A NORMAL MODE STUDY OF WOBBLE AND NUTATION

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Introduction

The observed eigenperiod of the Chandler Wobble is about 435.2 sidereal days while the theoretical eigenperiod of a rigid body having the same composition and geometry as the Earth is about 305 days. The attempt to reconcile these two numbers has led scientists to study theoretically the free wobble and nutation of various classes of rotating bodies.

Among the earliest such efforts were analytical studies of the influence of a fluid core filled with an incompressible, homogeneous liquid (Hough, 1895) and of the influence of the quasi-static elastic relaxation of an everywhere-solid planet (Hough, 1896; Love, 1909; Larmor, 1909). Jeffreys and Vicente (1957a, 1957b) and Molodensky (1961) pursued greatly generalized calculations intended to model more accurately the Earth's known complexities. In both cases the investigators sought to account for the Earth's radially varying properties and the presence of a fluid outer core. Shen and Mansinha (1976) pursued Molodensky's approach further by adopting a more general representation for motion in the fluid core.

There are qualitative and quantitative discrepancies between the results of Jeffreys and Vicente (1957a, 1957b) on the one hand, and those of Molodensky (1961) and Shen and Mansinha (1976) on the other. It is difficult to reconcile these differences, partly because of the inherent complexity of these theories and partly because of some expository difficulties in various of these papers.

The importance of having a confident theoretical understanding of the Earth's free wobble and nutation, however, is clear. That importance has increased as polar motion observation has become increasingly precise due to the introduction of new, sophisticated ranging techniques.

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Approach

We suppose that the undisturbed Earth appears to be at rest in a reference frame rotating rigidly with the fixed angular velocity, Ω_0. We suppose that the undisturbed Earth is in hydrostatic equilibrium everywhere and is nearly spherical; then (Jeffreys, 1959) surfaces of constant density and also equipotential surfaces and have a slight axisymmetric ellipticity which varies with radius in a manner described by Clairaut's equation. We suppose that the response of the Earth to any small disturbances about its equilibrium state is characterized by a linear, perfectly elastic, isotropic constitutive relation. Finally, we suppose that such material properties as the elastic parameters are constant on surfaces of constant density. Earth models consistent with these suppositions may have a fluid outer core, a solid inner core, and may have properties which depend in an arbitrary way on radius. The class of Earth models we consider here includes all current geophysically useful models.

The equations of motion which govern the infinitesimal elastic-gravitational deviation of such an Earth model away from its equilibrium configuration are well known and are given in Smith (1974). Solutions to these equations which have a simple-harmonic time dependence are known as "normal modes" or "free oscillations." The angular frequency of such an oscillation is called the "eigenfrequency" and the spatially-varying displacement field is called the "eigenfunction." The set of normal modes for a particular Earth model spans all of the free motions of which that model is capable.

In the non-rotating limit, that is, as we let the Earth's rotation rate go to zero, the Earth becomes perfectly spherical. The calculation of the normal modes of such a model becomes particularly straightforward (as a result of the perfect spherical symmetry) and, in fact, this approximation is used to compute the "seismic" normal modes exploited by geophysicists to study the composition of the Earth's interior. The non-rotating approximation is valid in that case because all of the "seismic" normal modes are very fast compared to one day.

The normal modes of a rotating Earth model are more numerous and complex than those of the associated non-rotating model. Two of these are of particular interest to astronomers because they are potentially capable of altering the Earth's instantaneous rotation axis; they are the Chandler Wobble and the Nearly Diurnal Free Wobble.

In order to study these normal modes for geophysically interesting, rotating Earth models we have had to extend somewhat the theory used to study the normal modes of non-rotating Earth models. Briefly, that extension proceeds as follows: In the non-rotating case, any normal mode can be associated with a single surface spherical harmonic. The problem of determining a normal mode eigenfunction becomes the problem of determining several scalar functions of radius which serve, together with the particular spherical harmonic, to everywhere determine the eigenfunction. The use of spherical harmonics reduces the tensor partial-differential equations of motion to simple ordinary-differential equations over radius. When the Earth model is allowed to rotate, however, we can no longer associate a single spherical harmonic with a particular normal mode. Instead, each normal mode eigenfunction is associated with an infinite series of spherical harmonics. Substitution of this series into the tensor partial-differential equations of motion produces an infinite set of coupled ordinary-differential equations over radius which simultaneously determines the scalar functions of radius associated with all of the spherical harmonics in the series.

