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THE GROUP OF AUTOMORPHISMS OF THE LIE ALGEBRA OF DERIVATIONS OF A FIELD OF RATIONAL FUNCTIONS

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Abstract. We prove that the group of automorphisms of the Lie algebra $\mathrm{Der}_K(Q_n)$ of derivations of the field of rational functions $Q_n = K(x_1, \dots, x_n)$ over a field of characteristic zero is canonically isomorphic to the group of automorphisms of the K-algebra Q_n.

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1. Introduction. In this paper, module means a left module, K is a field of characteristic zero and K^* is its group of units, and the following notation is fixed:

- $P_n := K[x_1, \dots, x_n] = \bigoplus_{\alpha\in\mathbb{N}^n} Kx^\alpha$ is a polynomial algebra over K where $x^\alpha := x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ and $Q_n := K(x_1, \dots, x_n)$ is the field of rational functions,
- $G_n := \mathrm{Aut}_{\mathrm{K-alg}}(P_n)$ and $\mathbb{Q}_n := \mathrm{Aut}_{\mathrm{K-alg}}(Q_n)$,
- $\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_n := \frac{\partial}{\partial x_n}$ are the partial derivatives (K-linear derivations) of P_n,
- $D_n := \mathrm{Der}_K(P_n) = \bigoplus_{i=1}^n P_n\partial_i \subseteq E_n := \mathrm{Der}_K(Q_n) = \bigoplus_{i=1}^n Q_n\partial_i$ are the Lie algebras of K-derivations of P_n and Q_n, respectively, where $[\partial, \delta] := \partial\delta - \delta\partial$,
- $\mathbb{G}_n := \mathrm{Aut}_{\mathrm{Lie}}(D_n)$ and $\mathbb{E}_n := \mathrm{Aut}_{\mathrm{Lie}}(E_n)$,
- $\delta_1 := \mathrm{ad}(\partial_1), \dots, \delta_n := \mathrm{ad}(\partial_n)$ are the inner derivations of the Lie algebras D_n and E_n where $\mathrm{ad}(a)(b) := [a, b]$,
- $\mathcal{D}_n := \bigoplus_{i=1}^n K\partial_i$,
- $\mathcal{H}_n := \bigoplus_{i=1}^n KH_i$ where $H_1 := x_1\partial_1, \dots, H_n := x_n\partial_n$,
- for each natural number $n \geq 2$, $\mathfrak{u}_n := K\partial_1 + P_1\partial_2 + \cdots + P_{n-1}\partial_n$ is the *Lie algebra of triangular polynomial derivations* (it is a Lie subalgebra of D_n) and $\mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{u}_n)$ is its group of automorphisms.

THEOREM 1.1. $\mathbb{G}_n = G_n$.

The above result is due to Rudakov [**10**] where a detailed sketch of a proof is given based on his algebro-geometric approach developed in [**9**] (where the groups of automorphisms of infinite dimensional Lie algebras of Cartan type are found). A short proof of Theorem 1.3 is given in [**4**]. The group of automorphisms of (infinite dimensional) algebras of generalized Cartan type were studied by Osborn, [**8**], and Zhao, [**12**]. The group of automorphisms of the *Virasoro* Lie algebra was found in [**5**]. A lot of information about derivations and automorphisms the interested reader can find in the following books [**7**, **6**, **11**].

The aim of the paper is to prove the following theorem.

THEOREM 1.2. $\mathbb{E}_n = \mathbb{Q}_n$.

Structure of the proof.

(i) $\mathbb{Q}_n \subseteq \mathbb{E}_n$ via the group monomorphism (Lemma 2.3 and (3))

$$\mathbb{Q}_n \to \mathbb{E}_n, \quad \sigma \mapsto \sigma : \partial \mapsto \sigma(\partial) := \sigma\partial\sigma^{-1}.$$

(ii) Let $\sigma \in \mathbb{E}_n$. Then $\partial_1' := \sigma(\partial_1), \dots, \partial_n' := \sigma(\partial_n)$ are commuting derivations of Q_n such that $E_n = \bigoplus_{i=1}^n Q_n\partial_i'$ (Lemma 2.12.(2)).

(iii) $\bigcap_{i=1}^n \ker_{Q_n}(\partial_i') = K$ (Lemma 2.12.(1)).

(iv) (crux) There exist elements $x_1', \dots, x_n' \in Q_n$ such that $\partial_i'(x_j') = \delta_{ij}$ for $i, j = 1, \dots, n$ (Lemma 2.12.(3)).

(v) $\sigma(x^\alpha\partial_i) = x'^\alpha\partial_i'$ for all $\alpha \in \mathbb{N}^n$ and $i = 1, \dots, n$ (Lemma 2.12.(6)).

(vi) The K-algebra homomorphism $\sigma' : Q_n \to Q_n$, $x_i \mapsto x_i'$, $i = 1, \dots, n$ is an automorphism such that $\sigma'(q\partial_i) = \sigma'(q)\partial_i'$ for all $q \in Q_n$ and $i = 1, \dots, n$ (Lemma 2.12.(7)).

(vii) $\mathrm{Fix}_{\mathbb{E}_n}(\partial_1, \dots, \partial_n, H_1, \dots, H_n) := \{\tau \in \mathbb{E}_n \mid \tau(\partial_i) = \partial_i), \tau(H_i) = H_i, 1 \le i \le n\} = \{e\}$ (Proposition 2.9.(1)). Hence, $\sigma = \sigma' \in \mathbb{Q}_n$, by (v) and (vi), i.e. $\mathbb{E}_n = \mathbb{Q}_n$. $\square$

THEOREM 1.3. (Theorem 5.3, [**3**]) *The group* $\mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{u}_n)$ *of automorphisms of the Lie algebra* $\mathfrak{u}_n$ *is isomorphic to an iterated semi-direct product of groups* $\mathbb{T}^n \ltimes (\mathrm{UAut}_K(P_n)_n \rtimes (\mathbb{F}_n' \times \mathbb{E}_n))$ *where* $\mathbb{T}^n$ *is an algebraic n-dimensional torus,* $\mathrm{UAut}_K(P_n)_n$ *is an explicit factor group of the group* $\mathrm{UAut}_K(P_n)$ *of unitriangular polynomial automorphisms,* $\mathbb{F}_n'$ *and* $\mathbb{E}_n$ *are explicit groups that are isomorphic, respectively, to the groups* $\mathbb{I}$ *and* $\mathbb{J}^{n-2}$ *where* $\mathbb{I} := (1 + t^2K[[t]], \cdot) \simeq K^{\mathbb{N}}$ *and* $\mathbb{J} := (tK[[t]], +) \simeq K^{\mathbb{N}}$.

Comparing the groups $\mathbb{G}_n$, $\mathbb{E}_n$ and $\mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{u}_n)$, we see that the group $\mathrm{UAut}_K(P_n)_n$ of polynomial automorphisms is a *tiny* part of the group $\mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{u}_n)$ but in contrast $\mathbb{G}_n = G_n$ and $\mathbb{E}_n = \mathbb{Q}_n$.

