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DIFFERENTIATION OF n-DIMENSIONAL ADDITIVE PROCESSES

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1. Introduction. Let $n \geqq 1$ be an integer and let $\mathbf{R}_n$ be the usual n-dimensional real vector space, considered together with all its usual structure. The usual n-dimensional Lebesgue measure on $\mathbf{R}_n$ is denoted by λ_n. The positive cone of $\mathbf{R}_n$ is $\mathbf{R}_n^+$ and the interior of $\mathbf{R}_n^+$ is $\mathbf{P}_n$. Hence $\mathbf{P}_n$ is the set of vectors with strictly positive coordinates. A subset of $\mathbf{R}_n$ is called an *interval* if it is the cartesian product of one dimensional bounded intervals. If $a, b \in \mathbf{R}_n$ then $[a, b]$ denotes the interval $\{u|a \leqq u \leqq b\}$. The closure of any interval I is of the form $[a, b]$; the initial point of I will be defined as the vector a. The class of all intervals contained in $\mathbf{R}_n^+$ is denoted by $\mathscr{I}_n$. Also, for each $u \in \mathbf{P}_n$, let $\mathscr{I}_n^u$ be the set of all intervals that are contained in the interval $[0, u]$ and that have non-empty interiors. Finally let $e_n \in \mathbf{P}_n$ be the vector with all coordinates equal to 1.

(1.1) *Continuous semigroups.* Let $(X, \mathscr{F}, \mu)$ be a measure space and let $L_1 = L_1(X, \mathscr{F}, \mu)$ be the usual Banach space of integrable functions $f: X \rightarrow \mathbf{R}$. The positive cone of L_1 is L_1^+. The main object of this paper is an n-dimensional (strongly) continuous semigroup $\{T_u\} = \{T_u\}_{u \in \mathbf{P}_n}$ of positive linear contractions on L_1. This means that:

(1.2) If $u \in \mathbf{P}_n$ then $T_u: L_1 \rightarrow L_1$ is a linear operator with norm not more than 1, such that $T_u L_1^+ \subset L_1^+$,

(1.3) If $u, v \in \mathbf{P}_n$ then $T_u T_v = T_{u+v}$.

(1.4) If $u \in \mathbf{P}_n$ and $f \in L_1$, then $\|T_v f - T_u f\| \rightarrow 0$ as $v \rightarrow u$ in $\mathbf{P}_n$.

Such a semigroup $\{T_u\}$ is said to be *continuous at the origin* if it satisfies the following additional condition:

(1.5) There is a positive linear contraction $T_0: L_1 \rightarrow L_1$ such that if $f \in L_1$ then $\|T_v f - T_0 f\| \rightarrow 0$ as $v \rightarrow 0$ in $\mathbf{P}_n$.

Note that the continuity of $\{T_u\}$ at the origin, together with the fact that T_u's are contractions, implies the uniform strong continuity of $\{T_u\}$ (i.e. given $\epsilon > 0$ and $f \in L_1$ there is a $\delta > 0$ such that $\|T_u f - T_v f\| < \epsilon$ whenever $u, v \in \mathbf{P}_n$ and $p(u, v) < \delta$, p being the euclidean distance in $\mathbf{R}_n$). Hence if $\{T_u\}$ is continuous at the origin then it can be extended to

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the closure of $\mathbf{P}_n$, which is $\mathbf{R}_n{}^+$. In this case the properties (1.2), (1.3) and (1.4) are also satisfied if $\mathbf{P}_n$ is replaced by $\mathbf{R}_n{}^+$.

In Section 2 it will be shown that for the purpose of this paper there is no loss of generality in assuming that $\{T_u\}$ is continuous at the origin and also that $T_0 = 1$ is the identity transformation. Hence, after the remarks in Section 2, these additional hypotheses will be made on the semigroup $\{T_u\}$.

(1.6) *Additive processes.* A set function $F: \mathscr{I}_n \to L_1$ will be called a *bounded additive process* (with respect to $\{T_u\}$) if it satisfies the following conditions:

$$(1.7) \qquad \sup\left\{\frac{\|F(I)\|}{\lambda_n(I)}\,\middle|\, I \in \mathscr{I}_n, \lambda_n(I) > 0\right\} = K(F) = K < \infty,$$

$$(1.8) \qquad T_uF(I) = F(u + I) \text{ for all } u \in \mathbf{P}_n \text{ and } I \in \mathscr{I}_n,$$

(1.9) If $I_1, \ldots, I_k \in \mathscr{I}_n$ are pairwise disjoint and if $I = \bigcup_{i=1}^k I_i \in \mathscr{I}_n$ then $F(I) = \sum_{i=1}^k F(I_i)$.

Note that (1.7) and (1.9) imply that if $\{I_i\}$ is a sequence of intervals converging to an interval I in the sense that $\lambda_n(I_i \Delta I) \to 0$, then $\|F(I_i) - F(I)\| \to 0$. In particular, (1.9) remains true for countably many intervals. Also, it is clear that F can be extended (in a unique way) to the class of bounded Borel subsets of $\mathbf{R}_n{}^+$. For the extended function we also have that $\|F(B_i) - F(B)\| \to 0$ whenever $\lambda_n(B_i \Delta B) \to 0$, where B_i and B are bounded Borel subsets of $\mathbf{R}_n{}^+$.

Finally we also note that if $\{T_u\}$ is continuous at the origin and if F is a bounded additive process then $T_uF(I) = F(u + I)$, $I \in \mathscr{I}_n$, is true not only for $u \in \mathbf{P}_n$ but for all $u \in \mathbf{R}_n{}^+$. Here, if $v \in \mathbf{R}_n{}^+ - \mathbf{P}_N$ then T_v of course denotes the value of the extended semigroup at this boundary point.

(1.10) *The main result.* Our purpose is to show that if F is a bounded additive process then

$$\frac{F[0, \alpha e_n]}{\lambda_n[0, \alpha e_n]} = \alpha^{-n} F[0, \alpha e_n]$$

converges a.e. as $\alpha \to 0^+$. Since, however, F's are members of L_1 and not actual functions, we can let α change only over a countable set. Following the convention in [**2**], we will consider only rational values of α and write $q - \lim_{\alpha\to 0}$ to indicate that the limit is taken as α approaches zero over the set of positive rational numbers. The main theorem is then stated as follows.

(1.11) THEOREM. *If* $F: \mathscr{I}_n \to L_1$ *is a bounded additive process then* $q - \lim_{\alpha\to 0} \alpha^{-n} F[0, \alpha e_n]$ *exists a.e.*

As in [**2**], one can also give a version of this theorem that deals with the unrestricted values of α, applied to a fixed representation of $F[0, \alpha e_n]$. This follows directly, however, from (1.11) and from the fact that if $h: \mathbf{R} \to \mathbf{R}$ is a monotone function then the existence of

$$q - \lim_{\alpha\to 0} (1/\alpha)h(\alpha)$$

is equivalent to the existence of $\lim_{\alpha\to 0} (1/\alpha)h(\alpha)$.

