti, Ghigi and Heinzner generalized the construction of Schur-Horn orbitopes to other (real) semisimple Lie groups, which they called *polar orbitopes*. In particular, they showed that polar orbitopes are facially exposed and faces are again polar orbitopes. Kobert and Scheiderer [17] gave explicit spectrahedral descriptions of polar orbitopes involving the fundamental representations of the associated Lie algebra. It would be interesting to generalize our spectrahedral representations of spectral polyhedra to this setting. A first step was taken in [7], where (5.1) was studied for polar representations.

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Spectral zonotopes

For $z \in \mathbb{R}^d$, we denote the segment with endpoints $-z$ and z by $[-z, z]$. A **zonotope** is a polytope of the form

$$Z \;=\; [-z_1, z_1] + [-z_2, z_2] + \cdots + [-z_m, z_m]\,, \tag{5.2}$$

where $z_1, \ldots, z_m \in \mathbb{R}^d$ and addition is Minkowski sum. Zonotopes are important in convex geometry as well as in combinatorics; see, for example, [3, 9, 12]. For $z \in \mathbb{R}^d$, we obtain a symmetric zonotope

$$Z(z) \;:=\; \sum_{\sigma \in \mathfrak{S}_d} \sigma[-z, z]$$

and for $z = e_1 - e_2 = (1, -1, 0, \ldots, 0)$, the resulting symmetric zonotope is $2(d-2)!\Pi(d-1, d-3, \ldots, -(d-3), -(d-1))$ and thus homothetic to the standard permutahedron $\Pi(1, 2, \ldots, d)$. For $z = e_1$, we obtain a dilate of the unit cube $[0, 1]^d$.

We define **spectral zonotopes** as convex bodies of the form

$$\Lambda(Z(z_1)) + \cdots + \Lambda(Z(z_m))\,,$$

where $Z(z_i)$ are symmetric zonotopes. This class of convex bodies includes the Schur-Horn orbitope $\mathcal{SH}((d-1, d-3, \ldots, -(d-1)))$ as well as symmetric matrices with spectral norm at most one. It follows from Corollary 2.6 that spectral zonotopes are spectral convex bodies, and, in particular, spectral zonotopes form a sub-semigroup (with respect to Minkowski sum) among spectral convex bodies. It would be very interesting to explore the combinatorial, geometric and algebraic properties of spectral zonotopes.

There are a number of remarkable characterizations of zonotopes; cf. [9]. In particular, zonotopes have a simple characterization in terms of their support functions: The support function of a zonotope Z as in (5.2) is given by $h_Z(c) = \sum_{i=1}^m |\langle z_i, c\rangle|$. We obtain the following characterization for spectral zonotopes.

Corollary 5.1. *A convex body $\Omega \subset \mathrm{S}_2\mathbb{R}^d$ is a spectral zonotope if and only if its support function is of the form*

$$h_\Omega(B) \;=\; \sum_{i=1}^m \sum_{\sigma \in \mathfrak{S}_d} |\langle \sigma z_i, \lambda(B)\rangle|\,,$$

for some $z_1, \ldots, z_m \in \mathbb{R}^d$.

The support function for $Z(e_1 - e_2)$ is

$$h_{Z(e_1-e_2)}(c) \;=\; 2(d-2)! \sum_{i<j} |c_i - c_j|.$$

From Proposition 2.3, we infer that the support function of the (standard) Schur-Horn orbitope is

$$h_{\mathcal{SH}(d-1,\ldots,-(d-1))}(B) \;=\; \sum_{i<j} |\lambda(B)_i - \lambda(B)_j| \;=\; \|\mathcal{M}_B\|_*. \tag{5.3}$$

Here, $\|\cdot\|_*$ is the **nuclear norm** – that is, the sum of the singular values – and, for fixed $B \in \mathrm{S}_2\mathbb{R}^d$, $\mathcal{M}_B$ is the linear map from $d \times d$ skew-symmetric matrices to traceless $d \times d$ symmetric matrices defined by $\mathcal{M}_B(X) = [B, X] = BX - XB$, which has non-zero singular values $|\lambda(B)_i - \lambda(B)_j|$ for $1 \le i < j \le d$. The $m_1 \times m_2$ nuclear norm ball has a spectrahedral representation of size $2^{\max\{m_1, m_2\}}$ [25, Theorem 1.2], and a projected spectrahedral representation of size $m_1 + m_2$. These observations show that $\mathcal{SH}(d-1,\ldots,-(d-1))^\circ = \{B \;:\; \|\mathcal{M}_B\|_* \leq 1\}$ has a spectrahedral representation of size $2^{\binom{d+1}{2}-1}$ and a projected spectrahedral representation of size d^2-1.

A convex body $K \subset \mathbb{R}^d$ is a (generalized) **zonoid** if it is the limit (in the Hausdorff metric) of zonotopes, or, equivalently, if its support function is of the form

$$h_K(c) \;=\; \int_{S^{d-1}} |\langle c, u\rangle| \, d\rho(u)\,, \tag{5.4}$$

for some (signed) even measure ρ; see [26, Ch. 3]. It was hoped that spectral zonotopes are zonoids, but this is not the case. Leif Nauendorf [21] showed that the Schur-Horn orbitopes $\mathcal{SH}(d-1,\ldots,-(d-1))$ are never zonoids for $d \geq 3$.

A convex body $K \subset \mathbb{R}^d$ is a symmetric zonoid if and only if the measure ρ in (5.4) is symmetric. We define **spectral zonoids** as those convex bodies with support functions of the form

$$h_\Omega(B) \;=\; \int_{S^{d-1}} |\langle \lambda(B), u\rangle| \, d\rho(u)\,,$$

where ρ is a symmetric even measure. Examples of spectral zonoids include the Schatten p-norm balls in $\mathrm{S}_2\mathbb{R}^d$ when $p \geq 2$. Further examples of spectral zonoids can be found in [1, Section 5.1] (in the Hermitian setting) and [11, Section 5] (in the setting where the singular values of general matrices play the role of eigenvalues of symmetric matrices).

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