Can. J. Math., Vol. XLI, No. 4, 1989, pp. pp. 592–611

REFLEXIVE BIMODULES

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If V_K is a finite dimensional vector space over a field K and L is a lattice of subspaces of V, then, following Halmos [**11**], alg L is defined to be (the K-algebra of) all K-endomorphisms of V which leave every subspace in L invariant. If $R \subseteq \operatorname{end}(V_K)$ is any subalgebra we define lat R to be (the sublattice of) all subspaces of V_K which are invariant under every transformation in R. Then $R \subseteq$ alg[lat R] and R is called a reflexive algebra when this is equality. Every finite dimensional algebra is isomorphic to a reflexive one ([**4**]) and these reflexive algebras have been studied by Azoff [**1**], Barker and Conklin [**3**] and Habibi and Gustafson [**9**] among others.

Our point of departure starts with the following observation. If $R \subseteq \operatorname{end}(V_K)$ is as above then $V = {}_RV_K$ is a bimodule. Moreover alg(lat R) consists of those endomorphisms in $\operatorname{end}(V_K)$ which leave invariant every R- submodule of ${}_RV$ and as such is determined by the bimodule ${}_RV_K$. This leads to our notion of $\operatorname{alglat}({}_RV_\Delta)$ for any bimodule ${}_RV_\Delta$ (e.g., R an algebra of operators on V_K and Δ any subalgebra of the endomorphism ring (or *commutant*) $\operatorname{end}({}_RV)$) and to the idea of a reflexive bimodule. Moreover, it provides extensions and new proofs of earlier results, and it encompasses other situations (for example Hadwin and Kerr [**10**] study the case where $R = \Delta$ is any commutative ring).

Throughout the paper we will be concerned with bimodules ${}_RM_\Delta$ where R and Δ are rings with unity, we will write all ring and module homomorphisms in the left of their arguments, and R-modules will generally be left R-modules.

1. The ring $\operatorname{alglat}({}_RM_\Delta)$.

Given a bimodule ${}_RM_\Delta$ we define

$$\begin{aligned}\operatorname{alglat}({}_RM_\Delta) &= \{\alpha \in \operatorname{end}(M_\Delta) | \alpha K \subseteq K \quad \text{for all } {}_RK \subseteq {}_RM\} \\ &= \{\alpha \in \operatorname{end}(M_\Delta) | \alpha m \in Rm \quad \text{for all } m \in M\}\end{aligned}$$

and we write this as alglat M when no confusion can result. Clearly $\operatorname{alglat}({}_RM_\Delta) = \operatorname{end}(M_\Delta)$ if ${}_RM$ is simple, and $\operatorname{alglat}({}_RR_R) \cong R$. Hadwin and Kerr [**10**] deals with $\operatorname{alglat}({}_RM_R)$ where R is commutative, while Habibi and Gustafson [**9**] considers $\operatorname{alglat}(K^n)$ where $K = \Delta$ is a field and R a certain type of subring of the matrix ring $M_n(K)$. In this section we present general

Received January 28, 1988. This research was supported by NSERC Grant A8075, by University of Calgary Visiting Scholar Awards 862328 and 862346, and by the University of Leicester Research Board.

notions and results regarding alglat$({}_RM_\Delta)$, and we extend results in [**1**], [**4**], [**8**] and [**10**]. We begin with

LEMMA 1.1. *Let M_i be an* R-Δ-*bimodule,* $1 \leq i \leq k$, *and let* $\alpha \in$ alglat $(M_1 \oplus \ldots \oplus M_k)$. *Then there exist* $\alpha_i \in$ alglat M_i, $1 \leq i \leq k$ *such that*

$$\alpha(m_1, \ldots, m_k) = (\alpha_1 m_1, \ldots, \alpha_k m_k) \quad \textit{for all} \quad (m_1, \ldots, m_k) \in M_1 \oplus \ldots \oplus M_k.$$

Furthermore if $i \neq j$ then

$$\alpha_j \gamma = \gamma \alpha_i \quad \textit{for all } R\textit{-homomorphisms } \gamma : N_i \longrightarrow M_j, N_i \subseteq M_i.$$

Proof. The first statement follows from the fact that, being submodules, the canonical images of the M_i in $M_1 \oplus \ldots \oplus M_k$ are stable under each $\alpha \in$ alglat$(M_1 \oplus \ldots \oplus M_k)$. Moreover if $\gamma : N_i \longrightarrow M_j$ is any R-homomorphism with $N_i \subseteq M_i$, let $m_i \in N_i$ and consider

$$x = (0, \ldots, m_i, \ldots, \gamma m_i, \ldots, 0) \in \bigoplus_{i=1}^{k} M_i.$$

Then $\alpha x = rx$ for some $r \in R$ so $\alpha_i m_i = r m_i$ and $\alpha_j(\gamma m_i) = r(\gamma m_i)$. Since γ is R-linear, it follows that $\alpha_j \gamma = \gamma \alpha_i$.

The map α in Lemma 1.1 will be denoted

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k).$$

Given ${}_RM_\Delta$ it is clear that alglat M is a subring of end(M_Δ) which contains the image of R under the canonical ring homomorphism

$$\gamma_M : R \longrightarrow \mathrm{end}(M_\Delta).$$

Thus

$$\lambda_M : R \longrightarrow \text{alglat } M \quad \text{where } \lambda_M(r) = r\cdot$$

and $r\cdot$ denotes multiplication by the element r. Then ${}_RM_\Delta$ is called a *reflexive bimodule* if

$$\text{alglat}({}_RM_\Delta) = \lambda_M(R),$$

that is, if the only Δ-linear maps $M_\Delta \longrightarrow M_\Delta$ which leave invariant every R-submodule of ${}_RM$ are given by multiplication by an element of R.

The following two results provide examples of reflexive and non- reflexive bimodules, and will prove useful in the sequel.

PROPOSITION 1.2. *Given $R = {}_R R_\Delta$, let*

$$R = e_1\Delta + e_2\Delta + \cdots + e_n\Delta + L$$

where $L = {}_R L_\Delta$ and the e_i are orthogonal idempotents with $1 \in \Sigma e_i \Delta$. Then $L \oplus R$ is reflexive.

Proof. Let $(\alpha, \beta) \in \operatorname{alglat}(L \oplus R)$. Then for each $i \in \{1, \ldots, n\}$ there exist $b_i \in R$ such that $\beta e_i = b_i e_i$, so letting

$$b = \sum_{i=1}^{n} b_i e_i$$

we have

$$b e_j = \beta e_j,\ j = 1, \ldots, n.$$

If $x \in L$ then $(\alpha, \beta)(x, x) \in R(x, x)$ so $\alpha x = \beta x$. Now, if $x \in L$ then $(\alpha, \beta)(x, 1) = r(x, 1), r \in R$, so $\alpha x = rx$ and $\beta(1) = r$. But $\beta(1) = b \cdot 1$ because $1 \in \Sigma e_i \Delta$ so $r = b$ and $\alpha = b\cdot$ on L. Thus $\beta = \alpha = b\cdot$ on L whence $\beta = b\cdot$ on $R = \Sigma e_i \Delta + L$. Finally then $(\alpha, \beta) = b\cdot$ on $L \oplus R$.