THEOREM 1.4 ([**1**]). *Every monomorphism of the Lie algebra* $\mathfrak{u}_n$ *is an automorphism.*

Not every epimorphism of the Lie algebra $\mathfrak{u}_n$ is an automorphism. Moreover, there are countably many distinct ideals $\{I_i \mid i \ge 0\}$ such that

$$I_0 = \{0\} \subset I_1 \subset I_2 \subset \cdots \subset I_i \subset \cdots$$

and the Lie algebras $\mathfrak{u}_n/I_i$ and $\mathfrak{u}_n$ are isomorphic (Theorem 5.1.(1), [**2**]).

2. Proof of Theorem 1.2. In this section, a proof of Theorem 1.2 is given. The proof is split into several statements that reflect "Structure of the proof of Theorem 1.2" given in the Introduction.

Let $\mathcal{G}$ be a Lie algebra and $\mathcal{H}$ be its Lie subalgebra. The *centralizer* $C_{\mathcal{G}}(\mathcal{H}) := \{x \in \mathcal{G} \mid [x, \mathcal{H}] = 0\}$ of $\mathcal{H}$ in $\mathcal{G}$ is a Lie subalgebra of $\mathcal{G}$. In particular, $Z(\mathcal{G}) := C_{\mathcal{G}}(\mathcal{G})$ is the *centre* of the Lie algebra $\mathcal{G}$. The *normalizer* $N_{\mathcal{G}}(\mathcal{H}) := \{x \in \mathcal{G} \mid [x, \mathcal{H}] \subseteq \mathcal{H}\}$ of $\mathcal{H}$ in $\mathcal{G}$ is a Lie subalgebra of $\mathcal{G}$, it is the largest Lie subalgebra of $\mathcal{G}$ that contains $\mathcal{H}$ as an ideal.

Let V be a vector space over K. A K-linear map $\delta : V \to V$ is called a *locally nilpotent map* if $V = \bigcup_{i \ge 1} \ker(\delta^i)$ or, equivalently, for every $v \in V$, $\delta^i(v) = 0$ for all $i \gg 1$. When δ is a locally nilpotent map in V we also say that δ *acts locally nilpotently* on V. Every *nilpotent* linear map δ, that is $\delta^n = 0$ for some $n \ge 1$, is a locally nilpotent map but not vice versa, in general. Each element $a \in \mathcal{G}$ determines the derivation of the

Lie algebra $\mathcal{G}$ by the rule $\mathrm{ad}(a) : \mathcal{G} \to \mathcal{G}, b \mapsto [a, b]$, which is called the *inner derivation* associated with a. The set $\mathrm{Inn}(\mathcal{G})$ of all the inner derivations of the Lie algebra $\mathcal{G}$ is a Lie subalgebra of the Lie algebra $(\mathrm{End}_K(\mathcal{G}), [\cdot, \cdot])$ where $[f, g] := fg - gf$. We have the short exact sequence of Lie algebras

$$0 \to Z(\mathcal{G}) \to \mathcal{G} \overset{\mathrm{ad}}{\to} \mathrm{Inn}(\mathcal{G}) \to 0,$$

that is $\mathrm{Inn}(\mathcal{G}) \simeq \mathcal{G}/Z(\mathcal{G})$ where $\mathrm{ad}([a, b]) = [\mathrm{ad}(a), \mathrm{ad}(b)]$ for all elements $a, b \in \mathcal{G}$. An element $a \in \mathcal{G}$ is called a *locally nilpotent element* (respectively, a *nilpotent element*) if so is the inner derivation $\mathrm{ad}(a)$ of the Lie algebra $\mathcal{G}$.

The Lie algebra E_n. Since

$$E_n = \bigoplus_{i=1}^{n} Q_n \partial_i = \bigoplus_{i=1}^{n} Q_n H_i \tag{1}$$

every element $\partial \in E_n$ is a unique sum $\partial = \sum_{i=1}^n a_i\partial_i = \sum_{i=1}^n b_i H_i$ where $a_i = x_i b_i \in Q_n$. The field Q_n is the union $\bigcup_{0 \neq f \in P_n} P_{n,f}$ where $P_{n,f}$ is the localization of P_n at the powers of f. The algebra Q_n is a localization of $P_{n,f}$. Hence $D_{n,f} := \mathrm{Der}_K(P_{n,f}) = \bigoplus_{i=1}^n P_{n,f}\partial_i \subseteq E_n$ and

$$E_n = \bigcup_{0 \neq f \in P_n} D_{n,f}.$$

Q_n is an E_n-module. The field Q_n is a (left) E_n-module: $E_n \times Q_n \to Q_n$, $(\partial, q) \mapsto \partial * q$. In more detail, if $\partial = \sum_{i=1}^n a_i\partial_i$ where $a_i \in Q_n$ then

$$\partial * q = \sum_{i=1}^{n} a_i \frac{\partial q}{\partial x_i}.$$

The E_n-module Q_n is not a simple module since K is an E_n-submodule of Q_n, and

$$\bigcap_{i=1}^{n} \ker_{Q_n}(\partial_i) = K. \tag{2}$$

LEMMA 2.1. *The E_n-module Q_n/K is simple with* $\mathrm{End}_{E_n}(Q_n/K) = K\mathrm{id}$ *where* id *is the identity map.*

Proof. We have to show that for each non-scalar rational function, say $pq^{-1} \in Q_n$, the E_n-submodule M of Q_n/K it generates coincides with the E_n-module Q_n/K. By (2), $a_i = \partial_i * (pq^{-1}) \neq 0$ for some i. Then for all elements $u \in Q_n$, $ua_i^{-1}\partial_i * (pq^{-1} + K) = u + K$. So, Q_n/K is a simple E_n-module. Let $f \in \mathrm{End}_{E_n}(Q_n/K)$. Then applying f to the equalities $\partial_i * (x_1 + K) = \delta_{i1}$ for $i = 1, \dots, n$, we obtain the equalities

$$\partial_i * f(x_1 + K) = \delta_{i1} \quad \text{for} \quad i = 1, \dots, n.$$

Hence, $f(x_1 + K) \in \bigcap_{i=2}^n \ker_{Q_n/K}(\partial_i) \cap \ker_{Q_n/K}(\partial_i^2) = (K(x_1)/K) \cap \ker_{Q_n/K}(\partial_i^2) = K(x_1 + K)$. So, $f(x_1 + K) = \lambda(x_1 + K)$ and so $f = \lambda\mathrm{id}$, by the simplicity of the E_n-module Q_n/K. $\square$

The Cartan subalgebra $\mathcal{H}_n$ of E_n. A nilpotent Lie subalgebra C of a Lie algebra $\mathcal{G}$ is called a *Cartan subalgebra* of $\mathcal{G}$ if it coincides with its normalizer. We use often the following obvious observation: *An abelian Lie subalgebra that coincides with its centralizer is a maximal abelian Lie subalgebra.*

LEMMA 2.2.
(1) *$\mathcal{H}_n$ is a Cartan subalgebra of E_n.*
(2) *$\mathcal{H}_n = C_{E_n}(\mathcal{H}_n)$ is a maximal abelian Lie subalgebra of E_n.*

Proof. 2. Clearly, $\mathcal{H}_n \subseteq C_{E_n}(\mathcal{H}_n)$. Let $\partial = \sum_{i=1}^n a_i H_i \in C_{E_n}(\mathcal{H}_n)$ where $a_i \in Q_n$. Then all $a_i \in \cap_{i=1}^n \ker_{Q_n}(H_i) = \cap_{i=1}^n \ker_{Q_n}(\partial_i) = K$, by (2), and so $\partial \in \mathcal{H}_n$. Therefore, $\mathcal{H}_n = C_{E_n}(\mathcal{H}_n)$ is a maximal abelian Lie subalgebra of E_n.

