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A GENERALISED EXCHANGE THEOREM FOR MATROID BASES

John Donald and Malcolm Tobey

Let b and c be bases of a matroid. Then for any integer r, there exists an injection σ from r-subsets I of b to r-subsets $\sigma(I)$ of c such that $b - I + \sigma(I)$ is a base for all I. This result has implications for the structure of matroid base graphs.

1. Generalised base exchange

Given bases b and c of a matroid M, and an element $i \in b$, there always exists an element $j \in c$ such that, with the obvious notational conventions, $b-i+j$ is a base of M. This *base exchange* property of matroids implies stronger exchange properties. For example, Brualdi proved in [1] that there always exists an injection σ from b to c such that for all i in b, $b-i+\sigma(i)$ is a base of M. In this note we generalise Brualdi's result to arbitrary finite subsets of b.

Brualdi's proof depends on Hall's theorem on distinct representatives ([1] or [5, p.505]); indeed, the exchange property appears tailor-made for that theorem. To introduce our ideas we sketch an alternative proof, also depending on Hall's theorem, that avoids using circuits. Hall's theorem states the following: finite sets $X(I)$ have distinct representatives, I ranging over any index set, if and only if the union of any finite number m of the $X(I)$ contains at least m elements.

To use Hall's theorem, first we define our indexed sets. Accordingly, for each $i \in b$ let $X(i)$ denote $\{j \in c \mid b - i + j \text{ is a base}\}$. The desired injection σ from b to c corresponds to a choice of distinct representatives for the $X(i)$. By Hall's theorem, it suffices to show that for any finite m the union X of m distinct $X(i)$ contains at least m elements. Let I denote the m elements of b involved, and for convenience assume I disjoint from c. Then there exists, by repeated exchange, a base $b - I + J$, J a subset of $c - b$ of size m. Again by repeated exchange we may delete the elements of J in any order as we move back toward b. In particular, for each j in J, there exists a base $b - i + j$ for some i in I. Since each of these distinct bases is in X, X contains at least m elements.

Through a further application of Hall's theorem, we generalise Brualdi's result to subsets of b.

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THEOREM 1. *Let b and c be bases of a matroid M with bases B. Then there exists an injection σ from r-subsets I of b to r-subsets J of c such that $b - I + \sigma(I)$ is always a base.*

PROOF OF THEOREM 1: We note that the case $r = 1$ is known ([1], or as sketched above) and use induction. Henceforth we assume $r > 1$.

For any r-subset I of b let $X(I)$ denote $\{J \mid c \supset J \text{ and } b - I + J \text{ in } B\}$. Any set of distinct representatives for the $X(I)$ will define the desired injection, so we try to use Hall's theorem.

Accordingly it suffices to verify the hypothesis of Hall's theorem: given any m r-subsets $I_1, \dots, I_m$ of b, and denoting the union of the associated $X(I_j)$ by X, we must have $|X| \geqslant m$. We now get a lower bound on the size of that union in two steps. First we label a bunch of elements of that union, and secondly we show that not too many labels denote the same subset.

Each I_j, $j = 1, \dots, m$, contains r $(r-1)$-subsets, which of course may be $(r-1)$-subsets of other I_k, $k \neq j$. Let $K_1, \dots, K_t$ be a listing, *without repetition*, of all the $(r-1)$-subsets of all the I_j, and suppose each K_p is contained in n_p distinct I_j. Let, by induction, π be a bijection between $(r-1)$-subsets K of b and $\pi(K)$ of c such that $b - K + \pi(K)$ is a base. Then each $b - K_p + \pi(K_p)$ is a base. Let S_p denote the set of I_j containing K_p, $p = 1, \dots, t$. By the case $r = 1$ applied separately to each of the bases $b - K_p + \pi(K_p)$ and c, there are n_p *distinct* y_{jp} in c such that for I_j in S_p, $b - I_j + \pi(K_p) + y_{jp}$ is a base. We now have a labelled collection of not necessarily distinct subsets $\pi(K_p) + y_{jp}$ belonging to X. The number of such labels is, of course, $\sum n_p$. It is also mr, because there are r distinct K_p contained in each of the m I_j.