PROPOSITION 1.3. *Let R be a ring with Jacobson radical $J = J(R)$, and suppose that Δ is a division subring of R such that $R = \Delta \oplus J$. If $J \neq 0$ then ${}_R R_\Delta$ is not reflexive.*

Proof. By hypothesis R is a local ring, so that every proper left ideal is contained in J. Thus the Δ-projection $\pi : R_\Delta \longrightarrow R_\Delta$ onto Δ with $\ker \pi = J$ belongs to $\operatorname{alglat}({}_R R_\Delta)$. Now suppose $\pi = p\cdot$ on R for some $p \in R$. Then $p = p \cdot 1 = \pi 1 = 1$ so $\pi = 1$. Hence $\Delta = \operatorname{im} \pi = R$, contrary to $J \neq 0$.

For example, let Δ be a division ring, let

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \middle| a, b \in \Delta \right\}$$

and let $M = \Delta^2$ viewed as columns, so ${}_R M_\Delta \cong_R R_\Delta$ using matrix multiplication. Then $M \oplus J(R)$ is reflexive, and M is not. In fact, as a subring of $\mathbf{M}_2(\Delta)$,

$$\operatorname{alglat}(M) = \mathbf{UTM}_2(\Delta),$$

the full ring of upper triangular matrices, because the (2,1)-entry of any member of $\operatorname{alglat}(M)$ clearly must be zero, and $\operatorname{alglat}(M)$ is a Δ-subspace properly containing R.

Given bimodules ${}_RM_\Delta$ and ${}_RN_\Delta$, and subrings

$$\lambda_M(R) \subseteq S \subseteq \operatorname{end}(M_\Delta) \quad \text{and} \quad \lambda_N(R) \subseteq T \subseteq \operatorname{end}(N_\Delta),$$

a ring homomorphism $\sigma : S \longrightarrow T$ will be called *R-compatible* in case the diagram

$$\begin{tikzcd} S \arrow[rr, "\theta"] & & T \\ & R \arrow[ul, "\lambda_M"] \arrow[ur, "\lambda_N"'] & \end{tikzcd}$$

commutes. If θ is injective we shal say it is an *R-compatible embedding*. In particular, if

$$\theta : \operatorname{alglat} M \longrightarrow \operatorname{alglat} N$$

is an R-compatible embedding and N is reflexive then M is too. If

$$f : {}_RK_\Delta \longrightarrow {}_RM_\Delta \quad \text{and} \quad g : {}_RM_\Delta \longrightarrow {}_RL_\Delta$$

are, respectively, a monomorphism and an epimorphism of bimodules, then since $\alpha(\operatorname{im} f) \subseteq \operatorname{im} f$ and $\alpha(\ker g) \subseteq \ker g$ for $\alpha \in \operatorname{alglat} M$, it follows that $f^{-1}\alpha f$ and $g\alpha g^{-1}$ are well defined elements of alglat K and alglat L, respectively. In fact

$$\alpha \longrightarrow f^{-1}\alpha f \quad \text{and} \quad \alpha \longrightarrow g\alpha g^{-1}$$

define R-compatible ring homomorphisms

$$\operatorname{alglat}(M) \longrightarrow \operatorname{alglat}(K) \quad \text{and} \quad \operatorname{alglat} M \longrightarrow \operatorname{alglat} L,$$

respectively. We shall refer to these as the *canonical ring homomorphisms induced by f and g*.

The remaining results in this section concern R-compatible embeddings. Given ${}_RM_\Delta$ write M^k for the direct sum of k copies of M. Then M^k is an E-Δ-bimodule where $E = \operatorname{end}(M_\Delta)$ and so, if $k \geqq 1$, we may define

$$\begin{aligned} \operatorname{alg}_k\operatorname{lat}(M) &= \{\alpha \in \operatorname{end}(M_\Delta) \mid \alpha X \subseteq X \quad \text{for all } {}_RX \subseteq {}_RM^k\} \\ &= \{\alpha \in \operatorname{end}(M_\Delta) \mid \text{Given}\{m_1, \ldots, m_k\} \subseteq M \text{ there exists} \\ &\qquad r \in R \text{ such that } \alpha m_i = rm_i \quad \text{for all } i\}. \end{aligned}$$

Thus $\operatorname{alg}_1\operatorname{lat} M = \operatorname{alglat} M$ and these $\operatorname{alg}_k\operatorname{lat} M$ are subrings of $\operatorname{alglat} M$ such that

$$\lambda_M(R) \subseteq \cdots \subseteq \operatorname{alg}_k\operatorname{lat} M \subseteq \operatorname{alg}_{k-1}\operatorname{lat} M \subseteq \cdots \subseteq \operatorname{alglat} M$$

(see [**8**], page 11).

LEMMA 1.4. *Given ${}_RM_\Delta$ and $k \geqq 1$, the map*

$$\theta : \mathrm{alg}_k\mathrm{lat}\, M \longrightarrow \mathrm{alglat}\,(M^k)$$

given by $\theta(\alpha) = (\alpha, \alpha, \ldots, \alpha)$ is an R-compatible ring isomorphism.

Proof. If $\alpha \in \mathrm{alg}_k\mathrm{lat}\, M$ and $(m_1, \ldots, m_k) \in M^k$ let $r \in R$ satisfy $\alpha m_i = rm_i$ for each i. Then

$$(\alpha, \ldots, \alpha)(m_1, \ldots, m_k) = r(m_1, \ldots, m_k)$$

and so $\theta(\alpha) \in \mathrm{alglat}\,(M^k)$. It is now clear that θ is a one-to-one ring homomorphism, compatible with R. Finally let $\beta \in \mathrm{alglat}\,(M^k)$. Then Lemma 1.1 gives $\beta = \{\alpha_1, \ldots, \alpha_k\}$ where each $\alpha_i \in \mathrm{alglat}\, M$. Moreover taking $\gamma = 1_M$ in Lemma 1.1 shows that $\alpha_i = \alpha_j$ for all $i \neq j$. This completes the proof.

Lemma 1.4 leads to the following extension of a result of Habibi [**8**] for finite dimensional algebras.