1. By statement 2, we have to show that $\mathcal{H}_n = N := N_{E_n}(\mathcal{H}_n)$. Let $\partial = \sum_{i=1}^n a_i H_i \in N$, we have to show that all $a_i \in K$. For all $j = 1, \ldots, n$, $\mathcal{H}_n \ni [H_j, \partial] = \sum_{i=1}^n H_j(a_i) H_i$, and so $H_j(a_i) \in K$ for all i and j. These inclusions hold if all $a_i \in K$, i.e. $\partial \in \mathcal{H}_n$. Suppose that $a_i \notin K$ for some i, we seek a contradiction. Then necessarily, $a_i \notin K(x_1, \ldots, \widehat{x_j}, \ldots, x_n)$ for some j. Since $Q_n = K(x_1, \ldots, \widehat{x_j}, \ldots, x_n)(x_j)$, the result follows from the following claim.

Claim: If $a \in K(x) \backslash K$ then $H(a) \notin K$ where $H := x\frac{d}{dx}$. The field $K(x)$ is a subfield of the series field $K((x)) := \{\sum_{i > -\infty} \lambda_i x^i \mid \lambda_i \in K\}$. Since $H(\sum_{i > -\infty} \lambda_i x^i) = \sum_{i > -\infty} i\lambda_i x^i$, the Claim is obvious. Then, by the Claim, $H_j(a_i) \notin K$, a contradiction. $\square$

LEMMA 2.3 ([**5**]). *Let R be a commutative ring such that there exists a derivation $\partial \in \mathrm{Der}(R)$ such that $r\partial \neq 0$ for all non-zero elements $r \in R$ (eg, $R = P_n, Q_n$ and $\delta = \partial_1$). Then the group homomorphism*

$$\mathrm{Aut}(R) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Der}(R)), \quad \sigma \mapsto \sigma : \delta \mapsto \sigma(\delta) := \sigma\delta\sigma^{-1},$$

is a monomorphism.

Proof. If an automorphism $\sigma \in \mathrm{Aut}(R)$ belongs to the kernel of the group homomorphism $\sigma \mapsto \sigma$ then, for all $r \in R$, $r\partial = \sigma(r\partial)\sigma^{-1} = \sigma(r)\sigma\partial\sigma^{-1} = \sigma(r)\partial$, i.e. $\sigma(r) = r$ for all $r \in R$. This means that σ is the identity automorphism. Therefore, the homomorphism $\sigma \mapsto \sigma$ is a monomorphism. $\square$

The $\mathbb{Q}_n$-module E_n. The Lie algebra E_n is a $\mathbb{Q}_n$-module,

$$\mathbb{Q}_n \times E_n \to E_n, \quad (\sigma, \partial) \mapsto \sigma(\partial) := \sigma\partial\sigma^{-1}.$$

By Lemma 2.3, the $\mathbb{Q}_n$-module E_n is faithful and the map

$$\mathbb{Q}_n \to \mathbb{E}_n, \quad \sigma \mapsto \sigma : \partial \mapsto \sigma(\partial) = \sigma\partial\sigma^{-1}, \tag{3}$$

is a group monomorphism. We *identify* the group $\mathbb{Q}_n$ with its image in $\mathbb{E}_n$, $\mathbb{Q}_n \subseteq \mathbb{E}_n$. Every automorphism $\sigma \in \mathbb{Q}_n$ is uniquely determined by the elements

$$x_1' := \sigma(x_1), \ \ldots, \ x_n' := \sigma(x_n).$$

Let $M_n(Q_n)$ be the algebra of $n \times n$ matrices over Q_n. The matrix $J(\sigma) := (J(\sigma)_{ij}) \in M_n(Q_n)$, where $J(\sigma)_{ij} = \frac{\partial x_j'}{\partial x_i}$, is called the *Jacobi matrix* of the automorphism (endomorphism) σ and its determinant $\mathcal{J}(\sigma) := \det J(\sigma)$ is called the *Jacobian* of σ. So, the jth column of $J(\sigma)$ is the *gradient* $\mathrm{grad}\, x_j' := (\frac{\partial x_j'}{\partial x_1}, \ldots, \frac{\partial x_j'}{\partial x_n})^T$ of the rational function x'_j. Then the derivations

$$\partial'_1 := \sigma\partial_1\sigma^{-1}, \ \ldots, \ \partial'_n := \sigma\partial_n\sigma^{-1}$$

are the partial derivatives of Q_n with respect to the variables $x'_1, \ldots, x'_n$,

$$\partial'_1 = \frac{\partial}{\partial x'_1}, \ \ldots, \ \partial'_n = \frac{\partial}{\partial x'_n}. \tag{4}$$

Every derivation $\partial \in E_n$ is a unique sum $\partial = \sum_{i=1}^n a_i\partial_i$ where $a_i = \partial * x_i \in Q_n$. Let $\partial := (\partial_1, \ldots, \partial_n)^T$ and $\partial' := (\partial'_1, \ldots, \partial'_n)^T$ where T stands for the transposition. Then

$$\partial' = J(\sigma)^{-1}\partial, \ \text{ i.e. } \ \partial'_i = \sum_{j=1}^n (J(\sigma)^{-1})_{ij}\partial_j \ \text{ for } \ i = 1, \ldots, n. \tag{5}$$

In more detail, if $\partial' = A\partial$ where $A = (a_{ij}) \in M_n(Q_n)$, i.e. $\partial'_i = \sum_{j=1}^n a_{ij}\partial_j$. Then for all $i, j = 1, \ldots, n$,

$$\delta_{ij} = \partial'_i * x'_j = \sum_{k=1}^n a_{ik}\frac{\partial x'_j}{\partial x_k}$$

where δ_{ij} is the Kronecker delta function. The equalities above can be written in the matrix form as $AJ(\sigma) = 1$ where 1 is the identity matrix. Therefore, $A = J(\sigma)^{-1}$.