Theorem (1.11) generalizes a theorem of Akcoglu-Krengel [**2**], [**3**] on the differentiation of one dimensional additive processes, an n-dimensional local ergodic theorem of Terrell [**6**] and also the Lebesgue differentiation theorem on $\mathbf{R}_n$.

In fact, let $n = 1$ and let $\{\tilde{F}_u\}$, $u \in \mathbf{P}_1 = (0, \infty)$ be a "bounded additive process" in the sense of [**2**]. This means that

$$\tilde{F}_u + T_u\tilde{F}_v = \tilde{F}_{u+v}$$

and that

$$\sup\left\{\frac{\|\tilde{F}_u\|}{u}\,\middle|\, 0 < u\right\} < \infty.$$

Then, letting $\tilde{F}_0 = 0$ and defining $F(I) = \tilde{F}_b - \tilde{F}_a$ for any type of subinterval I of $\mathbf{R}_1^+ = [0, \infty)$, with end points $0 \leqq a \leqq b$, we see that F becomes a bounded additive process in the sense of the present work. Then Theorem (1.11) gives the differentiation theorem of Akcoglu-Krengel.

To obtain Terrell's theorem from (1.11), let $f \in L_1$ and define $F: \mathscr{I}_n \to L_1$ as

$$(1.12)\quad F(I) = \int_I T_u f du,\ I \in \mathscr{I}_n.$$

It is easy to see that F is a bounded additive process. Hence (1.11) gives the existence of

$$q - \lim_{\alpha\to 0} \alpha^{-n} \int_{[0,\alpha e_n]} T_u f du \quad \text{a.e.}$$

This implies the existence of the unrestricted limit, if $f \in L_1^+$. This is Terrell's local ergodic theorem, actually generalized to semigroups that are not necessarily continuous at the origin.

Finally, to see the relation between (1.11) and the Lebesgue differentiation theorem, let $X = \mathbf{R}_n$ and $\mu = \lambda_n$. Let ν be a Borel measure on $\mathbf{R}_n$ with bounded total variation. For each $I \in \mathscr{I}_n$, define $F(I) \in L_1$ as

$$[F(I)](x) = \nu(x + I).$$

Then $F: \mathscr{I}_n \to L_1$ is a bounded additive process with respect to the translation semigroup $\{T_u\}$, $u \in \mathbf{R}_n^+$, defined as

$$(T_u f)(x) = f(x + u), f \in L_1, x \in \mathbf{R}_n.$$

(Actually it will follow from the remark in (4.12) that in this case any

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bounded additive process is of this form.) Then (1.11) gives that $\alpha^{-n}\nu([x, x + \alpha e_n])$ converges λ_n – a.e. as $\alpha \to 0^+$ through the rational numbers. For the unrestricted convergence, one may assume that ν is a positive measure, which makes $\nu([x, x + \alpha e_n])$ a monotone function of α.

(1.13) *An outline of the proof of* (1.11). Since (1.11) deals with a countable class of L_1 functions, there is no loss of generality in assuming that μ is σ-finite and that the σ-algebra $\mathscr{F}$ is generated by a countable class of sets $\mathscr{A}$. We will actually assume that μ is finite. Standard arguments show that this is also not a restriction in the proof of (1.11).

We will then start, in Section 2, by reducing the semigroup $\{T_u\}$ to a semigroup that is continuous at the origin. This will be done by generalizing some results of Akcoglu-Chacon [**1**] on the decomposition of X into (initially) conservative and dissipative parts, and on some properties of the conservative part.

Section 3 contains several results that are either essentially known or that can be obtained without too much effort. First we show that a bounded additive process is the difference of two positive bounded additive processes. This is done by a routine extension of the corresponding result given by Akcoglu-Krengel [**2**] in the one-dimensional case. Section 3 also contains an outline of a technique introduced by Dunford-Schwartz [**5**] and further developed by Terrell [**6**] to reduce the $2m$-dimensional case to the m-dimensional case. The original technique is slightly extended in order to deal with the additive processes.

In Section 4 it is shown that a positive additive process is the sum of two processes that are called the absolutely continuous and the singular parts. This decomposition is obtained by using an idea of Akcoglu-Sucheston [**4**], that was used to obtain a similar result for super-additive processes in [**4**]. The convergence of the absolutely continuous part is covered by Terrell's theorem [**6**]. Hence it remains to show the convergence for the singular part.

This part of the proof of (1.11) contains the main argument of this work. First we show that the singularity of a process is equivalent to a property, which is called the localization property. Then we note that if a $2m$-dimensional semigroup has the localization property then the reduced m-dimensional process has also the same property. This makes it possible to apply an induction argument over the number of dimensions, to obtain the convergence of singular processes, starting with the known one-dimensional case. Section 4 is concluded by giving a general form of the additive processes for the translation semigroup in $\mathbf{R}_n$.

2. The conservative and dissipative parts. Let $\{T_u\}$ be a semigroup satisfying (1.2), (1.3) and (1.4) and let F be a bounded additive process with respect to $\{T_u\}$. The continuity of $\{T_u\}$ at the origin is not assumed.

If $u \in \mathbf{P}_n$ then $\{T_{tu}\}$, $t \in (0, \infty)$ is a one dimensional semigroup. Let C^u and D^u be the initially conservative and dissipative parts of X with respect to this one dimensional semigroup, as given in [**1**]. Then $\chi_{D^u}T_{tu}f = 0$ for any $t > 0$ and $f \in L_1$, where χ denotes the characteristic function of its subscript.

(2.1) LEMMA. *C^u and D^u are independent of $u \in \mathbf{P}_n$.*

Proof. Given $u, v \in \mathbf{P}_n$, there is a real $\alpha > 0$ such that $w = \alpha u - v \in \mathbf{P}_n$. Hence,

$$\chi_{D^v}T_{t\alpha u}f = \chi_{D^v}T_{tv}T_{tw}f = 0$$

for all $t > 0$ and $f \in L_1$. This means that $D^u \supset D^v$, so by symmetry, $D^u = D^v$ and $C^u = C^v$.

We will now write $D = D^u$ and $C = C^u$ for some $u \in \mathbf{P}_n$ and define D and $C = X - D$ as the initially dissipative and conservative parts of X, with respect to the n dimensional semigroup $\{T_u\}$. Note that $\chi_D T_u f = 0$ for any $u \in \mathbf{P}_n$ and $f \in L_1$.

(2.2) LEMMA. *If F is a bounded additive process then $\chi_D F(I) = 0$ for all $I \in \mathscr{I}_n$.*

Proof. Let $a \in \mathbf{R}_n^+$ be the initial point of I. Hence $I = a + I'$ with $I' \in \mathscr{I}_n$. If $a \in \mathbf{P}_n$ then

$$\chi_D F(I) = \chi_D F(a + I') = \chi_D T_a F(I') = 0.$$

If $a \in \mathbf{R}_n^+ - \mathbf{P}_n$ then we can find a sequence of intervals I_i with initial points in $\mathbf{P}_n$, such that $\lambda(I \Delta I_i) \to 0$. This implies that $F(I_i) \to F(I)$ in L_1 and hence $\chi_D F(I) = 0$, since $\chi_D F(I_i) = 0$ for each i.