PROPOSITION 1.5. *Let $1 = e_1 + e_2 + \cdots + e_k$ in R where the e_i are orthogonal idempotents. Given ${}_RM_\Delta$ assume e_iM can be Δ-generated by n_i elements, $1 \leqq i \leqq k$. If $n = \max\{n_1, n_2, \ldots, n_k\}$ then M^n is reflexive.*

Proof. Let

$$e_iM = \sum_{j=1}^{n_i} m_{ij}\Delta \text{ where } e_im_{ij} = m_{ij} \text{ for each } j.$$

If $\beta \in$ alglat (M^n) then Lemma 1.4 gives $\beta = (\alpha, \alpha, \ldots, \alpha)$ for some $\alpha \in \mathrm{alg}_n\mathrm{lat}(M)$. Since $n_i \leqq n$ there exists $r_i \in R$ such that $\alpha m_{ij} = r_im_{ij}$ for each $j = 1, 2, \ldots, n_i$. Define

$$r = \sum_{t=1}^{k} r_te_t.$$

If $1 \leqq i \leqq k$ and $1 \leqq j \leqq n_i$ we have

$$rm_{ij} = r(e_im_{ij}) = \sum_{t=1}^{k} r_te_t(e_im_{ij}) = r_ie_im_{ij} = r_im_{ij} = \alpha m_{ij}.$$

It follows that $\alpha = r\cdot$ on e_iM, and hence on $M = \Sigma e_iM$. But then $\beta = (\alpha, \alpha, \ldots. \alpha) = r\cdot$ on M^n, as required.

In particular this shows that M^n is reflexive if M_Δ can be generated by n elements.

Given a left module ${}_RM$, write $E = \text{end}({}_RM)^{\text{op}}$ so that $M = {}_RM_E$. We write $\text{biend}({}_RM) = \text{end}(M_E)$, note that $\lambda_M(R) \subseteq \text{biend}({}_RM)$, and recall that ${}_RM$ is *balanced* if equality holds. (The *biendomorphism ring* $\text{biend}({}_RM)$ corresponds to the *second commutant* in operator theory.)

PROPOSITION 1.6. *Given ${}_RM_\Delta$ the map*

$$\varphi : \text{alglat}(M^2) \longrightarrow \text{biend}({}_RM)$$

given by $\varphi(\alpha, \alpha) = \alpha$ is an R*-compatible ring embedding. In particular, ${}_RM^2_\Delta$ is reflexive whenever ${}_RM$ is balanced.*

Proof. We have

$$\theta^{-1} : \text{alglat}(M^2) \longrightarrow \text{alg}_2\text{lat}\, M$$

where θ is the map in Lemma 1.4, and so it suffices to show

$$\text{alg}_2\text{lat}\, M \subseteq \text{biend}({}_RM).$$

If $\alpha \in \text{alg}_2\text{lat}\, M$, let $\lambda \in \text{end}({}_RM)$ and $m \in M$. Then there exists $r \in R$ such that $\alpha m = rm$ and $\alpha(\lambda m) = r(\lambda m)$. But λ is R-linear so $\alpha(\lambda m) = \lambda(rm) = \lambda(\alpha m)$, and hence $\alpha \in \text{biend}({}_RM)$ as well.

This last result was observed for ${}_RR_K$ by Brenner and Butler [**4**], and for ${}_RM_K$ by Azoff [**1**, Lemma 4.3] and Habibi [**8**] when R is a finite dimensional algebra over a field K.

The following notion is useful in obtaining R-compatible embeddings. Given bimodules ${}_RM_\Delta$ and ${}_RN_\Delta$ we say that M *controls* N if, for all $\alpha \in \text{end}(N_\Delta)$ and all $n \in N$, there exists

$$\{(n_i, m_i) \mid i \in I\} \subseteq N \times M$$

with the following property:

If $r_i \in R, i \in I$, and $r \in R$ satisfy

$$\alpha n_i = r_i n_i \quad \text{and} \quad r_i m_i = r m_i$$

for all $i \in I$, then $\alpha n = rn$.

Such a subset $\{(n_i, m_i) | i \in I\} \subseteq N \times M$ will be called a *connection for α and n*. It is easy to verify that if M controls N and M is a subquotient of U, then U controls N.

As in [**2**, Section 8] a collection of modules U generates (cogenerates) N in case N is an epimorph of a direct sum (resp. embeds in a direct product) of members of U.

LEMMA 1.7. *A bimodule ${}_RM_\Delta$ controls ${}_RN_\Delta$ in either of the following situations:*

(1) *${}_RN$ is generated by submodules of ${}_RM$;*

(2) *${}_RN$ is cogenerated by images of ${}_RM$.*

Proof. Suppose $\alpha \in \operatorname{end}(N_\Delta)$ and $n \in N$ are given. In the first case let

$$\pi : \bigoplus_{i \in I} M_i \longrightarrow N$$

be R-epic where $M_i \subseteq M$ for each i, and let

$$\sigma_j : M_j \longrightarrow \bigoplus_{i \in I} M_i$$

be the canonical maps. If $n = \sum \pi\sigma_i m_i$ and we write $n_i = \pi\sigma_i m_i$ for each i, then $\{(n_i, m_i) | i \in I\}$ is a connection. Indeed, if $\alpha n_i = r_i n_i$ and $r_i m_i = r m_i$ for all i, then $\alpha n_i = r n_i$ for each i so, since $n = \sum n_i$, $\alpha n = rn$.

In the second case let

$$\sigma : N \longrightarrow \prod_{i \in I} \varphi_i(M)$$

be R-monic where each φ_i is an R-epimorphism, and let

$$\pi_j : \prod_{i \in I} \varphi_i M \longrightarrow \varphi_j M$$

be the canonical projections. If $\pi_i \sigma n = \varphi_i m_i, m_i \in M$, then $\{(n, m_i) | i \in I\}$ is a connection. For if $\alpha n = r_i n$ and $r_i m_i = r m_i$ for all i then $\pi_i \sigma \alpha n = \pi_i \sigma(rn)$ for each i so $\sigma \alpha n = \sigma(rn)$. Hence $\alpha n = rn$ because σ is one-to-one.

PROPOSITION 1.8. *Let ${}_RM_\Delta$ and ${}_RN_\Delta$ be bimodules such that M controls N. If ${}_RL_\Delta$ is isomorphic to either a submodule or an image of ${}_RN_\Delta$, then the canonical ring homomorphism*

$$\operatorname{alglat}(N \oplus M) \longrightarrow \operatorname{alglat}(L \oplus M)$$

is injective. In particular if ${}_R(L \oplus M)_\Delta$ is reflexive then so is ${}_R(N \oplus M)_\Delta$.