The maximal abelian Lie subalgebra $\mathcal{D}_n$ of E_n. Suppose that a group G acts on a set S. For a non-empty subset T of S, $\mathrm{St}_G(T) := \{g \in G \mid gT = T\}$ is the *stabilizer* of the set T in G and $\mathrm{Fix}_G(T) := \{g \in G \mid gt = t \text{ for all } t \in T\}$ is the *fixator* of the set T in G. Clearly, $\mathrm{Fix}_G(T)$ is a *normal* subgroup of $\mathrm{St}_G(T)$. The set $\mathrm{Sh}_n := \{s_\lambda \in \mathbb{Q}_n \mid s_\lambda(x_1) = x_1 + \lambda_1, \ldots, s_\lambda(x_n) = x_n + \lambda_n\}$ is a subgroup of $\mathbb{Q}_n$. Then Sh_n is also a subgroup of $\mathbb{E}_n$ where $s_\lambda(q\partial_i) = s_\lambda(q)\partial_i$ for all elements $q \in Q_n$ and $i = 1, \ldots, n$.

LEMMA 2.4.

(1) $C_{E_n}(\mathcal{D}_n) = \mathcal{D}_n$ *and so* $\mathcal{D}_n$ *is a maximal abelian Lie subalgebra of* E_n.
(2) $\mathrm{Fix}_{\mathbb{Q}_n}(\mathcal{D}_n) = \mathrm{Fix}_{\mathbb{Q}_n}(\partial_1, \ldots, \partial_n) = \mathrm{Sh}_n$.
(3) $\mathrm{Fix}_{\mathbb{Q}_n} = (\partial_1, \ldots, \partial_n, H_1, \ldots, H_n) = \{e\}$.
(4) $C_{E_n}(\mathcal{D}_n + \mathcal{H}_n) = 0$.

Proof.

1. Statement 1 follows from (2): Clearly, $\mathcal{D}_n \subseteq C_{E_n}(\mathcal{D}_n)$. Let $\partial = \sum a_i\partial_i \in C_{E_n}(\mathcal{D}_n)$ where $a_i \in Q_n$. Then all elements $a_i \in \bigcap_{i=1}^n \ker_{Q_n}\partial_i = K$, by (2), and so $\partial \in \mathcal{D}_n$. So, $C_{E_n}(\mathcal{D}_n) = \mathcal{D}_n$ and as a result $\mathcal{D}_n$ is a maximal abelian Lie subalgebra of E_n.
2. Let $\sigma \in \mathrm{Fix}_{\mathbb{Q}_n}(\mathcal{D}_n)$ and $J(\sigma) = (J_{ij})$. By (5), $\partial = J(\sigma)\partial$, and so, for all $i, j = 1, \ldots, n$, $\delta_{ij} = \partial_i * x_j = J_{ij}$, i.e. $J(\sigma) = 1$, or equivalently, by (2),

$$x'_1 = x_1 + \lambda_1, \ldots, x'_n = x_n + \lambda_n$$

for some scalars $\lambda_i \in K$, and so $\sigma \in \mathrm{Sh}_n$ (since $x'_i - x_i \in \bigcap_{j=1}^n \ker_{Q_n}(\partial_j) = K$ for $i = 1, \ldots, n$).

3. Let $\sigma \in \mathrm{Fix}_{\mathbb{Q}_n} = (\partial_1, \dots, \partial_n, H_1, \dots, H_n)$. Then $\sigma \in \mathrm{Fix}_{\mathbb{Q}_n}(\partial_1, \dots, \partial_n) = \mathrm{Sh}_n$, by statement 2. So, $\sigma(x_1) = x_1 + \lambda_1, \dots, \sigma(x_n) = x_n + \lambda_n$ where $\lambda_i \in K$. Then $x_i\partial_i = \sigma(x_i\partial_i) = (x_i + \lambda_i)\partial_i$ for $i = 1, \dots, n$, and so $\lambda_1 = \cdots = \lambda_n = 0$. This means that $\sigma = e$. So, $\mathrm{Fix}_{\mathbb{Q}_n} = (\partial_1, \dots, \partial_n, H_1, \dots, H_n) = \{e\}$.
4. Statement 4 follows from statement 1 and Lemma 2.2.(2).

□

LEMMA 2.5. *Let A be a K-algebra,* $\mathrm{Der}_K(A)$ *be the Lie algebra of K-derivations of A and $\mathcal{D}(A)$ be the ring of differential operators on A. If the algebra $\mathcal{D}(A)$ is simple and generated by A and* $\mathrm{Der}_K(A)$ *then the $\mathcal{D}(A)$-module A is simple.*

Proof. Let $\mathfrak{a}$ be a non-zero $\mathcal{D}(A)$-submodule of A. So, $\mathfrak{a}$ is an ideal of A such that $\partial(\mathfrak{a}) \subseteq \mathfrak{a}$ for all $\partial \in \mathrm{Der}_K(A)$. The algebra $\mathcal{D} := \mathcal{D}(A)$ is generated by A and D. So, $\mathcal{D}\mathfrak{a} \subseteq \mathfrak{a}\mathcal{D}$ and $\mathfrak{a}\mathcal{D} \subseteq \mathcal{D}\mathfrak{a}$, i.e. $\mathcal{D}\mathfrak{a} = \mathfrak{a}\mathcal{D}$ is a non-zero ideal of the simple algebra $\mathcal{D}$. Hence, $1 \in \mathcal{D}\mathfrak{a}$ and so $1 = \sum_i a_i d_i$ for some elements $d_i \in \mathcal{D}$ and $a_i \in \mathfrak{a} \subseteq D$. Then

$$1 = 1 * 1 = \sum_i a_i d_i * 1 \in \mathfrak{a},$$

hence $\mathfrak{a} = A$, i.e. A is a simple $\mathcal{D}(A)$-module. □

THEOREM 2.6.

(1) *E_n is a simple Lie algebra.*
(2) $Z(E_n) = \{0\}$.
(3) $[E_n, E_n] = E_n$.

Proof. 1. (i) $n = 1$, i.e. $E_1 = K(x)\partial$ *is a simple Lie algebra*: We split the proof into several steps.

(a) $D_1 := K[x]\partial$ and $W_1 := K[x, x^{-1}]\partial$ are simple Lie subalgebras of E_1 (easy).
(b) For all $\lambda \in K$, $W_1(\lambda) := K[x, (x-\lambda)^{-1}]$ is a simple Lie subalgebra of E_1, by applying the K-automorphism $s_\lambda : x \mapsto x - \lambda$ of the K-algebra Q_1 to W_1, i.e. $s_\lambda(W_1) = W_1(\lambda)$.
(c) *For any non-empty subset $I \subset K$, $W_1(I) := W_1(I)_K := K[x, (x-\lambda)^{-1} \mid \lambda \in I]\partial$ is a simple Lie subalgebra of E_1*: Let $\mathfrak{a}$ be a non-zero ideal of $W_1(I)$ and $0 \neq a\partial \in \mathfrak{a}$. Then either $a\partial \in D_1$ or $0 \neq [p\partial, a\partial] \in D_1 \cap \mathfrak{a}$ for some $p \in P_1$. Since $D_1 \subseteq W_1(\lambda)$ for all $\lambda \in I$ and $W_1(\lambda)$ are simple Lie algebra, $\mathfrak{a} \cap W_1(\lambda) = W_1(\lambda)$. Hence $\mathfrak{a} = W_1(I)$ since