(2.3) These considerations show that to prove Theorem (1.11) we may restrict the semigroup $\{T_u\}$ to the conservative part C. Hence from now on we will assume that $C = X$, in addition to the previous assumptions (1.2), (1.3) and (1.4) on $\{T_u\}$.

The following theorem is a generalization of Theorem (4.1) in [**1**].

(2.4) THEOREM. *If $C = X$ then $\{T_u\}$ is continuous at the origin.*

Proof. We have to show that (1.5) is satisfied; i.e. that there is a positive contraction T_0 on L_1 such that if $f \in L_1$ then $\|T_u f - T_0 f\| \to 0$ as $u \to 0$ in $\mathbf{P}_n$.

Let $u \in \mathbf{P}_n$ be a fixed vector. Then the one dimensional semigroup $\{T_{tu}\}_{t>0}$ is continuous at the origin, by Theorem (4.1) of [**1**]. Let T_{0u} be the initial transformation for $\{T_{tu}\}_{t>0}$.

Now let v, w be any two vectors in $\mathbf{P}_n$ such that $u = \alpha v + \beta w$ with strictly positive scalars α and β. We will show that $T_{0u} = T_{0v}T_{0w}$, which will obviously imply $T_0 = T_{0u}$ is independent of $u \in \mathbf{P}_n$. Now if $f \in L_1$ then

$$\begin{aligned}\|T_{tu}f - T_{0v}T_{0w}f\| &= \|T_{t\alpha v}T_{t\beta w}f - T_{0v}T_{0w}f\| \\ &\leqq \|T_{t\alpha v}T_{t\beta w}f - T_{t\alpha v}T_{0w}f\| + \|T_{t\alpha v}T_{0w}f - T_{0v}T_{0w}f\| \\ &\leqq \|T_{t\beta w}f - T_{0w}f\| + \|T_{t\alpha v}T_{0w}f - T_{0v}T_{0w}f\| \to 0\end{aligned}$$

as $t \to 0^+$. Hence $T_{0u} = T_{0v}T_{0w} = T_0$. Note that $T_0^2 = T_0$ and that $T_uT_0 = T_0T_u = T_u$ for any $u \in \mathbf{P}_n$.

We will now show that if $f \in L_1$ then $\|T_vf - T_0f\| \to 0$ as $v \to 0$ in $\mathbf{P}_n$.

Let $u_1, \ldots, u_n \in \mathbf{P}_n$ be a basis for $\mathbf{R}_n$. Hence any $v \in \mathbf{R}_n$ can be written as

$$v = \sum_{i=1}^{n} \alpha_i(v)u_i,$$

where α_i's are bounded linear functionals. If $v \in \mathbf{P}_n$ is such that $\alpha_i = \alpha_i(v) \geqq 0$ then it is easy to see that

$$\|T_vf - T_0f\| \leqq \sum_{i=1}^{n} \|T_{\alpha_i u_i}f - T_0f\|.$$

This shows that if $v \to 0$ remaining in $\mathbf{Q}_n = \{\sum_{i=1}^n \alpha_i u_i | \alpha_i \geqq 0\}$ then $\|T_vf - T_0f\| \to 0$. In general, for any $v \in \mathbf{R}_n$ we let

$$v^+ = \sum_{i=1}^{n} u_i[0 \vee \alpha_i(v)] \quad \text{and} \quad v^- = v^+ - v.$$

Then v^+ and v^- are in $\mathbf{Q}_n$. Now if $v \to 0$ remaining in $\mathbf{P}_n$ then both $v^+ \to 0$ and $v^- \to 0$, remaining in $\mathbf{Q}_n$. Hence,

$$\begin{aligned}\|T_vf - T_0f\| &\leqq \|T_vT_0f - T_vT_{v^-}f\| + \|T_{v^+}f - T_0f\| \\ &\leqq \|T_{v^-}f - T_0f\| + \|T_{v^+}f - T_0f\| \to 0.\end{aligned}$$

This completes the proof.

We conclude this section by observing that there is no loss of generality in assuming that T_0 is the identity operator 1. In fact, as shown in (3.8) of [**2**], by a change of measure μ to an equivalent measure one can assume that T_0 is a conditional expectation with respect to a sub σ-algebra $\mathscr{F}' \subset \mathscr{F}$. As in the proof of Lemma (2.2) one then notes that if F is a bounded additive process then $F(I)$ is measurable with respect to $\mathscr{F}'$ for each $I \in \mathscr{I}_n$. Since T_u maps $\mathscr{F}'$-measurable functions to $\mathscr{F}'$-measurable functions, we may then assume that $\mathscr{F}' = \mathscr{F}$ and hence $T_0 = 1$.

3. Some properties of F. From now on we assume that $\{T_u\}$ is a strongly continuous semigroup of positive L_1 contractions, defined for all $u \in \mathbf{R}_n^+$ and that $T_0 = 1$. A bounded additive process $F: \mathscr{I}_n \to L_1$ with respect to $\{T_u\}$ will then satisfy $T_uF(I) = F(u + I)$ for all $u \in \mathbf{R}_n^+$

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and for all $I \in \mathscr{I}_n$. As in (1.7), we let

$$K = K(F) = \sup\left\{\frac{\|F(I)\|}{\lambda_n(I)}\,\middle|\, I \in \mathscr{I}_n,\ \lambda_n(I) > 0\right\}.$$

Also recall that if $u \in \mathbf{P}_n$ then

$$\mathscr{I}_n{}^u = \{I | I \in \mathscr{I}_n,\ I \subset [0, u],\ \lambda_n(I) > 0\}.$$

Note that given any $J \in \mathscr{I}_n$ and any $\epsilon > 0$ one can find a $u \in \mathbf{P}_n$ such that if $I \in \mathscr{I}_n{}^u$ then there is an interval I', which is the disjoint union of intervals of the form $(a_i + I)$, $a_i \in \mathbf{R}_n{}^+$, for which $\lambda_n(J\Delta I') < \epsilon$. This choice of u is the choice of a 'small' vector. We will have the same situation in several other occasions: a certain property is satisfied for all $I \in \mathscr{I}_n{}^u$ if u is sufficiently close to the origin.

(3.1) LEMMA. *For each $\epsilon > 0$ there is a $u \in \mathbf{P}_n$ such that*

$$\|F(I)\| > (K - \epsilon)\lambda_n(I) \text{ for all } I \in \mathscr{I}_n{}^u.$$

Proof. First note that if an interval I' is the disjoint union of intervals of the form $(a_i + I)$, where $a_i \in \mathbf{R}_n{}^+$ and I is another fixed interval, then

$$\|F(I')\|/\lambda_n(I') \leqq \|F(I)\|/\lambda_n(I).$$

This follows from the fact that

$$\|F(a_i + I)\| = \|T_{a_i}F(I)\| \leqq \|F(I)\|$$

and from the finite additivity of F.