We cannot, in general, solve ordinary differential equations of infinite order so we instead truncate the spherical harmonic series after a few terms. The level at which truncation occurs can be justified a priori, as is discussed in Smith (1977), and it can also be justified a posteriori, as we do below, by comparing results computed using this approach with several analytic and quasi-analytic solutions for wobble and nutation. The calculations discussed here were based upon a three-term spherical harmonic series. This series consisted of a toroidal term of degree one, a spheroidal term of degree two, and a toroidal term of degree three.

Validation

A direct way to assess the validity of this approach is to apply the technique to calculations for which the answer is already known. We give here the results of two comparisons which are wholly independent of both the theory and the computer codes upon which our results depend.

The first test is based upon the famous Love-Larmor formula which predicts the Chandler Wobble eigenfrequency of an everywhere solid but elastic body in terms of its Love number, k_2. To this comparison a modern geophysical Earth model (due to J. F. Gilbert and A. M. Dziewonski, personal communication) was modified slightly to have a very "mushy," but solid, outer core. The Love number, k_2, was computed through independent means to be $k_2 = 0.301$. The Love-Larmor formula predicts that this model will have a Chandler Wobble period of 453.66 sidereal days. For the same model, the theory used here gave 452.53 sidereal days.

The second test exploited a result due to Hough (1895) who was able to compute analytically both the Chandler Wobble and Nearly Diurnal Free Wobble eigenperiods for a model consisting of a rigid outer shell and an ellipsoidal core filled with an incompressible, homogeneous fluid. Although "rigid" and "incompressible" are singular limits of elastic behavior, we were able to use a model which had an extremely stiff mantle and which had a nearly incompressible fluid core. For the perfectly rigid and incompressible limit, Hough's results predicted that the model would have a Chandler Wobble eigenperiod of 173.54 sidereal days and a Nearly Diurnal Free Wobble eigenperiod of 0.99153 sidereal days. The corresponding results of our calculations were 174.81 and 0.99157 sidereal days.

The excellent agreement between the results given by our approach and the independent results of classical analytical models provides strong justification for accepting the results of our calculations for more complex models of the Earth. The results of the latter are given below.

Some additional test cases demonstrate the limits of this approach for certain other types of long-period normal modes. These limitations are discussed in Smith (1977).

Numerical Results for Realistic Earth Models

Although quite a lot is now known about the internal structure of the Earth, the existing seismological data are not adequate to accurately determine the extent to which the fluid outer core is convectively stable or unstable. Those data do yield good estimates of the average value of density and the elastic constants over various radial regions. Convective stability, however, is closely tied to the local radial derivative of the density field and the resolution of our current observational data is not adequate to determine this derivative accurately. (See Smith, 1976, and Smith, 1977 for more discussion of this point.) Simple stability considerations suggest that the fluid core is either convectively stable or at least neutrally stable. This means that the squared Brunt-Vaisala frequency,

$$N^2 = - g\{\frac{\rho g}{\lambda} \quad \frac{\partial_r \rho}{\rho}\}$$

is everywhere non-negative. Here ρ is density, r is radius, g is gravitational acceleration, and λ is incompressibility. This expression assumes chemical homogeneity.

In order to assess the effect of our current uncertainties about convective stability in the fluid outer core upon theoretical predictions of the Earth's free wobble and nutation eigenfrequencies, we performed calculations for a <u>suite</u> of Earth models, rather than just for a single model. This suite was based upon a single, modern model constructed by J. F. Gilbert and A. M. Dziewonski to fit a large set of seismological data, principally free oscillations. The density

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distribution in the core was slightly modified to provide three models which were virtually identical to the original save that each had a core density distribution which gave a constant, preselected value for N^2. The three models used here corresponded to $N^2 = 0$, $N^2 = 8.1 \times 10^{-9}$, and $N^2 = 3.38 \times 10^{-7}$.