Proof. Suppose that $g : N \longrightarrow L$ is an R-Δ-epimorphism. Then the canonical ring homomorphism

$$\operatorname{alglat}(N \oplus M) \longrightarrow \operatorname{alglat}(L \oplus M)$$

is given by

$$(\alpha, \beta) \longrightarrow (g\alpha g^{-1}, \beta).$$

If (α, β) is in the kernel, then $\beta = 0$, so $(\alpha, 0) \in \operatorname{alglat}(N \oplus M)$. Thus, if $n \in N$ and $\{(n_i, m_i) | i \in I\}$ is a connection for α and n then there exist $r_i \in R$ with

$$(\alpha, 0)(n_i, m_i) = (r_i n_i, r_i m_i).$$

But then $r_i m_i = 0 \cdot m_i$ for all $i \in I$, so by definition of a connection $\alpha n = 0 \cdot n = 0$. So in this case the canonical ring homomorphism is an embedding. The other case is entirely similar.

COROLLARY 1.9. *If ${}_RM_\Delta$ is a reflexive bimodule and ${}_RN_\Delta$ is a bimodule such that ${}_RN$ is either generated by submodules or cogenerated by images of ${}_RM$, then $N \oplus M$ is reflexive.*

We conclude this section by providing a generalization of results of Hadwin and Kerr [10] dealing with a commutative ring R and $\operatorname{alglat}({}_RM_R)$ where $rm = mr$. In this setting we say M is reflexive if ${}_RM_R$ is a reflexive bimodule. (In later sections we consider $\operatorname{alglat}({}_RM_K)$ where R is a K-algebra.) Recalling that an artinian selfinjective ring (e.g. a group algebra) is called QF (or quasi-Frobenius) we have

THEOREM 1.10. *The following statements about a commutative artinian ring R are equivalent:*
(a) *Every faithful 2-generated R-module is reflexive;*
(b) *R is QF;*
(c) *Every faithful R-module is reflexive.*

Proof. Since R is a direct sum of local rings, we may assume that R is in fact local.

(a) $\Rightarrow$ (b). Let R be any commutative local ring and suppose that p and $q \in J = J(R)$ generate distinct minimal idels of R, so that $J = \operatorname{ann}(p) = \operatorname{ann}(q)$, and p and q are independent over R/J. Let

$$K = \left\{ \begin{pmatrix} up + vq \\ uq \end{pmatrix} \middle| u, v \in R \right\} \subseteq {}_RR^2,$$

and let

$$M = R^2/K.$$

Then, by the independence of p and q we see that ${}_RM$ is faithful. To see that M is not reflexive, let

$$A = \begin{pmatrix} 1 & p \\ q & 1 \end{pmatrix} \in M_2(R).$$

Then $AK \subseteq K$ so

$$\begin{pmatrix} r \\ s \end{pmatrix} + K \longrightarrow A \begin{pmatrix} r \\ s \end{pmatrix} + K$$

defines $\alpha \in \mathrm{end}(M_R)$. If $s \in J$, let $t_1 = 1 - p$. Then it is easy to check that

$$A\binom{r}{s} - t_1\binom{r}{s} \in K.$$

If $s \notin J$ and $t_2 = (s^{-1}r - 1)q + 1$ then

$$A\binom{r}{s} - t_2\binom{r}{s} \in K.$$

Thus $\alpha \in \mathrm{alglat}(M)$. Moreover a simple computation shows that

$$A\begin{pmatrix}1\\0\end{pmatrix} - t\begin{pmatrix}1\\0\end{pmatrix} \in K$$

and

$$A\begin{pmatrix}0\\1\end{pmatrix} - t\begin{pmatrix}0\\1\end{pmatrix} \in K$$

would violate the independence of p and q, so $\alpha \notin \lambda_M(R)$. Thus if every faithful 2-generated R-module is reflexive then R contains at most one minimal ideal. Since it is well known that an artinian local ring is QF if and only if its left and right socle are simple (see [**7**], for example), this proves that (a) implies (b).

(b) $\Rightarrow$ (c) Since ${}_RR_R$ is always reflexive, and ${}_RR$ is isomorphic to a direct summand of every faithful R-module whenever R is a commutative (hence basic) QF-ring, this implication follows from Corollary 1.9.

The results of Hadwin and Kerr [**10**, Theorems 9 and 12] are contained in

COROLLARY 1.11. *Let R be a commutative semiprimary ring. Then the following are equivalent:*

(a) *Every 2-generated R-module is reflexive;*

(b) *R is uniserial;*

(c) *Every R-module is reflexive.*

Proof. A commutative uniserial ring is a direct sum of commutative local semiprimary rings $R_i, i = 1, \ldots, n$, with radicals J_i such that J_i^k/J_i^{k+1} is simple or zero for $k = 0, 1, \ldots$. On the other hand a ring is uniserial if and only if all of its factor rings are QF see [**6**, pp. 235–238], so this corollary follows.

Also presented in [**6**, pp. 235–238] are characterizations of uniserial rings among semiprimary rings as principal ideal rings and, when commutative, as those whose finitely generated modules are direct sums of cyclic modules. The algebra generated by an operator on a finite dimensional space is uniserial, as is any proper quotient of a PID.

2. Morita invariance. The main aim of this section is to show that the property of being reflexive over a K-algebra is preserved under Morita equivalence, where K is any commutative ring. This result is a consequence of two lemmas and a proposition which hold in greater generality; the proposition being that tensoring with a finitely generated projective module P_R transfers reflexivity from ${}_RM_\Delta$ to ${}_T(P \otimes {}_RM)_\Delta$ where $T = \operatorname{end}(P_R)$.

LEMMA 2.1. *Let ${}_RM_\Delta$ be an R-Δ-bimodule and $A = \operatorname{alglat} M$. If $e = e^2 \in R$ then $e\Delta e \cong B$, where $B = \operatorname{alglat} eM$ and eM is regarded as an eRe-Δ-bimodule.*

Proof. Let L be an eRe-submodule of eM and let $\alpha \in A$. Then

$$(e\alpha e)L \subseteq e\alpha(ReL) \subseteq e(ReL) \subseteq L$$

and so we have $\lambda(e\alpha e) \in B$ with

$$\lambda(e\alpha e) \cdot x = e\alpha e x \quad \text{for all } x \in eM.$$

Thus $\lambda : eAe \longrightarrow B$ is a ring monomorphism.

Now let $\beta \in B$ and define $\mu(\beta) \in A$ by $\mu(\beta)m = \beta em$. Since $\beta(em) = erem$ for some $r \in R$ it follows that $\mu(\beta)m = \text{erem} \in Rm$. Also $e\mu(\beta)em = e\beta em = \beta em$, so

$$e\mu(\beta)e \cdot x = \beta x \quad \text{for all } x \in eM.$$

Thus $\lambda(e\mu(\beta)e) = \beta$ and λ is an isomorphism.