Now the proof of the lemma is obtained easily, first finding $J \in \mathscr{I}_n$ with $\|F(J)\|/\lambda_n(J) > K - \epsilon/2$ and then finding a $u \in \mathbf{P}_n$ such that if $I \in \mathscr{I}_n{}^u$ then there is an interval I', which is a disjoint union of intervals $(a_i + I)$, $a_i \in \mathbf{R}_n{}^+$, for which

$$\left|\frac{\|F(J)\|}{\lambda_n(J)} - \frac{\|F(I')\|}{\lambda_n(I')}\right| < \frac{\epsilon}{2}.$$

(3.2) LEMMA. *Given any $J \in \mathscr{I}_n$ and any $\epsilon > 0$ there is a $u \in \mathbf{P}_n$ such that if $I \in \mathscr{I}_n{}^u$ then*

$$\left\|F(J) - \int_J T_v \frac{F(I)}{\lambda_n(I)}\, dv\right\| < \epsilon.$$

Proof. As will be discussed in the next paragraph in more detail, the integral above is with respect to the measure λ_n and is defined as the L_1-limit of the corresponding Riemann sums. Given $J \in \mathscr{I}_n$ and $\epsilon > 0$, choose $u \in \mathbf{P}_n$ such that for any $I \in \mathscr{I}_n{}^u$ the following two conditions are satisfied:

(i) $\left\|F(J) - \dfrac{1}{\lambda_n(I)}\displaystyle\int_I T_v F(J)\, dv\right\| < \dfrac{\epsilon}{2},$

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(ii) there is an interval $I' \subset J$ such that I' is a disjoint union of intervals $a_i + I$, with $a_i \in \mathbf{R}_n^+$, $i = 1, \ldots, m$, and such that $\lambda_n(J - I') < \epsilon/4K$.

Then, for any given $I \in \mathscr{I}_n^u$ we have

$$\left\|\int_J T_v \frac{F(I)}{\lambda_n(I)} dv - \int_{I'} T_v \frac{F(I)}{\lambda_n(I)} dv\right\| = \left\|\int_{J-I'} T_v \frac{F(I)}{\lambda_n(I)} dv\right\| < \frac{\epsilon}{4},$$

and also that

$$\int_{I'} T_v \frac{F(I)}{\lambda_n(I)} dv = \sum_{i=1}^m \int_{a_i+I} T_v \frac{F(I)}{\lambda_n(I)} dv = \sum_{i=1}^m \int_I T_{a_i+v} \frac{F(I)}{\lambda_n(I)} dv$$
$$= \frac{1}{\lambda_n(I)} \int_I T_v \sum_{i=1}^m F(a_i + I) dv = \frac{1}{\lambda_n(I)} \int_I T_v F(I') dv.$$

But $\|F(J) - F(I')\| \leqq K\lambda_n(J - I') < \epsilon/4$, which implies that

$$\left\|\frac{1}{\lambda_n(I)} \int_I T_v F(I') dv - \frac{1}{\lambda_n(I)} \int_I T_v F(J) dv\right\| < \epsilon/4.$$

Hence

$$\left\|\int_J T_v \frac{F(I)}{\lambda_n(I)} dv - \frac{1}{\lambda_n(I)} \int_I T_v F(J) dv\right\| < \epsilon/2,$$

which, together with the condition (i) above, completes the proof.

(3.3) *Some L_1-valued integrals.* Let $g: \mathbf{R}_n \to L_1$ and $\phi: \mathbf{R}_n \to \mathbf{R}$ be continuous functions, where L_1 is considered with its norm topology. If $I \subset \mathbf{R}_n$ and $J \subset \mathbf{R}_n^+$ are (bounded) intervals then

$$\int_I g(u) du$$

and

$$\int_J \phi(v) F(dv)$$

are defined, in the usual manner, as the L_1-limits of Riemann sums $\sum_i g(u_i)\lambda_n(I_i)$ and $\sum_j \phi(v_j)F(J_j)$, respectively. Here $\{I_i\}$ and $\{J_j\}$ are finite partitions of I and J into intervals, respectively, and $u_i \in I_i$, $v_j \in J_j$. If

$$\int_{\mathbf{R}_n} \|g(u)\| du < \infty \quad \text{and} \quad \int_{\mathbf{R}_{n+}} |\phi(v)| dv < \infty$$

then

$$\int_{\mathbf{R}_n} g(u) du \quad \text{and} \quad \int_{\mathbf{R}_{n+}} \phi(v) F(dv)$$

are also well defined, in the usual way, since

$$\left\|\int_I g(u)du\right\| \leqq \int_I \|g(u)\|du$$

and

$$\left\|\int_J \phi(v)F(dv)\right\| \leqq K\int_J |\phi(v)|dv.$$

In our applications ϕ will always vanish on $\mathbf{R}_n - \mathbf{R}_n{}^+$ and we will write

$$\int_{\mathbf{R}_n} \phi(v)F(dv)$$

instead of

$$\int_{\mathbf{R}_n{}^+} \phi(v)F(dv).$$

Note that

$$T_u \int_{\mathbf{R}_n} \phi(v)F(dv) = \int_{\mathbf{R}_n} \phi(v-u)F(dv) \quad \text{for all } u \in \mathbf{R}_n{}^+.$$

Let $\psi: \mathbf{R}_n \times \mathbf{R}_n \to \mathbf{R}$ be a continuous function that vanishes on $\mathbf{R}_n \times (\mathbf{R}_n - \mathbf{R}_n{}^+)$ and let I and J be intervals in $\mathbf{R}_n$. Then the iterated integrals

$$\int_I \left(\int_J \psi(u,v)F(dv)\right)du \quad \text{and} \quad \int_J \left(\int_I \psi(u,v)du\right)F(dv)$$

are both well defined and equal to each other. Also, the norm of the resulting L_1 function is bounded by

$$K\int_{I\times J} |\psi(u,v)|du\,dv.$$

Hence, if

$$\int_{\mathbf{R}_n\times\mathbf{R}_n} |\psi(u,v)|du\,dv < \infty$$

and if both

$$g(u) = \int_{\mathbf{R}_n} \psi(u,v)F(dv) \quad \text{and} \quad \phi(v) = \int_{\mathbf{R}_n} \psi(u,v)du$$

define continuous functions $g: \mathbf{R}_n \to L_1$ and $\phi: \mathbf{R}_n \to \mathbf{R}$, then we will also have that

$$\int_{\mathbf{R}_n} \left(\int_{\mathbf{R}_n} \psi(u,v)F(dv)\right)du = \int_{\mathbf{R}_n} \left(\int_{\mathbf{R}_n} \psi(u,v)du\right)F(dv).$$

We note that the restriction of these definitions to continuous functions is not necessary and, as it is well known, can be removed easily. We omit this, however, as we will deal only with continuous functions.

Finally, as already mentioned in the introduction, if $f \in L_1$ then

$$F(I) = \int_I T_u f\, du, \quad I \in \mathscr{I}_n$$

defines a bounded additive process $F: \mathscr{I}_n \to L_1$. More generally let $g.: \mathbf{P}_n \to L_1$ be a function such that $\|g.\|: \mathbf{P}_n \to \mathbf{R}$ is a bounded function and such that $T_v g_u = g_{u+v}$ for all $v \in \mathbf{R}_n^+$ and for all $u \in \mathbf{P}_n$. It is clear that this function is continuous on $\mathbf{P}_n$, but not necessarily on $\mathbf{R}_n^+$. From the boundedness of the norm function it follows easily, however, that

$$G(I) = \int_I g_u du$$

is well defined for all $I \in \mathscr{I}_n$ and that $G: \mathscr{I}_n \to L_1$ is a bounded additive process.