$$W_1(I) = \sum_{\lambda \in I} W_1(\lambda),$$

i.e. $W_1(I)$ is a simple Lie algebra.
(d) If K is an algebraically closed field then E_1 is a simple Lie algebra since $E_1 = W_1(K)$.
The algebra E_1 is the union $\bigcup_{0 \neq f \in P_1} W_1[f^{-1}]$ of the Lie algebras $W_1[f^{-1}] := P_{1,f}\partial$ where $P_{1,f}$ is the localization of P_1 at the powers of the element f. Let $\mathfrak{a}$ be the ideal of E_1 generated by a non-zero element $a = pq^{-1}\partial$ for some $pq^{-1} \in Q_1$ where $p, q \in P_1$. Clearly, $a \in W_1[(fq)^{-1}]$ for all non-zero elements $f \in P_1$ and $E_1 = \bigcup_{0 \neq f \in P_1} W_1[(fg)^{-1}]$. So, to finish the proof of (i) it suffices to show that all the algebras $W_1[f^{-1}]$ are simple.

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(e) $A := W_1[f^{-1}]$ *is a simple Lie algebra for all* $0 \neq f \in P_1$: Let $K' := K(\nu_1, \dots, \nu_s)$ be the subfield of the algebraic closure $\overline{K}$ of K generated by the roots $\nu_1, \dots, \nu_s$ of the polynomial f and $G = \mathrm{Gal}(K'/K)$ be the Galois group of the finite Galois field extension K'/K (since $\mathrm{char}(K) = 0$). Let $K' = \oplus_{i=1}^d K\theta_i$ for some elements $\theta_i \in K'$ and $\theta_1 = 1$. By (c),

$$A' := K'[x, f^{-1}]\partial = W_1(\nu_1, \dots, \nu_s)_{K'}$$

is a simple Lie K'-algebra. Let $a \in A\backslash\{0\}$, $\mathfrak{a}$ and $\mathfrak{a}'$ be the ideals in A and A', respectively, that are generated by the element a. Then $\mathfrak{a}' = A'$, by (c). Notice that $A' = \sum_{i=1}^d \theta_i A$ and for $a' = \sum_{i=1}^d \theta_i a_i$, $b = \sum_{i=1}^d \theta_i b_i \in A'$ where $a_i, b_i \in A$, $[a', b] = \sum_{i=1}^d \theta_i\theta_j[a_i, b_j]$. Moreover, every element in $A' = \mathfrak{a}'$ is a linear combination of several commutators in A' (where $c = \sum_{i=1}^d \theta_k c_k \in A'$ and $c_k \in A$),

$$[a', [b, \dots [c, a] \dots] = \sum \theta_i \cdots \theta_j \theta_k [a_i, [b_j, \dots [c_k, a]] \dots]. \tag{6}$$

The *symmetrization map* $\mathrm{Sym} : K' \to K$, $\lambda \mapsto |G|^{-1} \sum_{g \in G} g(\lambda)$, is a surjection such that $\mathrm{Sym}(\mu) = \mu$ for all $\mu \in K$. Clearly, $K'(x)/K(x)$ is a Galois field extension with the Galois group G where the elements of G act trivially on the element x. So, the symmetrization map Sym can be extended to the surjection $K'(x) \to K(x)$ by the same rule, and then to the surjection $A' \to A, f\partial \mapsto \mathrm{Sym}(f)\partial$.

Each element $e \in A \subseteq A'$, can be expressed as a finite sum of elements in (6). Then applying Sym, we see that e is a linear combination of elements (commutators) from $\mathfrak{a}$, i.e. A is a simple Lie algebra.

(ii) E_n *is a simple Lie algebra for* $n \geq 2$: Let $a \in E_n\backslash\{0\}$ and $\mathfrak{a} = (a)$ be the ideal in E_n generated by the element $a = \sum_{i=1}^n a_i\partial_i$ where $a_i \in Q_n$.

(a) $\mathfrak{a} \cap D_n \neq 0$: If $a \in D_n$ then there is nothing to prove. Suppose that $a \notin D_n$.

(a1) Suppose that $a_i \in K(x_i)$ for all i. Then $a_i \notin K[x_i]$ for some i (since $a \notin D_n$), and so

$$\mathfrak{a} \ni [H_i, a] = H_i(a_i)\partial_i \in K(x_i)\partial_i\backslash\{0\}.$$

By (i), $\partial_i \in \mathfrak{a} \cap D_n$.

(a2) Suppose that $a_i \notin K(x_i)$ for some i. Then $\partial_j(a_i) \neq 0$ for some $j \neq i$. Let $q \in P_n$ be the common denominator of the fractions $a_1, \dots, a_n$, that is $a_1 = p_1q^{-1}, \dots, a_n = p_nq^{-1}$ for some elements $p_i \in P_n$. For all $n \geq 2$,

$$D_n \cap \mathfrak{a} \ni [q^n\partial_j, a] = q^n\partial_j(a_i)\partial_i + \sum_{k \neq i} (\dots)\partial_k \neq 0.$$

(b) By (a), $\mathfrak{a} \cap D_n = D_n$ since D_n is a simple Lie algebra, **[4]**.

(c) $\mathfrak{a} \supseteq K(x_i)\partial_i$ *for* $i = 1, \dots, n$: In view of symmetry it suffices to prove that $\mathfrak{a} \supseteq K(x_1)\partial_1$. Notice that for all $u \in Q_n$ and $i = 2, \dots, n$,

$$\mathfrak{a} \ni [u\partial_1, x_1\partial_i] = u\partial_i - x_1\partial_i(u)\partial_1.$$

Therefore, $\mathfrak{a} + Q_n\partial_1 = E_n$. The field of rational functions $Q_n = Q_n(K)$ can be seen as the field of rational functions $Q_n(K) = Q_{n-1}(K')$ where $K' = K(x_1)$.

Then

$$E'_{n-1} := \mathrm{Der}_{K'}(Q_{n-1}(K')) = \bigoplus_{i=2}^{n} Q_{n-1}(K')\partial_i = \bigoplus_{i=2}^{n} Q_n\partial_i.$$

By Lemma 2.5, the E'_{n-1}-module $Q'_{n-1}/K' = Q_n/K(x_1)$ is simple. The Lie algebra E'_{n-1} is a Lie subalgebra of E_n, and E_n can be seen as a left E'_{n-1}-module with respect to the adjoint action. The ideal $\mathfrak{a}$ of E_n is an E'_{n-1}-submodule of E_n. The Lie algebra $K(x_1)\partial_1$ is simple and $\mathfrak{a} \cap K(x_1)\partial_1$ is a non-zero ideal of it (by (b)). Therefore, $K(x_1)\partial_1 \subseteq \mathfrak{a}$. The E'_{n-1}-module $E_n/\mathfrak{a} = (\mathfrak{a} + Q_n\partial_1)/\mathfrak{a} \simeq Q_n\partial_1/\mathfrak{a} \cap Q_n\partial_1$ is an epimorphic image of the simple E'_{n-1}-module $Q_n/K(x_1)$ via

$$\varphi : Q_n/K(x_1) \to Q_n\partial_1/\mathfrak{a} \cap Q_n\partial_1, \quad u + K(x_1) \mapsto u\partial_1 + \mathfrak{a} \cap Q_n\partial_1,$$

with $0 \neq (P_n + K(x_1))/K(x_1) \subseteq \ker(\varphi)$. Therefore, $Q_n\partial_1 = \mathfrak{a} \cap Q_n\partial_1 \subseteq \mathfrak{a}$, and so $E_n = \mathfrak{a} + Q_n\partial_1 = \mathfrak{a}$. So, E_n is a simple Lie algebra.
2 and 3. Statements 2 and 3 follow from statement 1 (since, for all simple Lie algebras $\mathcal{G}$, $Z(\mathcal{G}) = 0$ and $[\mathcal{G}, \mathcal{G}] = \mathcal{G}$).