LEMMA 2.2. *Let M be an R-Δ-bimodule and write $A = \operatorname{alglat} M, S = \mathbf{M}_p(R)$ the p by p matrix ring. Let M^p be the direct sum of p copies of M written as column vectors. Then*

$$\operatorname{alglat}({}_SM^p_\Delta) = \mathbf{M}_p(A).$$

Proof. Any S-submodule of M^p is of the form L^p for some ${}_RL \subseteq {}_RM$ so

$$\mathbf{M}_p(A) \subseteq \operatorname{alglat}({}_SM^p_\Delta).$$

On the other hand, let $\alpha = (\alpha_{ij}) \in \operatorname{alglat}(M^p)$ and $m \in M$. If $\sigma_k : M \longrightarrow M^p$ is the injection then

$$\alpha\sigma_k(m) = s\sigma_k(m)$$

where $s \in S = \mathbf{M}_p(R)$. If $s = (r_{ij})$ this means

$$(\alpha_{ik}(m))^T = (r_{ik}m)^T$$

so $\alpha_{ik}(m) \in Rm$ for all i and k. This shows $\alpha \in \mathbf{M}_p(A)$ as required.

PROPOSITION 2.3. *If P_R is a finitely generated projective module, with $T =$ end(P_R), and if ${}_RM_\Delta$ is reflexive, then ${}_T(P \otimes_R M)_\Delta$ is reflexive.*

Proof. If $I = \ell_R(M)$ then P/PI is projective over R/I and

$${}_T(P \otimes M)_\Delta \cong {}_T(P/PI \otimes_{R/I} M)_\Delta,$$

so we may assume that alglat$M = R$. There is a natural number p such that $R^p = P \oplus P'$. Let $e \in$ end$(R^p{}_R)$ be the projection onto P. Write $S = \mathbf{M}_p(R) \cong$ end$(R^p{}_R)$ and $T = eSe$. Then

$${}_TeM^p \cong {}_TeS \otimes_S M^p \cong {}_T(eS \otimes_S R^p) \otimes_R M \cong {}_TeR^P \otimes_R M = {}_TP \otimes_R M.$$

Now Lemma 2.1 gives

$$\text{alglat}_S M^p_\Delta = \mathbf{M}_p(\text{alglat}_R M_\Delta) = \mathbf{M}_p(R),$$

since ${}_RM_\Delta$ is reflexive, and so

$$\text{alglat}({}_{eSe}eM^p_\Delta) \cong e\,\text{alglat}({}_SM^p_\Delta)e \cong eSe$$

by Lemma 2.2. Hence ${}_TP \otimes_R M_\Delta \cong {}_TeM^p_\Delta$ is reflexive.

PROPOSITION 2.4. *Let K be a commutative ring, let R and S be K-algebras, and let $F : R$-*Mod $\longrightarrow$ *S-*Mod *be a Morita equivalence with inverse G, both of which are K-linear on morphisms. If ${}_RM_K$ is reflexive, then ${}_SFM_K$ is reflexive.*

3. Split algebras. If R is a (finite dimensional) algebra over a field K then, following [**1**], [**8**], and [**9**], if ${}_RM$ is an R-module we write alglat M for alglat ${}_RM_K$ and say that M is *reflexive* in case ${}_RM_K$ is reflexive. The K-algebra R is *split* in case the endomorphism ring of every simple R-module consists of K-scalar multiplications by elements of K; equivalently, in case for each simple R-module ${}_RS, R/\ell_R(S)$ is K-algebra isomorphic to a full matrix ring over K. If R is basic with radical $J(R)$, then R is split if and only if

$$R_K = e_1K \oplus \ldots \oplus e_nK \oplus J(R)$$

where $e_1, \ldots, e_n$ is a basic set of idempotents for R. Of course if K is algebraically closed then R is automatically split. (See [**2**, Section 27] for basic rings and sets of idempotents; every artinian ring is Monita equivalent to a basic ring.)

PROPOSITION 3.1. *A finite dimensional K-algebra R is split if and only if each of its simple modules is reflexive.*

Proof. If ${}_RS$ is simple then alglat$(S) \cong M_n(K)$ where $n = |S : K|$, so

$$R/\ell_R(S) \cong \lambda_S(R) = \text{alglat}(S)$$

if and only if

$$R/\ell_R(S) \cong M_n(K)$$

as K-algebras.

It follows from [**9**, Theorem 2] and is implicit in [**5**] that if R is a split uniserial algebra then $R \oplus J(R)$ is reflexive. More generally we have

PROPOSITION 3.2. *If R is any split algebra then $R \oplus J(R)$ is reflexive.*

Proof. This is just a special case of 1.2 if R is a basic split algebra. But if $e = e_1 + \ldots + e_n$ is the sum of a basic set of idempotents for a split K-algebra R, then

$$_RM \longrightarrow {}_{eRe}eM = eR \otimes_R M$$

is a Morita equivalence of K-algebras, and, setting $J = J(R), eRe \oplus eJe$ is reflexive over eRe. But this eRe-module corresponds to $Re \oplus Je$ which must be reflexive, a generator and a direct summand of $R \oplus J$, so $R \oplus J$ is reflexive by Corollary 1.9.

Indecomposable projective modules being reflexive is sufficient (but not necessary) to cause an algebra to split, as well shall see using the next lemma.

LEMMA 3.3. *Let R be a finite dimensional K-algebra and suppose that $_RM$ has a unique maximal R-submodule L. If M is reflexive, then so also is M/L.*

Proof. Let $\alpha \in \operatorname{alglat} M/L$ and write $M_K = L \oplus C$. Define $\bar{\alpha} : M \longrightarrow M$ by $\bar{\alpha}(\ell + c) = c_\alpha$, where $\ell \in L, c \in C$ and $\alpha(c + L) = c_\alpha + L$ with $c_\alpha \in C$. Then $\bar{\alpha}$ is a well-defined K-map. If $_RN \subseteq {}_RM$ and $N \subseteq L$, then $\bar{\alpha}N = 0$; if $N \nsubseteq L$, then $N = M$, by the uniqueness of L, whence $\bar{\alpha}N \subseteq N$. Therefore $\bar{\alpha} \in \operatorname{alglat} M$ and since M is reflexive there is $r \in R$ with $\bar{\alpha}(\ell + c) = r(\ell + c)$ for all $\ell \in L, c \in C$. Hence $rc - c_\alpha = r\ell \in L$ and so $c_\alpha + L = rc + L$. Therefore $\alpha(c + L) = r(c + L)$ and M/L is reflexive.

LEMMA 3.4. *Let R be a finite dimensional local algebra over a field K. If $_RR$ is reflexive then $R = K$.*

Proof. If $_RR$ is reflexive then so is $R/J(R)$ by Lemma 3.3. But then R is split by Proposition 3.1, so $R_K = 1K \oplus J(R)$ and, by Proposition 1.3, $J(R) = 0$.

PROPOSITION 3.5. *Let R be a finite dimensional algebra. If every indecomposable projective R-module is reflexive then R is a split algebra and $eJ(R)e = 0$ for every primitive idempotent $e \in R$.*

Proof. If $_RRe$ is reflexive then so is $_{eRe}eRe$ by Lemma 2.1. But then, being local, $eRe = eK$ by Lemma 3.4, and the proposition follows.