(3.4) *Reduction of the dimension.* We will now assume that the dimension n of the semigroup is an even integer $n = 2m$. This is no loss of generality for the following reason. If n is an odd integer than starting with the n-dimensional semigroup $\{T_u\}$, $u \in \mathbf{R}_n^+$, and the additive process $F: \mathscr{I}_n \to L_1$ we define an $n' = n + 1$ dimensional semigroup $\{T'_{(u,\alpha)}\}$, $(u, \alpha) \in \mathbf{R}_n^+ \times \mathbf{R}_1^+ = \mathbf{R}_{n+1}^+$ and a corresponding additive process $F': \mathscr{I}_n \times \mathscr{I}_1 = \mathscr{I}_{n+1} \to L_1$, as

$$T'_{(u,\alpha)} = T_u,\ F'(I, J) = F(I)\lambda_1(J).$$

If the main theorem, Theorem (1.11), could be proved for F' then it would also follow for F. Hence we may assume that $n = 2m$ is an even integer.

Starting with the $2m$-dimensional semigroup $\{T_u\}$, $u \in \mathbf{R}_{2m}^+$ and an additive process $F: \mathscr{I}_{2m} \to L_1$ we will define an m-dimensional semigroup $\{S_t\}$, $t \in \mathbf{R}_m^+$, and an additive process $G: \mathscr{I}_m \to L_1$. For the definition of $\{S_t\}$ we will follow exactly the technique introduced by Dunford-Schwartz [**5**] and further developed by Terrell [**6**]; hence we omit the details. For the definition of G we will use a slight variation of the same technique.

For each $\alpha > 0$ and $\beta \in \mathbf{R}$, let

$$\eta_\alpha(\beta) = \begin{cases} \dfrac{\alpha}{2\sqrt{\pi\beta^3}}\, e^{-\alpha^2/4\beta} & \text{if } \beta > 0 \\ 0 & \text{if } \beta \leqq 0. \end{cases}$$

If $t = (t_1, \ldots, t_m) \in \mathbf{P}_m$ and $u = (u_1, \ldots, u_{2m}) \in \mathbf{R}_{2m}$ then let

$$\phi_t(u) = \prod_{i=1}^m \eta_{ti}(u_{2i-1})\eta_{ti}(u_{2i}).$$

For each fixed $t \in \mathbf{P}_m$, $\phi_t: \mathbf{R}_{2m} \to \mathbf{R}$ is a non-negative continuous function that vanishes on $\mathbf{R}_{2m} - \mathbf{R}_{2m}^+$. Also

$$\int_{\mathbf{R}_{2m}} \phi_t(u)du = 1 \quad \text{and} \quad \int_{\mathbf{R}_{2m}} \phi_t(v-u)\phi_s(u)du = \phi_{t+s}(v)$$

for each $t, s \in \mathbf{P}_m$ and $v \in \mathbf{R}_{2m}$.

Finally, for any $v \in \mathbf{P}_{2m}$ and for any $\epsilon > 0$ there is a $t^0 \in \mathbf{P}_m$ such that

$$\int_{[0,v]} \phi_t(u)du > 1 - \epsilon$$

for all $t \in \mathbf{P}_m$ with $t \leqq t^0$.

For $f \in L_1$ and $t \in \mathbf{P}_m$ we now define

$$S_t f = \int_{\mathbf{R}_{2m}} \phi_t(u) T_u f du.$$

Then $\{S_t\}$, $t \in \mathbf{P}_m$, is a (strongly) continuous m-dimensional semigroup of positive linear contractions of L_1. It is also continuous at the origin, with $S_0 = 1$. Hence it can be extended to $\mathbf{R}_m^+$ to obtain the semigroup $\{S_t\}$, $t \in \mathbf{R}_m^+$.

For $s \in \mathbf{P}_m$ we now define

$$g_s = \int_{\mathbf{R}_{2m}} \phi_s(v) F(dv).$$

From the remarks in (3.3) it follows that

$$\begin{aligned} S_t g_s &= \int_{\mathbf{R}_{2m}} \phi_t(u) T_u \left[\int_{\mathbf{R}_{2m}} \phi_s(v) F(dv) \right] du \\ &= \int_{\mathbf{R}_{2m}} \phi_t(u) \left[\int_{\mathbf{R}_{2m}} \phi_s(v-u) F(dv) \right] du \\ &= \int_{\mathbf{R}_{2m}} \left[\int_{\mathbf{R}_{2m}} \phi_t(u)\phi_s(v-u) du \right] F(dv) \\ &= \int_{\mathbf{R}_{2m}} \phi_{t+s}(v) F(dv) = g_{t+s} \end{aligned}$$

for all $t \in \mathbf{R}_m^+$ and $s \in \mathbf{P}_m$. It is also clear that

$$\|g_s\| \leqq K \int_{\mathbf{R}_{2m}} \phi_s(v) dv = K.$$

Hence

$$G(I) = \int_I g_s ds,$$

$I \in \mathscr{I}_m$ defines a bounded additive process $G: \mathscr{I}_m \to L_1$ with respect to

$\{S_t\}$, $t \in \mathbf{R}_m^+$. Note that if F is a positive process, i.e. if $F(I) \in L_1^+$ for all $I \in \mathscr{I}_{2m}$, then G is also a positive process.

(3.5) LEMMA. *There exists a constant $d > 0$, depending only on the dimension m, such that if F is a positive process then*

$$d\epsilon^{-2m}F[0, \epsilon e_{2m}] \leqq \sqrt{\epsilon}^{-m}G[0, \sqrt{\epsilon}\, e_m]$$

for all $\epsilon > 0$.

Proof. This essentially follows from Lemma (2.3) in [**6**]. There it was shown that there is a constant $\delta > 0$, such that if $\epsilon > 0$ then

$$\frac{1}{\sqrt{\epsilon}}\int_0^{\sqrt{\epsilon}} \eta_\alpha(\beta_1)\eta_\alpha(\beta_2)d\alpha > \frac{\delta}{\epsilon^2} \quad \text{whenever } 0 < \beta_1, \beta_2 < \epsilon.$$

Hence, if $\epsilon > 0$ then

$$\frac{1}{\sqrt{\epsilon}^m}\int_{[0,\sqrt{\epsilon}e_m]} \phi_s(u)ds > \frac{\delta^m}{\epsilon^{2m}}$$

whenever $u \in \mathbf{P}_{2m}$ and $u < \epsilon e_{2m}$.