□

LEMMA 2.7. *For all non-zero elements $q \in Q_n$ and $i = 1, \dots, n$, $C_{E_n}(qP_n\partial_i) = \{0\}$.*

Proof. Let $c \in C_{E_n}(qP_n\partial_i)$. Then for all elements $p \in P_n$,

$$0 = [c, qp\partial_i] = c(p) \cdot q\partial_i + p[c, q\partial_i] = c(p) \cdot q\partial_i.$$

Then $c(p) = 0$ for all $p \in P_n$, and so $c = 0$. □

PROPOSITION 2.8. ([**4**]) $\mathrm{Fix}_{\mathbb{G}_n}(\partial_1, \dots, \partial_n, H_1, \dots, H_n) = \{e\}$.

Let $d_1, \dots, d_n$ be a commuting linear maps acting in a vector space E. Let $\mathrm{Nil}_E(d_1, \dots, d_n) := \{e \in E \mid d_i^j e = 0$ for all $i = 1, \dots, n$ and some $j = j(e)\}$. Let $\mathrm{Nil}_{E_n}(\mathcal{D}_n) := \mathrm{Nil}_{E_n}(\delta_1, \dots, \delta_n)$. Clearly, $\mathrm{Nil}_{E_n}(\mathcal{D}_n) = D_n$ is a Lie subalgebra of E_n.

PROPOSITION 2.9.
(1) $\mathrm{Fix}_{\mathbb{E}_n}(\partial_1, \dots, \partial_n, H_1, \dots, H_n) = \{e\}$.
(2) $\mathrm{Fix}_{\mathbb{E}_n}(\partial_1, \dots, \partial_n) = \mathrm{Sh}_n$.

Proof.
1. Let $\sigma \in F := \mathrm{Fix}_{\mathbb{E}_n}(\partial_1, \dots, \partial_n, H_1, \dots, H_n)$. We have to show that $\sigma = e$. Then $\sigma^{-1} \in F$ and $\sigma^{\pm 1}(\mathrm{Nil}_{E_n}(\mathcal{D}_n)) \subseteq \mathrm{Nil}_{E_n}(\mathcal{D}_n)$, i.e. $\sigma(D_n) = D_n$ since $\mathrm{Nil}_{E_n}(\mathcal{D}_n) = D_n$. So, $\sigma|_{D_n} \in \mathrm{Fix}_{\mathbb{G}_n}(\partial_1, \dots, \partial_n, H_1, \dots, H_n) = \{e\}$ (Proposition 2.8), i.e. $\sigma(\partial) = \partial$ for all $\partial \in D_n$. Let $0 \neq \delta \in E_n$. Then $\delta = q^{-1}\partial$ for some $0 \neq q \in P_n$ and $\partial \in D_n$. Now, $[q^2p\partial_i, \delta] = \partial' \in D_n$ for all $p \in P_n$. Applying σ to the equality yields the equality $[q^2p\partial_i, \sigma(\delta)] = \partial'$. By taking the difference, we obtain $\sigma(\delta) - \delta \in C_{E_n}(q^2P_n\partial_i) = \{0\}$, by Lemma 2.7, hence $\sigma = e$.
2. Clearly, $\mathrm{Sh}_n \subseteq F := \mathrm{Fix}_{\mathbb{E}_n}(\partial_1, \dots, \partial_n)$. Let $\sigma \in F$ and $H'_1 := \sigma(H_1), \dots, H'_n := \sigma(H_n)$. Applying the automorphism σ to the commutation relations $[\partial_i, H_j] = \delta_{ij}\partial_i$ gives the relations $[\partial_i, H'_j] = \delta_{ij}\partial_i$. By taking the difference, we see that $[\partial_i, H'_j - H_j] = 0$ for all i and j. Therefore, $H'_i = H_i + d_i$ for some elements

$d_i \in C_{E_n}(\mathcal{D}_n) = \mathcal{D}_n$ (Lemma 2.4.(1)), and so $d_i = \sum_{j=1}^n \lambda_{ij}\partial_j$ for some elements $\lambda_{ij} \in K$. The elements $H'_1, \dots, H'_n$ commute, hence

$$[H_j, d_i] = [H_i, d_j] \text{ for all } i, j,$$

or equivalently,

$$\lambda_{ij}\partial_j = \lambda_{ji}\partial_i \text{ for all } i, j.$$

This means that $\lambda_{ij} = 0$ for all $i \neq j$, i.e.

$$H'_i = H_i + \lambda_{ii}\partial_i = (x_i + \lambda_{ii})\partial_i = s_\lambda(H_i)$$

where $s_\lambda \in \mathrm{Sh}_n$, $s_\lambda(x_i) = x_i + \lambda_{ii}$ for all i. Then $s_\lambda^{-1}\sigma \in \mathrm{Fix}_{\mathbb{E}_n}(\partial_1, \dots, \partial_n, H_1, \dots, H_n) = \{e\}$ (statement 1), and so $\sigma = s_\lambda \in \mathrm{Sh}_n$. $\square$

The automorphism ν. Let ν be the K-automorphism of Q_n given by the rule $\nu(x_i) = x_i^{-1}$ for $i = 1, \dots, n$. Then

$$\nu(\partial_i) = -x_i H_i, \quad \nu(H_i) = -H_i, \quad \nu(x_i H_i) = -\partial_i, \quad i = 1, \dots, n. \tag{7}$$

By (7), the elements $X_1 := x_1 H_1, \dots, X_n := x_n H_n$ commute and the next lemma follows from Lemma 2.4 and Proposition 2.9 since $\mathcal{X}_n := \nu(\mathcal{D}_n) = \bigoplus_{i=1}^n KX_i$.

LEMMA 2.10.

(1) *$C_{E_n}(\mathcal{X}_n) = \mathcal{X}_n$ is a maximal abelian Lie subalgebra of E_n.*
(2) $\mathrm{Fix}_{\mathbb{Q}_n}(X_1, \dots, X_n) = \mathrm{Fix}_{\mathbb{E}_n}(X_1, \dots, X_n) = \mathrm{Sh}_n$.
(3) $\mathrm{Fix}_{\mathbb{Q}_n}(X_1, \dots, X_n, H_1, \dots, H_n) = \mathrm{Fix}_{\mathbb{E}_n}(X_1, \dots, X_n, H_1, \dots, H_n) = \{e\}$.