4. Reflexive projective modules. This section contains results which show that all projective modules are reflexive over split hereditary algebras and incidence algebras; two important classes of algebras that satisfy the conclusion of Proposition 3.5.

A ring is *hereditary* in case submodules of projective modules are projective. If R is artinian this occures if $J(R)$ is projective (see [**15**]).

THEOREM 4.1. *If R is a finite dimensional split hereditary algebra over a field K then every projective R-module is reflexive.*

Proof. Since we have seen in Proposition 2.4 that "reflexive" (as well as "projective") is a Morita invariant, we may assume that R is a split basic hereditary algebra. Let $e_1, \dots, e_n$ be a basic set of idempotents for R, and suppose that ${}_RP$ is finitely generated and projective. Then

$$_KP = e_1P \oplus \dots \oplus e_nP,$$

so P has a K-basis

$$X = X_1 \dot{\cup} \dots \dot{\cup} X_n$$

such that

$$x = e_ix \quad \text{for all } x \in X_i$$

(where some of the X_i may be empty). If $0 \neq \Sigma_{x \in X_i} k_x x \in JX_i$, with $k_x \in K$ we would have $e_iJe_iy \neq 0$ for some $y \in X_i$. But this is impossible because, since R is artinian and hereditary, $e_iJe_i = 0$. Since R is basic and split, this means

$$RX_i = Re_iX_i = KX_i \oplus JX_i$$

as K-spaces and RX_i/JX_i has a projective cover consisting of

$$\text{composition length } (RX_i/JX_i) = |RX_i/JX_i : K| = \text{card } X_i$$

copies of Re_i/Je_i. But RX_i is projective so, by uniqueness of projective covers (see [**2**, 17.17 and 27.13]), we must have

$$RX_i = \bigoplus_{x \in X_i} Re_ix.$$

Now let $\alpha \in \text{alglat}(P)$, and choose $x_i \in X_i$ for each $X_i \neq \phi$. Then, since $re_i \longrightarrow re_ix_i$ is an isomorphism of Re_i onto Re_ix_i, there is a unique element $r_ie_i \in Re_i$ such that

$$\alpha(x_i) = r_ie_ix_i.$$

If $x_i \neq x \in X_i$ then for some $s \in R$

$$r_i e_i x_i + \alpha(x) = \alpha(x_i + x) = s e_i x_i + s e_i x$$

so, since the sum $Re_i x_i + Re_i x$ is direct, $se_i = r_i e_i$ and

$$\alpha(x) = r_i e_i x \quad \text{for all } x \in X_i.$$

Now letting

$$r = \sum_{i=1}^{n} r_i e_i$$

we have for any $x \in X_j, j = 1, \dots, n$,

$$\alpha(x) = r_j e_j x = rx.$$

Hence $\alpha = \lambda_P(r)$ because $X = \cup X_i$ is a K-basis for P. Finally, since any projective module ${}_RQ$ is a direct sum $P \oplus P'$ where P is finitely generated and P' is a direct sum of copies of direct summands of P, an application of Corollary 1.9 completes the proof.

If $X = \{x_1, \dots, x_n\}$ is a finite set of cardinality n *pre-ordered* by a reflexive, transitive relation $\underline{\alpha}$ then the *incidence ring* $I(X, \Delta)$ is the subring of the matrix ring $M_n(\Delta)$ defined by

$$\begin{aligned} I(X, \Delta) &= \{(d_{ij}) \in M_n(\Delta) | d_{ij} = 0 \text{ unless } x_i \ \underline{\alpha}\ x_j\} \\ &= \Sigma\{e_{ij}\Delta | x_i \ \underline{\alpha}\ x_j\} \end{aligned}$$

where $e_{ij}, i, j \in \{1, \dots, n\}$, are the matrix units in $M_n(\Delta)$. Such a ring is called a tic-tac-toe ring by Mitchell in [**12**]; the notion appears to have been introduced by Rota in [**14**].

An upper triangular matrix ring over a division ring is both hereditary and an incidence ring, but if $\dim(V_K) \geqq 2$ then the ring of matrics $\begin{bmatrix} K & V \\ & K \end{bmatrix}$ is hereditary but not an incidence ring. If α is a non-trivial partial order and Δ is a division ring, then $I(X, \Delta)$ is hereditary if and only if the interval $[x_i, x_j]$ is a chain whenever $x_i \ \underline{\alpha}\ x_j$ (see [**12**, Theorem IX. 10.9]). We write $e_i = e_{ii}$ to obtain a complete set of orthogonal idempotents $e_1, \dots, e_n$ for $I(X, \Delta)$, and note that

$$e_i I(X, \Delta) e_i \cong \Delta \quad \text{for } i = 1, \dots, n.$$

If K is a field then $I(X, K)$ is called an *incidence algebra*, and is clearly a split K-algebra. We shall employ the following lemma to prove a result showing that every projective module over an incidence algebra is reflexive.

LEMMA 4.2. *Let R be a subring of $M_n(\Delta)$ and let $K = {}_RK_\Delta \subseteq M_n(\Delta)$ and $N = {}_RN_\Delta \subseteq M_n(\Delta)$ be bimodules of the following form:*

$$K = \sum\{e_{ij}\Delta | (i,j) \in X\} \quad \textit{and} \quad N = \sum\{e_{ij}\Delta | (i,j) \in Y\}$$

where X and Y are nonempty sets of pairs of indices (i, j) with $1 \leqq i, j \leqq n$. If both ${}_RK_\Delta$ and ${}_RN_\Delta$ are reflexive then ${}_R(K \oplus N)_\Delta$ is reflexive.

Proof. Let $(\alpha, \beta) \in \operatorname{alglat}(K \oplus N)$. Since N is reflexive, there exists $b \in R$ such that $\beta = b\cdot$ on N. But then $\alpha - b\cdot = a\cdot$ on K (because K is reflexive) and

$$(a\cdot, 0) = (\alpha, \beta) - (b\cdot, b\cdot) \in \operatorname{alglat}(K \oplus N).$$

Hence it suffices to show that $(a\cdot, 0) = c\cdot$ on $K \oplus N$ for some $c \in R$, that is $(a - c)K = 0$ and $cN = 0$. If we write

$$a = (a_{ij}) = \sum a_{ij}e_{ij},$$

the key observation is

(1) If $(p, s) \in X$ and $(p, t) \in Y$ for some p, s, t, then $a_{ip} = 0$ for all i.

Here we have

$$(a, 0)(e_{ps}, e_{pt}) = r(e_{ps}, e_{pt}) \quad \text{for some } r = (r_{ij}) \in R$$

so $ae_{ps} = re_{ps}$ and $0 = re_{pt}$. These give

$$\sum_i a_{ip}e_{is} = \sum_i r_{ip}e_{is} \quad \text{and} \quad 0 = \sum_i r_{ip}e_{it},$$

so $a_{ip} = r_{ip} = 0$ for all i.