Now,

$$\begin{aligned}
\frac{1}{\sqrt{\epsilon}^m}G[0,\sqrt{\epsilon}\,e_m] &= \frac{1}{\sqrt{e}^m}\int_{[0,\sqrt{\epsilon}e_m]} g_s ds \\
&= \frac{1}{\sqrt{\epsilon}^m}\int_{[0,\sqrt{\epsilon}e_m]}\left[\int_{\mathbf{R}_{2m}} \phi_s(v)F(dv)\right]ds \\
&= \frac{1}{\sqrt{\epsilon}^m}\int_{\mathbf{R}_{2m}}\left[\int_{[0,\sqrt{\epsilon}e_m]} \phi_s(v)ds\right]F(dv) \\
&\geqq \frac{1}{\sqrt{\epsilon}^m}\int_{[0,\epsilon e_{2m}]}\left[\int_{[0,\sqrt{\epsilon}e_m]} \phi_s(v)ds\right]F(dv) \\
&\geqq \frac{\delta^m}{\epsilon^{2m}}F[0, \epsilon e_{2m}].
\end{aligned}$$

Therefore it is enough to take $d = \delta^m$.

(3.6) *Decomposition of F into positive parts.* A bounded additive process F: $\mathscr{I}_n \to L_1$ can be written as the difference of two positive bounded additive processes F_i: $\mathscr{I}_n \to L_1^+$, $i = 1, 2$, as $F = F_1 - F_2$. In fact, let $I \in \mathscr{I}_n$ and let $P = \{I_1, \ldots, I_k\}$ be a partition of I into finitely many intervals. If $a_i \in \mathbf{R}_n^+$ is the initial point of I_i, then define

$$F_1^P(I) = \sum_{i=1}^k T_{a_i}[F(I_i - a_i)]^+ \text{ and } F_2^P(I) = \sum_{i=1}^k T_{a_i}[F(I_i - a_i)]^-.$$

Then it is clear that $F(I) = F_1^P(I) - F_2^P(I)$ and also that $\|F_i^P(I)\| \leqq K\lambda_n(I)$, $i = 1, 2$.

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(3.7) LEMMA. *If* $P = \{I_1, \ldots, I_k\}$ *and* $Q = \{J_1, \ldots, J_l\}$ *are two partitions of* I *into intervals such that* $P < Q$ *then* $F_i^P \leqq F_i^Q$, $i = 1, 2$.

Proof. Let a_i and b_j be the initial points of I_i and J_j. Let $M_i = \{j | 1 \leqq j \leqq l, J_j \subset I_i\}$, $i = 1, \ldots, k$. Then

$$F(I_i - a_i) = \sum_{j \in M_i} F(J_j - a_i)$$

and consequently

$$[F(I_i - a_i)]^+ \leqq \sum_{j \in M_i} [F(J_j - a_i)]^+.$$

Since $b_j - a_i \in \mathbf{R}_n^+$ for all $j \in M_i$ we then have that

$$\begin{aligned} T_{a_i}[F(I_i - a_i)]^+ &\leqq \sum_{j \in M_i} T_{a_i}[T_{b_j - a_i} F(J_j - b_j)]^+ \\ &\leqq \sum_{j \in M_i} T_{b_j}[F(J_j - b_j)]^+. \end{aligned}$$

Hence $F_1^P(I) \leqq F_1^Q(I)$, and also $F_2^P(I) \leqq F_2^Q(I)$.

From this lemma it is clear that $F_1(I) = \lim_P F_1^P(I)$ exists in L_1, where the limit is taken over the directed set of partitions P of I into intervals. It is also clear that the function $F_1: \mathscr{I}_n \to L_1^+$ so defined is a positive bounded additive process. Similarly one obtains the positive bounded additive process $F_2: \mathscr{I}_n \to L_1^+$ and then F is expressed as $F = F_1 - F_2$. Hence, to prove the main convergence theorem (1.11) it is enough to consider only positive bounded additive processes.

We also note that this decomposition has the additional property that $K(F) = K(F_1) + K(F_2)$. We omit the routine proof, as we are not going to need this property.

4. Singular and absolutely continuous processes. As before, we consider an n-dimensional continuous semigroup $\{T_u\}$, $u \in \mathbf{R}_n^+$, of positive L_1 contractions with $T_0 = 1$. We assume that $F: \mathscr{I}_n \to L_1^+$ is a positive bounded additive process with respect to $\{T_u\}$. Such a process will be called *absolutely continuous* if there is an $f \in L_1^+$ such that

$$F(I) = \int_I T_u f \, du \quad \text{for all } I \in \mathscr{I}_n,$$

and it will be called *singular* if it does not dominate any absolutely continuous nonzero positive process. If $F: \mathscr{I}_n \to L_1^+$ is absolutely continuous then Terrell's theorem [**6**] gives that

$$q - \lim_{\alpha \to 0} \alpha^{-n} F[0, \alpha e_n]$$

exists a.e. This theorem, combined with the following result, enables us to restrict our attention to singular processes in the proof of Theorem (1.11).

(4.1) THEOREM. *A positive bounded additive process is the sum of an absolutely continuous and a singular process.*

Proof. Let $i \in L_1^+$ be the function which is equal to i everywhere, $i = 1, 2, \ldots$, and let $\mathscr{A}$ be a countable class of measurable sets that generates $\mathscr{F}$. Consider a fixed sequence α_k of strictly positive numbers converging to zero, and let

$$I_k = [0, \alpha_k e_n] \text{ and } f_k = F(I_k)/\lambda_n(I_k).$$

By passing to a subsequence, if necessary, we may assume that

$$\lim_{k\to\infty} \int_A [f_k \wedge i] d\mu$$

exists for each $A \in \mathscr{A}$ and for each integer $i = 1, 2, \ldots$. Let ρ_i be the weak limit of $f_k \wedge i$ as $k \to \infty$. Then $\rho_i \leqq \rho_{i+1}$ and

$$\|\rho_i\| \leqq K = \lim_{k\to\infty} \|f_k\|.$$

Hence $\lim_{i\to\infty} \rho_i = \rho$ exists a.e. and also in L_1-norm. Let

$$F'(I) = \int_I T_u \rho du$$

for each $I \in \mathscr{I}_n$. To conclude the proof we will show that $F'' = F - F'$ is positive and singular.

For a fixed $I \in \mathscr{I}_n$ let $A_I: L_1 \to L_1$ be defined as

$$A_I f = \int_I T_u f du, \quad f \in L_1,$$

which is a positive linear bounded operator. Then, by (3.2), $A_I f_k \to F(I)$ in L_1-norm. Since $f_k \wedge i \to \rho_i$ weakly, we also have that $A_I(f_k \wedge i) \to A_I \rho_i$ weakly. Hence $A_I \rho_i \leqq F(I)$ for all i, which implies that $A_I \rho = F'(I) \leqq F(I)$. Hence $F'' = F - F'$ is a positive bounded additive process.