Suppose two labels denote the same set:

$$(*) \qquad \pi(K_p) + y_{jp} = \pi(K_q) + z_{kq}.$$

If $p = q$, then by construction $y_{jp} = z_{kq}$. If $p \neq q$, then by the injective property of π, $\pi(K_p) \neq \pi(K_q)$. Consequently z_{kq} is in $\pi(K_p)$, so there are at most $r - 1$ choices for z_{kq}, whence there are at most $r - 1$ choices for $\pi(K_q) + z_{kq}$, again by the injective property of π. Thus there can be at most r equalities of the form (*) for fixed $\pi(K_p) + y_{jp}$. Because there are mr such labels, the labels must denote at least m distinct subsets. Thus $|X| \geqslant m$. □

Curiously, our proof does not establish the existence of an injection from b to c by which the injection from r-subsets to r-subsets is induced. Obviously a *particular* injection of r-subsets need not result from an injection of the underlying elements, so we may ask whether there necessarily exists *any* injection of r-subsets induced in this way. The associated underlying injection would have to vary with r, as shown by the

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example in [1]. For that example, however, underlying injections exist, separately for each r, the only new case being $r = 2$.

2. Implications for base graphs

Given a matroid M with collection of finite bases B, we say that bases b and c of B are *adjacent* if they differ by one element; that is, if $|b - c| = |c - b| = 1$. Nonadjacent bases are *independent*. More generally we define the *distance* $d(b, c)$ between arbitrary bases b and c of B to be the number of elements by which they differ: $d(b, c) = |b - c| = |c - b|$. We denote by $N_s(b)$ the set of bases a distance s from the base b.

The matroid property imposes a certain combinatorial complexity on the sets $N_s(b)$. In particular if $d(b, c) = d$, then we can ask about the *joint neighbourhood* $N_r(b) \cap N_{d-r}(c)$ consisting of certain bases between b and c. For example, it is nearly obvious that for $d = 2$, $N_1(b) \cap N_1(c)$ contains a pair of independent bases. This fact along with other graph-theoretical properties has played a role in various attempts to characterise the adjacency properties of matroid bases [2, 3, 4]. Less obviously, but equivalent to the bijection result of [1], $N_1(b) \cap N_{d-1}(c)$ contains a set of d independent bases, for arbitrary finite d. Theorem 1 directly yields a further generalisation.

Theorem 2. *Let M be a matroid with bases b and c, and suppose the distance $d(b, c) = d$. Then $N_r(b) \cap N_{d-r}(c)$ contains a collection of $C(d, r)$ independent bases, where $C(d, r)$ denotes the rth binomial coefficient, "d choose r".*

Proof of Theorem 2: If $I_1 \neq I_2$ are r-subsets of b and σ is the bijection of Theorem 1, then $\sigma(I_1) \neq \sigma(I_2)$. Thus $b - I_1 + \sigma(I_1)$ and $b - I_2 + \sigma(I_2)$ are nonadjacent. □

One may neatly visualise the bases B of a matroid M as providing the vertices of a graph $B(M)$, the *matroid base graph* of M, whose adjacencies are the same as the adjacencies of the bases [2, 4]. From that point of view, a particularly simple matroid results from the d-fold product of a 2-base matroid. Its base graph is the d-cube, for which Theorem 2 is tight, albeit trivial. At an opposite extreme one has the complete matroid $B_{d,d}$ consisting of all d-subsets of a $2d$-set. For example, in $B_{4,4}$ bases b and c at a distance 4 admit 12 nonadjacent bases in $N_2(b) \cap N_2(c)$. If the union of shortest paths between bases a distance d apart always contained a d-cube, Theorem 2 would be an immediate corollary. Examples like that in [1] show, however, that there need be no such d-cube. We have no direct explanation for the presence of so much independence in the joint neighbourhoods of matroid base graphs.

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Department of Mathematical Sciences
San Diego State University
San Diego, CA 92182
United States of America

Mathematics Department
Southwest State University
Marshall, MN 56258
United States of America

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