Now define the element c as follows:

$$c = \sum\{ae_k | (k, j) \notin Y \quad \text{for all } j = 1, 2, \ldots, n\}$$

where we take $c = 0$ if, for all k, $(k, j) \in Y$ for some j. Then we have

(2) $cN = 0$.

This is clear if $c = 0$. Otherwise, if $(k, j) \notin Y$ for all j we must show $e_kN = 0$, that is $e_ke_{ij} = 0$ for all $(i, j) \in Y$. But $k \neq i$ because $(k, j) \notin Y$.

(3) $(a - c)K = 0$.

To verify (3) and thus complete the proof, if $(p,s) \in X$, we must show $ae_{ps} = ce_{ps}$. We have

$$ce_{ps} = \sum\{ae_k e_{ps} | (k,j) \notin Y \quad \text{for all } j\}$$
$$= \begin{cases} 0 & \text{if } (p,j) \in Y \text{ for some } j \\ ae_{ps} & \text{if } (p,j) \notin Y \text{ for all } j. \end{cases}$$

Thus it suffices to show $ae_{ps} = 0$ if $(p,j) \in Y$ for some j. But

$$ae_{ps} = \sum_i a_{ip} e_{is}$$

so this follows from (1).

THEOREM 4.3. *Let $R = I(X, \Delta)$ with idempotents $e_i = e_{ii} (i = 1, \ldots, n)$. Let P be a direct sum of copies of bimodules of the form ${}_RRe_{i\Delta}$ with $i \in \{1, \ldots, n\}$. Then ${}_RP_\Delta$ is reflexive.*

Proof. Since

$$Re_j = \sum_{x_i \underline{\alpha} x_j} e_i R e_j = \sum_{x_i \underline{\alpha} x_j} e_{ij} \Delta,$$

it follows from Proposition 1.5 that each Re_j is reflexive. Thus by Lemma 4.2, if $i_1, \ldots, i_k \in \{1, \ldots, n\}$ then the R-Δ bimodule $\oplus_{i=1}^k Re_{i_k}$ is reflexive, so even if the direct sum is infinite, an application of Corollary 1.9 shows that ${}_RP_\Delta$ is reflexive.

If Δ is a local ring and $R = I(X, \Delta)$, then, since $e_i R e_i \cong \Delta$, R is semiperfect and every projective left R-module is a direct sum of copies of the Re_i (see [**2**, Section 27]). Thus we have

COROLLARY 4.4. *If Δ is a local ring then every projective $I(X, \Delta)$-module is a reflexive R-Δ-bimodule.*

COROLLARY 4.5. *If K is a field then every projective module over the incidence algebra $I(X, K)$ is reflexive.*

5. Serial algebras. In this final section, we examine the results of Habibi and Gustafson [**9**]. In view of Proposition 2.3 showing that reflexivity is a Morita invariant, the Habibi-Gustafson results can be stated for arbitrary (rather than basic) split serial algebras: Let R be a split indecomposable serial algebra with $J = J(R)$. Then [**9**, Theorem 1] every faithful R-module is reflexive, provided that R has more than one isomorphism class of indecomposable injective projective modules; and [**9**, Theorem 2] if R has, up to isomorphism, n indecomposable projective modules and P is the only one of these that is injective, then $P \oplus J^n P$ is reflexive, and it is a subquotient of every reflexive R-module.

An artinian ring is called QF-2 (see [**16**], [**7**]) in case its left and right indecomposable projective modules all have simple socles (so both serial and QF rings are QF-2). A QF-2 ring R has a unique module ${}_RU$ that is minimal faithful in the sense that it is a direct summand of every faithful module; U is a direct sum of (one copy of each of) the indecomposable injective projective left R-modules. Habibi and Gustafson proved [**9**, Theorem 1] by first showing that under its hypothesis ${}_RU$ is reflexive, and then observing that a similar proof works for any finitely generated faithful R-module M because $M \cong U \oplus I_1 \oplus \ldots \oplus I_t$ where the I_j are subquotients of the indecomposable components of U. In this connection we have.

PROPOSITION 5.1. *Let R be a QF-2 algebra with minimal faithful module ${}_RU$. If ${}_RU$ is reflexive then so is every faithful R-module.*

Proof. Since U is faithful and each indecomposable projective has simple socle, U must contain a copy of each Re_i, where $e_1, \ldots, e_n$ is a basic set of idempotents. But then very faithful module is of the form $M \cong U \oplus N$ where N is generated by submodules of U, so Corollary 1.9 applies.

Concluding this section, we reprove results of Habibi [**8**] and Habibi and Gustafson [**9**] on basic indecomposable serial algebras over an algebraically closed field K. In such an algebra R,

$$1 = \sum_{i=1}^{n} e_i$$

where $\{e_1, e_2, \ldots, e_n\}$ is a set of primitive orthogonal idempotents, and the e_i's can be ordered so that if $\dim(Re_i) = c_i$, then c_1 is minimal, $c_i \geqq 2$ for $i > 1, c_{i+1} \leqq 1 + c_i$ for $1 \leqq i < n$, and $c_1 \leqq 1 + c_n$. Thse conditions arise out of epimorphisms $Re_i \longrightarrow Je_{i+1}$ for $1 \leqq i < n$, and $Re_n \longrightarrow Je_1 \longrightarrow 0$, where J is the Jacobson radical of R. Also every R-module is isomorphic to a direct sum of the modules $Re_i/J^k e_i$. The modules Re_i for which $c_{i+1} \neq 1 + c_i$ are called *chain ends*; they are the indecomposable injective projective R-modules [**13**]. In the case when R has only one chain end, $c_{i+1} = 1 + c_i$ for $1 \leqq i < n$. We shall provide simple proofs that $Re_n \oplus J^n e_n$ is a subquotient of every faithful reflexive R-module (stated without proof in [**9**]), and that every R-module which contains $Re_n \oplus J^n e_n$ as a subquotient is reflexive (implicit in [**8**, Corollary 3.3.2]). In this discussion K has been algebraically closed, but it is sufficient for the algebra R to be split. Also Proposition 2.4 allows us to drop "basic" from the hypothesis.

PROPOSITION 5.2. [**9**, Theorem 2]. *Let R be a split serial K-algebra over a field K with only one chain end Re_n. Then $Re_n \oplus J^n e_n$, where $J = J(R)$, is a subquotient of every faithful reflexive R-module.*

Proof. Let $c = c_n$, the composition length of Re_n. The composition series for Re_n is then

$$Re_n \supset Je_n \supset \ldots \supset J^{c-1}e_n \supset J^c e_n = 0$$

with simple factors $S_n, S_{n-1}, \dots, S_1, S_n, \dots$ where $S_i = Re_i/Je_i$, $1 \leqq i \leqq n$.