Let

$$f_k' = F'(I_k)/\lambda_n(I_k) \text{ and } f_k'' = F''(I_k)/\lambda_n(I_k) = f_k - f_k'.$$

We will now show that $\|f_k'' \wedge 1\| \to 0$ as $k \to \infty$. Since $f_k' \to \rho$ in L_1 we also have that $f_k' \wedge j \to \rho \wedge j$ in L_1. Note that, since $f_k' \leqq f_k$, this implies that $\rho \wedge j \leqq \rho_j$ and consequently $\rho \wedge j = \rho_j$. Now, given $\epsilon > 0$ find j such that $\|\rho - \rho_j\| < \epsilon$. Then

$$\begin{aligned} \overline{\lim}_{k\to\infty} \|f_k'' \wedge 1\| &= \overline{\lim}_{k\to\infty} \|(f_k - f_k') \wedge 1\| \\ \leqq \overline{\lim}_{k\to\infty} \|[f_k - (f_k' \wedge j)] \wedge 1\| &\leqq \overline{\lim}_{k\to\infty} \|f_k \wedge (j+1) - f_k' \wedge j\| \\ &= \|\rho_{j+1} - \rho_j\| < \epsilon. \end{aligned}$$

Now if $g \in L_1^+$ and

$$G(I) = \int_I T_u g du \leqq F''(I)$$

for all $I \in \mathscr{I}_n$, then letting $g_k = G(I_k)/\lambda_n(I_k)$, we see that $g_k \to g$ in L_1. Since $g_k \leqq f_k''$, the above result shows that $\|g \wedge 1\| = 0$ and consequently $g = 0$. Hence F'' is a positive singular process. We may also add that the sequences f_k' and g_k above are uniformly integrable, since T_u is continuous at the origin.

(4.2) *Localization property of singular processes.* The last argument of the previous proof shows that the singularity of a (positive) process F is equivalent to the following property: For each $\epsilon > 0$ there is a $u \in \mathbf{P}_n$ such that

$$\left\| \frac{F(I)}{\lambda_n(I)} \wedge 1 \right\| < \epsilon \quad \text{for all } I \in \mathscr{I}_n^u.$$

This is also equivalent to the following property: For each $\epsilon > 0$ there is a $u \in \mathbf{P}_n$ such that if $I \in \mathscr{I}_n^u$ then one can find a set $E \in \mathscr{F}$ with $\mu(E) < \epsilon$ and

$$\int_{E^c} \frac{F(I)}{\lambda_n(I)} d\mu < \epsilon.$$

We will now show that actually the singularity of F is equivalent to a much stronger property which states that the set E above can be chosen depending only on u and not on $I \in \mathscr{I}_n^u$.

(4.3) *Definition.* A function $f: \mathscr{I}_n \to L_1^+$ is said to have the *localization property* if for each $\epsilon > 0$ there is a set $E \in \mathscr{F}$, with $\mu(E) < \epsilon$, and a vector $u \in \mathbf{P}_n$ such that

$$\int_{E^c} f(I) d\mu < \epsilon \quad \text{for all } I \in \mathscr{I}_n^u.$$

(4.4) THEOREM. *Let F: $\mathscr{I}_n \to L_1^+$ be a bounded additive process and let $f(I) = F(I)/\lambda_n(I)$ if $\lambda_n(I) > 0$ and $f(I) = 0$ if $\lambda(I) = 0$, $I \in \mathscr{I}_n$. Then F is a singular process if and only if f has the localization property.*

If f has the localization property then F must be a singular process. This follows from the remarks already made in (4.2). For the other part of the proof we first obtain the following lemma.

(4.5) LEMMA. *If a function f: $\mathscr{I}_n \to L_1^+$ does not have the localization property then there is a number $\rho > 0$ and a set $B \in \mathscr{F}$ with $\mu(B) > 0$, such that if $G \subset B$ with $\mu(G) > \frac{1}{2}\mu(B)$ and if $u \in \mathbf{P}_n$ then there is an*

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$I \in \mathscr{I}_n^u$ *satisfying*

$$\int_G f(I)d\mu > \rho.$$

Proof. Let $r > 0$ and $E \in \mathscr{F}$. Call E an r-admissible set if there is a $u \in \mathbf{P}_n$ such that

$$\int_{E^c} f(I)d\mu < r$$

for all $I \in \mathscr{I}_n^u$. For each $r > 0$ let

$$\eta_r = \inf \{\mu(E) | E \text{ is an } r\text{-admissible set}\}.$$

It is clear that if $0 < r \leqq r'$ then $0 \leqq \eta_{r'} \leqq \eta_r \leqq \mu(X)$. Hence $\lim_{r\to 0^+} \eta_r = \eta \geqq 0$ exists and if $\eta = 0$ then f has the localization property. Therefore $\eta > 0$. Choose $r_0 > 0$ such that

$$\frac{9}{10}\eta < \eta_r \leqq \eta$$

for all r, $0 < r < r_0$. Then choose an $r_0/4$-admissible B such that

$$\frac{9}{10}\eta < \eta_{r_0/4} \leqq \mu(B) < \frac{11}{10}\eta.$$

If $G \subset B$ and $\mu(G) > \frac{1}{2}\mu(B)$ then

$$\mu(B - G) < \frac{9}{10}\eta$$

and consequently $B - G$ can not be $r_0/2$-admissible. Hence given $u \in \mathbf{P}_n$ there must exist an $I \in \mathscr{I}_n^u$ such that

$$\int_G f(I)d\mu > \frac{r_0}{4}.$$

The proof is then obtained with $\rho = r_0/4$.

(4.6) *Proof of Theorem* (4.4). Let F and f be as in Theorem (4.4) and assume that f does not have the localization property. Obtain the number $\rho > 0$ and the set $B \in \mathscr{F}$ from Lemma (4.5). We may assume that $\rho < \frac{1}{2}K = \frac{1}{2}K(F)$. If F is a singular process then we can find an interval $I_0 \in \mathscr{I}_n$, $\lambda_n(I_0) > 0$, and a set $H \in \mathscr{F}$ such that all of the following conditions (4.7), (4.8) and (4.9) are satisfied. Here $A: L_1 \to L_1$ denotes the averaging operator

$$Af = \frac{1}{\lambda_n(I_0)} \int_{I_0} T_u f\, du,$$

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which is a positive linear contraction.

(4.7) $\quad K - \frac{\rho}{100} < \|f(I_0)\|,$

(4.8) $\quad \mu(H) < \frac{1}{100}\mu(B) \quad \text{and} \quad \int_{H^c} f(I_0)d\mu < \frac{\rho}{100},$

(4.9) $\quad \|\chi_B - A\chi_B\| < \frac{1}{100}\mu(B).$

Here we must use Lemma (3.1) and also the fact that $T_u \to 1$ strongly as $u \to 0$ in $\mathbf{P}_n$. Now, using Lemmas (3.1) and (3.2) we find $u \in \mathbf{P}_n$ such that if $I \in \mathscr{I}_n{}^u$ then

$$K - \frac{\rho}{100} < \|f(I)\| \quad \text{and} \quad \|f(I_0) - Af(I)\| < \frac{\rho}{100}.$$

Hence

$$\int_{H^c} Af(I)d\mu < \frac{2\rho}{100} \quad \text{and} \quad \|Af(I)\| > \|f(I)\| - \frac{2\rho}{100}$$

for all $I \in \mathscr{I}_n{}^u$.