Table 1 compares the eigenperiods of the Chandler Wobble and the Nearly Diurnal Free Wobble which we computed for each of our three models with results reported by Molodensky (1961) for two other earth models. (Molodensky's Chandler eigenperiods were obtained from his reported results by undoing the correction he applied for the effect of the oceans. One of the nearly diurnal eigenperiods is missing from our results as a result of a computing error; we did not believe the result sufficiently interesting to justify recomputing it.)

Note first that the Chandler Wobble eigenperiod over all of Molodensky's models and ours shows only a slight variation and the Nearly Diurnal Free Wobble eigenperiod varies even less. One conclusion we can draw from these results is that improved knowledge of the actual eigenperiods of these normal modes is not likely to greatly increase our knowledge of the Earth's interior.

TABLE 1

Summary and Comparison of Numerical Results

MODE	SOURCE	EIGENPERIOD (Sidereal Days)
Chandler Wobble	Molodensky I	400.87
	Molodensky II	401.55
	Smith $N^2 = 0$	403.6
	Smith $N^2 = 8.1 \times 10^{-9}$	403.5
	Smith $N^2 = 3.38 \times 10^{-7}$	405.2
Nearly Diurnal Free Wobble	Molodensky I	0.9978
	Molodensky II	0.9978
	Smith $N^2 = 8.1 \times 10^{-9}$	0.9978
	Smith $N^2 = 3.38 \times 10^{-7}$	0.9979

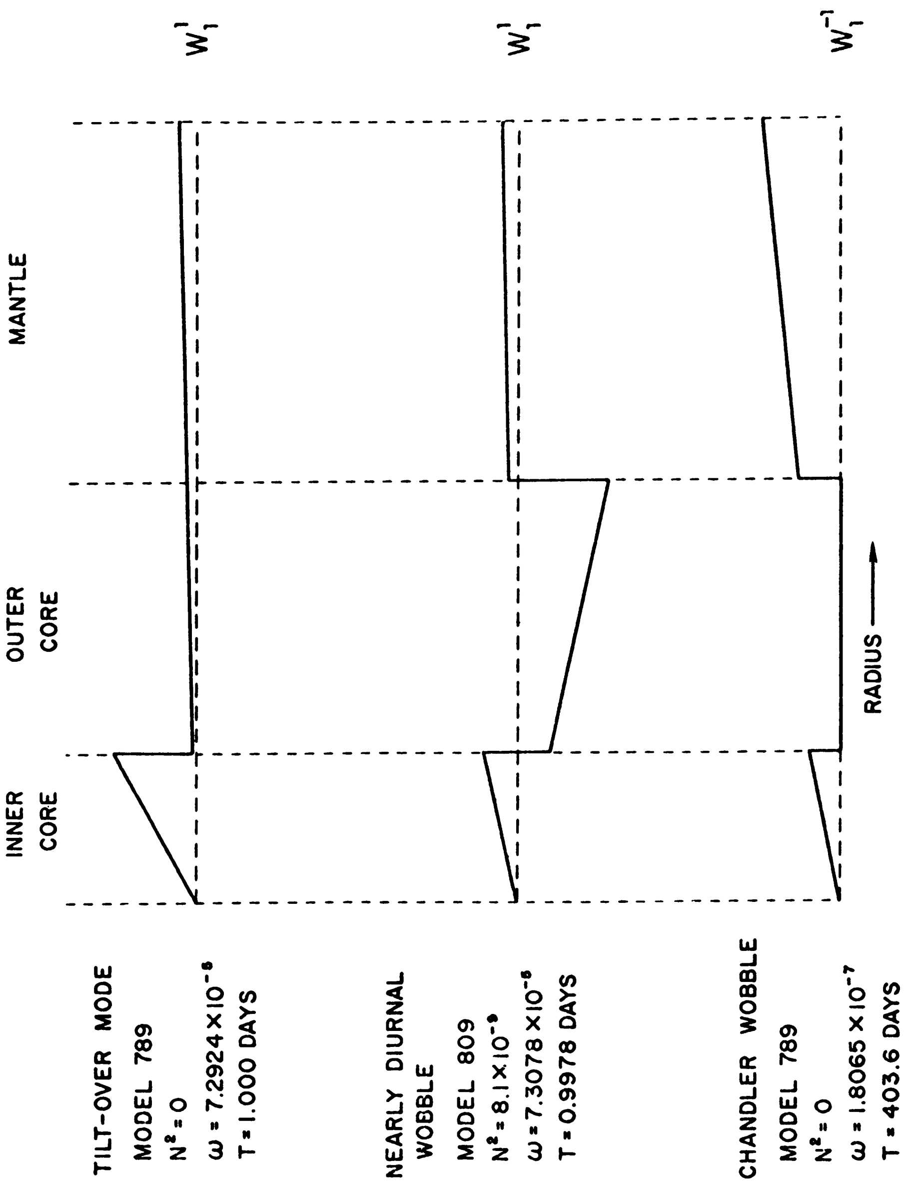
The model having $N^2 = 3.38 \times 10^{-7}$ is a rather extreme variation of core density. For the remaining two models, the Chandler eigenperiod is constant to within 0.1 sidereal days. In Smith (1977) we estimate that the nominal accuracy of these calculations is about ± 2 sidereal days. Dahlen (1976) estimated the effect of the oceans upon the Chandler period by modelling the oceans as a thin, self-gravitating, irregular, equilibrium sheet of fluid. He estimates that the oceans act to lengthen the Chandler period by 27.6 sidereal days. His results together with ours predict a Chandler period for the Earth of about 431.2 ± 2 sidereal days. This compares favorably with recent estimates by Wilson and Haubrich (1976) of the observed period of 435.2 ± 5.2 sidereal days at the 90% confidence level. So far as we can tell, then, theory and observation are currently in agreement and of comparable precision.