The following lemma is well known and easy to prove.

LEMMA 2.11. *Let ∂ be a locally nilpotent derivation of a commutative K-algebra A such that $\partial(x) = 1$ for some element $x \in A$. Then $A = A^\partial[x]$ is a polynomial algebra over the ring $A^\partial := \ker(\partial)$ of constants of the derivation ∂ in the variable x.*

The next lemma is the core of the proof of Theorem 1.2.

LEMMA 2.12. *Let $\sigma \in \mathbb{E}_n$, $\partial'_1 := \sigma(\partial_1), \dots, \partial'_n := \sigma(\partial_n)$ and $\delta'_1 := \mathrm{ad}(\partial'_1), \dots, \delta'_n := \mathrm{ad}(\partial'_n)$. Then*

(1) *$\partial'_1, \dots, \partial'_n$ are commuting derivations of Q_n such that $\bigcap_{i=1}^n \ker_{Q_n}(\partial'_i) = K$.*
(2) $E_n = \bigoplus_{i=1}^n Q_n \partial'_i$.
(3) *For each $i = 1, \dots, n$, $\sigma(x_i\partial_i) = x'_i\partial'_i$ for some elements $x'_i \in Q_n$. The elements $x'_1, \dots, x'_n$ are algebraically independent and $\partial'_i(x'_j) = \delta_{ij}$ for $i, j = 1, \dots, n$.*
(4) *$\mathrm{Nil}_{Q_n}(\partial'_1, \dots, \partial'_n) = P'_n$ where $P'_n := K[x'_1, \dots, x'_n]$.*
(5) $\mathrm{Nil}_{E_n}(\delta'_1, \dots, \delta'_n) = \bigoplus_{i=1}^n P'_n\partial'_i$.
(6) *$\sigma(x^\alpha\partial_i) = x'^\alpha\partial'_i$ for all $\alpha \in \mathbb{N}^n$ and $i = 1, \dots, n$.*
(7) *$\sigma' : Q_n \to Q_n$, $x_i \mapsto x'_i$, $i = 1, \dots, n$ is a K-algebra homomorphism (statement 3) such that $\sigma'(a\partial_i) = \sigma'(a)\sigma(\partial_i)$.*
(8) *The K-algebra homomorphism σ' is an automorphism.*

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Proof.

1. The elements $\partial_1, \dots, \partial_n$ are commuting derivations, hence so are $\partial'_1, \dots, \partial'_n$. Let $\lambda \in \bigcap_{i=1}^n \ker_{Q_n}(\partial'_i)$. Then

$$\lambda\partial'_1 \in C_{E_n}(\partial'_1, \dots, \partial'_n) = \sigma(C_{E_n}(\partial_1, \dots, \partial_n)) = \sigma(C_{E_n}(\mathcal{D}_n)) = \sigma(\mathcal{D}_n)$$
$$= \sigma(\bigoplus_{i=1}^n K\partial_i) = \bigoplus_{i=1}^n K\partial'_i,$$

since $C_{E_n}(\mathcal{D}_n) = \mathcal{D}_n$, Lemma 2.4.(1). Then $\lambda \in K$ since otherwise the infinite dimensional space $\bigoplus_{i\geq 0} K\lambda^i\partial'_1$ would be a subspace of the finite dimensional space $\sigma(\mathcal{D}_n)$.

2. It suffices to show that the elements $\partial'_1, \dots, \partial'_n$ of the n-dimensional (left) vector space E_n over the field Q_n are Q_n-linearly independent (the key reason for that is statement 1). Let $V = \sum_{i=1}^n Q_n\partial'_i$. Suppose that $m := \dim_{Q_n}(V) < n$, we seek a contradiction. Up to order, let $\partial'_1, \dots, \partial'_m$ be a Q_n-basis of V. Then $\partial'_{m+1} = \sum_{i=1}^m a_i\partial'_i$ for some elements $a_i \in Q_n$. By applying δ'_j $(j = 1, \dots, n)$, we see that $0 = \sum_{i=1}^m \partial'_j(a)\partial'_i$, and so $a_i \in \bigcap_{i=1}^n \ker_{Q_n}(\partial'_j) = K$, by statement 1. This means that the elements $\partial'_1, \dots, \partial'_m$ are K-linearly dependent, a contradiction.

3. Let $H'_i := \sigma(x_i\partial_i)$ for $i = 1, \dots, n$. By statement 2, $H'_i = \sum_{i=1}^n a_{ij}\partial'_j$ for some elements $a_{ij} \in Q_n$. Applying the automorphism σ to the relations $\delta_{ij}\partial_j = [\partial_j, H_i]$ yields the relations

$$\delta_{ij}\partial'_i = \sum_{i=1}^n \partial'_j(a_{ik})\partial'_k.$$

Let $x'_i := a_{ii}$. Then $\partial'_j(x'_i) = \delta_{ji}$ and $\partial'_j(a_{ik}) = 0$ for all $k \neq i$. By statement 1, $a_{ik} \in K$ for all $i \neq k$. Now,

$$H'_i := x'_i\partial'_i + \sum_{j\neq i} a_{ij}\partial'_j.$$

The elements $H'_1, \dots, H'_n$ commute, hence for all $i \neq j$, $0 = [H'_i, H'_j] = -a_{ji}\partial'_i + a_{ij}\partial'_j$, and so $a_{ij} = 0$. Therefore, $H'_i = x'_i\partial'_i$.
The equalities $\partial'_i(x'_j) = \delta_{ij}$ imply that the elements $x'_1, \dots, x'_n \in Q_n$ are algebraically independent over K: Suppose that $f(x'_1, \dots, x'_n) = 0$ for some non-zero polynomial $f(x_1, \dots, x_n) \in K[x_1, \dots, x_n]$. We can assume that the (total) degree $\deg(f)$ is the least possible. Clearly, $f \notin K$, hence $\frac{\partial f}{\partial x_i} \neq 0$ for some i and $\deg(\frac{\partial f}{\partial x_i}) < \deg(f)$ and so $0 \neq \frac{\partial f}{\partial x_i}(x'_1, \dots, x'_n) = \partial'_i(f(x'_1, \dots, x'_n)) = \partial'_i(0) = 0$, a contradiction.

4. Let $\mathcal{D}'_n = \sum_{i=1}^n K\partial'_i$ and $N = \mathrm{Nil}_{Q_n}(\mathcal{D}'_n)$. By statement 3 and Lemma 2.11,

$$N = N^{\mathcal{D}'_n}[x'_1, \dots, x'_n] = K[x'_1, \dots, x'_n]$$

since $K \subseteq N^{\mathcal{D}'_n} \subseteq Q_n^{\mathcal{D}'_n} = K$ (by statement 1).