Suppose that ${}_RM_K$ is a faithful reflexive bimodule. Then Re_n is a factor of M since M is faithful, so write $M = Re_n \oplus X$. Now suppose that X does not have J^ne_n as a subquotient; this means, in particular, that $c > n$. The module J^ne_n has composition length c-n with factors, in order, $S_n, S_{n-1}, \dots, S_1, S_n, \dots$. Therefore the indecomposable modules which do not have J^ne_n as a subquotient are just the subquotients of $Je_n/J^{c-1}e_n$.

For convenience identify Re_n with K^c. Then as Habibi and Gustafson pointed out in [**9**], using the n letters $a, b, \dots, x$, R has a lower triangular representation

$$\begin{bmatrix} a_1 & & & & & & & & 0 \\ a_2 & b_1 & & & & & & & \\ \cdot & \cdot & \cdot & & & & & & \\ \cdot & \cdot & & \cdot & & & & & \\ \cdot & \cdot & & & \cdot & & & & \\ \cdot & \cdot & & & & x_1 & & & \\ \cdot & \cdot & & & & \cdot & a_1 & & \\ \cdot & \cdot & & & & \cdot & \cdot & & \\ \cdot & \cdot & & & & \cdot & \cdot & & \\ \cdot & \cdot & & & & \cdot & \cdot & & \\ a_c & b_{c-1} & \cdot & \cdot & \cdot & x_{c-n+1} & a_{c-n} & \cdot\ \cdot & \cdot \end{bmatrix}$$

on Re_n where e_n has $a_1 = 1$ and all other entries 0, etc. Let $t \in \mathbf{M}_c(K)$ with $t_{c,n+1}$ and all other $t_{ij} = 0$. Then

$$t \cdot Je_n/J^{c-1}e_n = 0,$$

so t annihilates every subquotient of $Je_n/J^{c-1}e_n$. If $v \in K^c$ then there exists

$$r \in \ell_R(Je_n/J^{c-1}e_n)$$

such that $tv = rv$. Indeed, if $v_1 = 0$ choose $r \in R$ with $a_{c-n} = 1$ and all other entries 0, and if $v_1 \neq 0$ let $r \in R$ have $a_c = v_{n+1}v_1^{-1}$ and all other entries 0.

Since $t_{c,n+1} \neq 0 = t_{c-n,1}, t \notin R$, and furthermore, for all $v \notin Re_n$, there is $r \in \ell_R(X)$ with $tv = rv$. Therefore

$$(t, 0) \in \text{alglat}\,(Re_n \oplus X)$$

since

$$(t, 0)(v, x) = (tv, 0) = (rv, rx) = r(v, x),$$

but $t \notin R$, so $Re_n \oplus X$ is not reflexive.

It is worth noting that, in the setting of 5.2, alglat (Re_n) is the full ring of $c \times c$ lower triangular matrices over K. Indeed it is clear that any $\alpha \in \text{alglat}\,(Re_n) \subseteq$

$M_c(K)$ must be lower triangular, and that each matrix unit $e_{ii}, i = 1, 2, \ldots, c$ belongs to alglat (Re_n), so

$$\text{alglat}(Re_n) = \sum_{i=1}^{c} Re_{ii} = \mathbf{LTM}_c(K).$$

The converse of Proposition 5.2 follows easily from Proposition 1.8.

PROPOSITION 5.3. *If R is a split serial algebra with only one chain end Re_n, and ${}_RM$ has $Re_n \oplus J^n e_n$ as a subquotient, then M is a reflexive faithful R-module.*

Proof. According to [**9**, Theorem 2], $Re_n \oplus J^n e_n$ is reflexive. If $K \leqq {}_RM$ and $h : K \longrightarrow Re_n \oplus J^n e_n$ is an epimorphism, then, since Re_n is projective, $K = P \oplus L$ with $P \cong Re_n$ and

$$g = h_{|L} : L \longrightarrow J^n e_n$$

is an epimorphism. Then, since P is injective, $M = N \oplus P$ with L isomorphic to a submodule of N via a monomorphism $f : L \longrightarrow N$. Since each Re_i embeds in $Re_n \cong P, P$ controls every R-module by Lemma 1.7. Thus by Proposition 1.8 the canonical ring homomorphisms

$$\text{alglat}\,(M) = \text{alglat}\,(N \oplus P) \longrightarrow \text{alglat}\,(L \oplus P) \longrightarrow \text{alglat}\,(J^n e_n \oplus Re_n)$$

are R-compatible injections so, since $J^n e_n \oplus Re_n$ is reflexive, so is M.

REFERENCES

1. E. Azoff, *K-reflexivity in finite dimensional spaces*, Duke Math. J. *40* (1973), 821–830.

2. F. W. Anderson and K. R. Fuller, *Rings and categories of modules* (Springer-Verlag, New York, 1973).

3. G. P. Barker and J. J. Conklin, *Reflexive algebras of matrices*, Rocky Mountain J. of Math *15* (1985), 107–116.

4. S. Brenner and M. C. R. Butler, *Endomorphism rings of vector spaces and torsion free abelian groups*, J. London Math. Soc. *40* (1965), 183–187.

5. J. A. Deddens and P. Fillmore, *Reflexive linear transformations*, Lin. Alg. Appl. *10* (1975), 89–93.

6. C. Faith, *Algebra II ring theory* (Springer-Verlag, Berlin Heidelberg New York, 1976).

7. K. R. Fuller, *The structure of* QF-3 *rings*, Trans. Amer. Math Soc. *134* (1968), 343–354.

8. J. F. Habibi, *Reflexive modules over algebras*, Dissertation, Texas Tech. University (1986).

9. J. F. Habibi and W. H. Gustafson, *Reflexive serial algebras*, Lin. Alg. Appl. *99* (1988), 217–223.

10. D. Hadwin and J. W. Kerr, *Scalar-reflexive rings*, Proc. Amer. Math. Soc. *103* (1988), 1–8.

11. P. R. Halmos, *Reflexive lattices of subspaces*, J. London Math. Soc. *4* (1971), 257–263.

12. B. Mitchell, *Theory of categories* (Academic Press, New York, 1965).

13. I. Murase, *On the structure of generalized uniserial rings I*, Sci. Papers Coll. Gen. Ed., Univ. Tokyo *13* (1963), 1–22.

14. G. C. Rota, *On the foundations of combinatorial theory (I. Theory of Möbius functions)*, Z. Wahrschienlichkietstheorie und Verw. Gebiete *2* (1964), 340–368.

15. J. J. Rotman, *An introduction to homological algebra* (New York-San Francisco-London; Academic Press, 1969).

16. R. M. Thrall, *Some generalizations of quasi-Frobenius algebras*, Trans. Amer. Math. Soc. *64* (1948), 173–183.

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