Therefore, if $I \in \mathscr{I}_n{}^u$ and if $g \in L_1{}^+$ is any function with $g \leqq f(I)$ and $\|g\| = \rho$, then

$$\|Ag\| > \frac{98}{100}\rho \quad \text{and} \quad \int_{H^c} Agd\mu < \frac{2\rho}{100}$$

and, consequently,

$$\int_H Agdu > \frac{96}{100}\rho = \frac{96}{100}\|g\|.$$

Now call a set $E \in \mathscr{F}$ a bad set if

$$\int_H A\chi_E d\mu > \frac{96}{100}\mu(E).$$

We will show that if $G \subset B$ and if $\mu(G) > \frac{1}{2}\mu(B)$ then G has a bad subset of nonzero measure. In fact, we can find an interval $I \in \mathscr{I}_n{}^u$ such that

$$\int_G f(I)d\mu > \rho.$$

Hence we can find a simple function $g = \sum_{i=1}^k \alpha_i\chi_{E_i}$ with $\alpha_i > 0$, $E_i \subset G$, $\mu(E_i) > 0$, such that $g \leqq f(I)$ and $\|g\| = \rho$. Hence

$$\sum_{i=1}^k \alpha_i \int_H A\chi_{E_i}d\mu > \sum_{i=1}^k \alpha_i \frac{96}{100}\mu(E_i),$$

which implies that at least one E_i is a bad set. Since an increasing union of bad sets is also a bad set, we see that B must have a bad subset E of measure $\mu(E) \geqq \frac{1}{2}\mu(B)$. Then

$$\int_H A\chi_B d\mu \geqq \int_H A\chi_E d\mu > \frac{96}{100}\mu(E) \geqq \frac{48}{100}\mu(B).$$

But this contradicts (4.8) and (4.9). Therefore if F is singular then f must have the localization property.

(4.10) *Singularity of the reduced process.* Let $F: \mathscr{I}_{2m} \to L_1$ be a positive bounded additive process with respect to a $2m$-dimensional semigroup $\{T_u\}$, $u \in \mathbf{R}_{2m}^+$, and let $G: \mathscr{I}_m \to L_1$ be the reduced process with respect to the m-dimensional semigroup $\{S_t\}$, $t \in \mathbf{R}_m^+$ as defined in (3.4). We would like to show that if F is singular then G is also singular.

Now we have

$$G(I) = \int_I g_s ds, \quad I \in \mathscr{I}_m,$$

where

$$g_s = \int_{\mathbf{R}_{2m}} \phi_s(v) F(dv),$$

by the definitions in (3.4). Since F is singular, for each $\epsilon > 0$ we can find a set $E \in \mathscr{F}$ and a vector $u \in \mathbf{P}_{2m}$ such that $\mu(E) < \epsilon$ and such that

$$\int_{E^c} F(I) d\mu < \epsilon\lambda_{2m}(I) \quad \text{for all } I \in \mathscr{I}_{2m}{}^u,$$

by Theorem (4.4). This implies that

$$\int_{E^c} \left[\int_{[0,u]} \phi_s(v) F(dv)\right] d\mu < \epsilon \int_{[0,u]} \phi_s(v) dv < \epsilon.$$

Now, by the definition of $\phi_s(v)$, we can find a $t \in \mathbf{P}_m$ such that if $s \in \mathbf{P}_m$ and if $s \leqq t$ then

$$\int_{\mathbf{R}_{2m}-[0,u]} \phi_s(v) dv < \frac{\epsilon}{K}.$$

Therefore, if $s \in \mathbf{P}_m$ and $s \leqq t$, then

$$\int_{E^c} g_s d\mu = \int_{E^c} \left[\int_{[0,u]} \phi_s(v) F(dv)\right] d\mu + \int_{E^c} \left[\int_{\mathbf{R}_{2m}-[0,u]} \phi_s(v) F(dv)\right] d\mu < \epsilon + \left\|\int_{\mathbf{R}_{2m}-[0,u]} \phi_s(v) F(dv)\right\| < 2\epsilon.$$

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This implies that

$$\int_{E^c} \frac{G(I)}{\lambda_m(I)} d\mu < 2\epsilon \quad \text{for all } I \in \mathscr{I}_m{}^t.$$

Hence G is also a singular process.

(4.11) *Proof of the main theorem.* To prove the existence of $q - \lim_{\alpha\to 0} \alpha^{-n} F[0, \alpha e_n]$ a.e. we may assume that F is a singular process. If $n = 1$ then this theorem is proved in [**2**]. In this case the limit is zero. By Lemma (3.5), if this limit exists and is zero for the m-dimensional case, then the same is also true for the $2m$ dimensional case. This completes the proof.

(4.12) *Singular processes for the translation group.* Let $(X, \mathscr{F}, \mu)$ be $\mathbf{R}_n$ with the Borel sets and the Lebesgue measure λ_n. Let $\{T_u\}$, $u \in \mathbf{R}_n{}^+$ be the translation semigroup, defined as

$$(T_u F)(x) = f(x + u), f \in L_1, x \in \mathbf{R}_n.$$

We would like to note that any singular bounded additive process F with respect to $\{T_u\}$ is of the form $(F(I))(x) = \nu(x + I)$, where $x \in \mathbf{R}_n$, $I \in \mathscr{I}_n$ and ν is a Borel measure on $\mathbf{R}_n$, singular with respect to λ_n.

In fact, let $\alpha_k > 0$ be a sequence converging to zero and let

$$f_k = F[0, \alpha_k e_n]/\lambda_n[0, \alpha_k e_n].$$

Then f_k is a bounded sequence in $L_1{}^+$. Choosing a subsequence we may assume that

$$\lim_{k\to\infty} \int_{\mathbf{R}_n} f_k \xi d\lambda_n = \psi(\xi)$$

exists for each bounded continuous function ξ: $\mathbf{R}_n \to \mathbf{R}$. Then there is a finite Borel measure ν on $\mathbf{R}_n$ such that

$$\psi(\xi) = \int_{\mathbf{R}_n} \xi d\nu.$$

This measure ν must be singular with respect to λ_n, since $f_k \to 0 \; \lambda_n -$ a.e. Now, by (3.2),

$$\lim_{k\to\infty} \int_I T_u f_k du = F(I), \quad \text{in } L_1,$$

for each $I \in \mathscr{I}_n$. Hence

$$\lim_{k\to\infty} \int_{\mathbf{R}_n} \xi \left[\int_I T_u f_k du \right] d\lambda_n = \int_{\mathbf{R}_n} \xi F(I) d\lambda_n$$

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for each bounded and continuous $\xi: \mathbf{R}_n \to \mathbf{R}$. But

$$\int_{\mathbf{R}_n} \xi\left[\int_I T_u f_k du\right] d\lambda_n = \int_{\mathbf{R}_n} \xi(x)\left[\int_I f_k(x+u)du\right]dx$$
$$= \int_I \left[\int_{\mathbf{R}_n} \xi(x-u) f_k(x)dx\right]du$$

and this converges to

$$\int_I \left[\int_{\mathbf{R}_n} \xi(x-u)\nu(dx)\right]du,$$

which is equal to

$$\int_{\mathbf{R}_n} \xi(x)\nu(x+I)dx.$$

Hence $(F(I))(x) = \nu(x+I)$ for λ_n – a.a. $x \in \mathbf{R}_n$. We note that similar considerations are valid whenever $\{T_u\}$ is induced by a measurable flow of X. In this case, however, one should deal with the points in $L_\infty{}^*$ that do not correspond to points in L_1, instead of singular measures. We omit the details.

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