5. Let $\partial = \sum_{i=1}^n a_i\partial_i' \in N := \mathrm{Nil}_{E_n}(\delta_1', \dots, \delta_n')$ where $a_i \in Q_n$ (statement 2). For all $\alpha \in \mathbb{N}^n$,

$$\delta'^{\alpha}(\partial) = \sum_{i=1}^n \partial'^{\alpha}(a_i)\partial_i'$$

where $\delta'^{\alpha} := \prod_{i=1}^n \delta_i'^{\alpha_i}$, $\delta_i' = \mathrm{ad}(\partial_i')$ and $\partial'^{\alpha} := \prod_{i=1}^n \partial_i'^{\alpha_i}$. So, $\delta'^{\alpha}(a_i) = 0$ iff $\partial'^{\alpha}(a_i) = 0$ for $i = 1, \dots, n$ (statement 2). Now, statement 5 follows from statement 4.
6. By statement 3,

$$\partial_i'(x_j') = \delta_{ij} \quad \text{and} \quad \sigma(H_i) = \sigma(x_i\partial_i) = x_i'\partial_i' := H_i'.$$

We prove statement 6 by induction on $|\alpha|$. The initial cases when $|\alpha| = 0, 1$ are obvious (statement 3). So, let $|\alpha| \geq 2$ and we assume that statement 6 holds for all α' with $|\alpha'| < |\alpha|$. Then

$$\begin{aligned}[\partial_j', \sigma(x^{\alpha}\partial_i) - x'^{\alpha}\partial_i'] &= \sigma([\partial_j, x^{\alpha}\partial_i]) - \alpha_j x'^{\alpha - e_j}\partial_i' = \sigma(\alpha_j x^{\alpha-e_j}\partial_i) - \alpha_j x'^{\alpha-e_j}\partial_i' \\ &= \alpha_j x'^{\alpha-e_j}\partial_i' - \alpha_j x'^{\alpha-e_j}\partial_i' = 0.\end{aligned}$$

Hence, $\sigma(x^{\alpha}\partial_i) - x'^{\alpha}\partial_i' \in C_{E_n}(\mathcal{D}_n') = \mathcal{D}_n'$, Lemma 2.4.(1). Therefore, $\sigma(x^{\alpha}\partial_i) = x'^{\alpha}\partial_i' + \sum \lambda_{ij}\partial_j'$ for some scalars $\lambda_{ij} = \lambda_{ij}(\alpha) \in K$. We have to show that all $\lambda_{ij} = 0$. Applying the automorphism σ to the equalities $(\alpha_j - \delta_{ij})x^{\alpha}\partial_i = [H_j, x^{\alpha}\partial_i]$ we have (notice that $x^{\alpha}\partial_i \neq H_i$ since $|\alpha| \geq 2$)

$$\begin{aligned}(\alpha_j - \delta_{ij})(x'^{\alpha}\partial_i' + \sum_{k=1}^n \lambda_{ik}\partial_k') &= \sigma((\alpha_j - \delta_{ij})x^{\alpha}\partial_i) \\ &= \sigma([H_j, x^{\alpha}\partial_i]) = [H_j', x'^{\alpha}\partial_i' + \sum_{k=1}^n \lambda_{ik}\partial_k'] \\ &= (\alpha_j - \delta_{ij})x'^{\alpha}\partial_i' - \lambda_{ij}\partial_j',\end{aligned}$$

and so $(\alpha_j - \delta_{ij} + 1)\lambda_{ij} = 0$ and $(\alpha_j - \delta_{ij})\lambda_{ik} = 0$ for all $k \neq j$.
If $n = 1$ then $\alpha_1 \geq 2$ and the first equation $(\alpha_j - \delta_{ij} + 1)\lambda_{ij} = 0$ takes the form $(\alpha_1 - 1 + 1)\lambda_{11} = 0$, and so $\lambda_{11} = 0$. We can assume that $n \geq 2$. For all $i \neq j$, the first equation $(\alpha_j - \delta_{ij} + 1)\lambda_{ij} = 0$ yields $\lambda_{ij} = 0$. Fix an index j such that $\alpha_j \geq 1$. Then, for $i = j$ the first equation $(\alpha_j - \delta_{ij} + 1)\lambda_{ij} = 0$ yields $\lambda_{jj} = 0$. Finally for all $i = k \neq j$, the second equation $(\alpha_j - \delta_{ij})\lambda_{ik} = 0$ yields $\lambda_{ii} = 0$. This means that all $\lambda_{st} = 0$.
7. By statement 3, σ' is a K-algebra homomorphism such that $\mathrm{im}(\sigma') = Q_n' := K(x_1', \dots, x_n')$. By statement 3, for all elements $a \in Q_n$,

$$\partial_i'\sigma'(a) = \sigma'\partial_i(a)$$

since ∂_i' acts as $\frac{\partial}{\partial x_i'}$ on Q_n'.
Let $a = pq^{-1} \neq 0$ where $p, q \in P_n$. Then, for all $r \in q^2 P_n$, $[a\partial_i, r\partial_i] = (a\partial_i(r) - \partial_i(a)r)\partial_i \in P_n\partial_i$. By applying σ and using statement 6, we have the equality

$$[\sigma(a\partial_i), \sigma'(r)\partial_i'] = \sigma'(a\partial_i(r) - \partial_i(a)r)\partial_i'.$$

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On the other hand,

$$\begin{aligned}[\sigma'(a)\partial_i', \sigma'(r)\partial_i'] &= (\sigma'(a)\partial_i'\sigma'(r) - \partial_i'\sigma'(a)\sigma'(r))\partial_i' \\ &= (\sigma'(a)\sigma'\partial_i(r) - \sigma'\partial_i(a)\sigma'(r))\partial_i' \\ &= \sigma'(a\partial_i(r) - \partial_i(a)r)\partial_i'.\end{aligned}$$

Hence,

$$\begin{aligned}\sigma(a\partial_i) - \sigma'(a)\partial_i' \in C_{E_n}(\sigma'(q^2P_n)\partial_i') &= C_{E_n}(\sigma(q^2P_n\partial_i)) \\ &= \sigma(C_{E_n}(q^2P_n\partial_i)) = \sigma(0) = 0,\end{aligned}$$

by Lemma 2.7. Therefore, $\sigma(a\partial_i) = \sigma'(a)\sigma(\partial_i)$.

8. Since $\sigma(Q_n\partial_i) = \sigma'(Q_n)\partial_i'$ for all $i = 1, \dots, n$ (statement 7), we must have $\sigma'(Q_n) = Q_n$, by statement 2, and so $\sigma' \in \mathbb{Q}_n$. □

Proof of Theorem 1.2. Let $\sigma \in \mathbb{E}_n$. By Lemma 2.12.(8), we have the automorphism $\sigma' \in \mathbb{Q}_n$ such that, by Lemma 2.12.(3,6), $\sigma'^{-1}\sigma \in \mathrm{Fix}_{\mathbb{E}_n}(\partial_1, \dots, \partial_n, H_1, \dots, H_n) = \{e\}$ (Proposition 2.9). Therefore, $\sigma = \sigma'$ and so $\mathbb{E}_n = \mathbb{Q}_n$. □

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