Figure 1 portrays the dominant spherical harmonic component, as a function of radius, of the particle displacement eigenfunctions associated with the Chandler Wobble, the Nearly Diurnal Free Wobble, and a third normal mode, the Tiltover mode, discussed below. This figure is included because it succinctly describes the principal features of these free motions.

The quantity shown is the amplitude, versus radius, of the rigid rotation of the thin spherical shell centered at that radius about an axis in the equatorial plane. More technically, it is the amplitude of the order one, degree one toroidal spherical harmonic component of the normal mode's particle displacement eigenfunction. Any region, such as the mantle, in which this quantity is exactly proportional to radius is undergoing a net rigid rotation. There are other components of the displacement eigenfunction which represent non-rigid motion of portions of the Earth, such as the elastic deformation arising from centripetal forces; these will not be discussed here. The simple linear variations depicted in Figure 1 did not come from a priori assumptions involving rigid rotations; rather they arose from solving a complex system of partial differential equations. The presence of a large rigid rotational component in the answer was the result of the physics involved, not the kinematics.

For the Chandler Wobble, we see that the motion principally consists of a rigid rotation of the mantle about an axis in the equatorial plane. As seen from the invariably rotating frame, the axis of mantle rotation itself rotates in the equator through 360° every Chandler period. The core essentially does not rotate at all. (For reasons discussed in Smith (1977) the inner core rotations shown for all of these modes are unreliable.)

For the Nearly Diurnal Wobble, the mantle and core undergo opposing rotations. The ratio of mantle to core rotation is approximately in the inverse ratio of their respective moments of inertia.



$\ell = 1$ TOROIDAL DISPLACEMENT EIGENFUNCTIONS

Figure 1

The third mode shown, the Tiltover mode, consists of a rigid rotation of the entire Earth and has an eigenperiod of exactly one sidereal day. This normal mode describes the steady rotation of the Earth about an axis slightly displaced from that of our reference frame. It is shown here for two reasons: First, the existence of this motion serves as a check on our calculations since we know theoretically that it should be present. Second, the forced excitation of this normal mode is important in describing the forced nutations.

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