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LIFTINGS OF TENSOR FIELDS AND CONNECTIONS TO TANGENT BUNDLES OF HIGHER ORDER

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§ Introduction

In the previous papers [1], [2] we have studied the prolongations of G-structures to tangent bundles of arbitrary order, and in [3] we have considered the prolongations of connections to the tangential fibre bundles of higher order. The purpose of the present paper is to study the liftings of tensor fields and affine connections to tangent bundles of higher order. In fact, most of results in [4], [5] will be generalized in a natural fashion and some of the formulas concerning vertical and complete lifts in [5] will be unified and generalized in our formulas concerning (λ)-lifts (cf. § **3**).

The crucial starting point of our procedure is the following fact (§ **1**): For any vector field X on a manifold M and for any integer $\lambda = 0, 1, \cdots r$, there exists one and only one vector field $X^{(\lambda)}$ (called the (λ)-lift of X) on the tangent bundle $\overset{r}{T}M$ of order r satisfying the following equality

$$X^{(\lambda)} f^{(\mu)} = (Xf)^{(\lambda+\mu-r)}$$

for any differentiable function f on M and any $\mu = 0, 1, \cdots, r$, where $f^{(\mu)}$ is a function naturally lifted to $\overset{r}{T}M$ from M.

Second, we construct (λ)-lifts of differential 1-forms in § **2**. After preparing several nice equalities between the (λ)-lifts of functions, vector fields and 1-forms on M, we shall construct in § **3** the (λ)-lifts of any tensor fields on M to $\overset{r}{T}M$ for $\lambda = 0, 1, \cdots, r$. In the case $r = 1$, i.e. in the case of usual tangent bundle, our (0)-lifts coincide with the vertical lifts, while our (1)-lifts coincide with the complete lifts in [5]. Therefore, our (r)-lifts of tensor fields can be considered as a generalization of complete lifts of tensor fields.

As an application of our (λ)-lifts, we consider in § **4** the case of almost complex structures, and we shall prove that if M is a homogeneous (almost)

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complex manifold then $\overset{r}{T}M$ is canonically a homogeneous (almost) complex manifold.

In § 5, we shall consider the lifting of affine connections. We shall prove similar results to [5], and, in fact, some formulas are unified in a nice equality. As one of its consequences, we obtain locally affine symmetric spaces $\overset{r}{T}M$ from any locally affine symmetric space M for any integer r.

In § 6, we shall enumerate, without proof, some results concerning the liftings of Riemannian structures and others.

In this paper, all manifolds and mappings (functions) are assumed to be differentiable of class C^∞, unless otherwise stated.

We shall fix a positive integer r throughout the paper.

§ 1. (λ)-lifts of functions and vector fields.

Let M be an n-dimensional manifold. We denote by $C^\infty(M)$ the algebra of all differentiable functions on M. Let $(\overset{r}{T}M, \overset{r}{\pi})$ be the tangent bundle of order r to the manifold M (cf. [2]). We shall define the (λ)-lifting $L_\lambda : C^\infty(M) \to C^\infty(\overset{r}{T}M)$ of functions as follows:

DEFINITION 1.1. For any $f \in C^\infty(M)$, we define $L_\lambda(f) = f^{(\lambda)} \in C^\infty(\overset{r}{T}M)$ for $\lambda = 0, 1, \cdots, r$ as follows:

$$f^{(\lambda)}([\varphi]_r) = \frac{1}{\lambda!}\left[\frac{d^\lambda(f \circ \varphi)}{dt^\lambda}\right]_{t=0} \tag{1.1}$$

for $[\varphi]_r \in \overset{r}{T}M$, where $\varphi : R \to M$ is a differentiable map. For the sake of convenience, we define $f^{(\lambda)} = 0$ for any negative integer λ. We shall call $f^{(\lambda)}$ the (λ)-lift of f. Clearly, $f^{(0)} = f \circ \overset{r}{\pi}$ holds. We see readily that $f^{(\lambda)}$ is a well-defined differentiable function on $\overset{r}{T}M$, i.e. the value $f^{(\lambda)}([\varphi]_r)$ of (1.1) is independent of the choice of the representative φ.

LEMMA 1.2. *The (λ)-lifting $L_\lambda : C^\infty(M) \to C^\infty(\overset{r}{T}M)$ is linear and satisfies the following equality*

$$(f \cdot g)^{(\lambda)} = \sum_{\mu=0}^{\lambda} f^{(\mu)} \cdot g^{(\lambda-\mu)} \tag{1.2}$$

for every $f, g \in C^\infty(M)$.

Proof. The linearity of the map L_λ is clear from the equality (1.1). The equality (1.2) is verified as follows:

$$
\begin{aligned}
(f\cdot g)^{\lambda}([\varphi]_r) &= \frac{1}{\lambda!}\left[\frac{d^{\lambda}((f\cdot g)\circ\varphi)}{dt^{\lambda}}\right]_0 \\
&= \frac{1}{\lambda!}\left[\frac{d^{\lambda}(f\circ\varphi)\cdot(g\circ\varphi)}{dt^{\lambda}}\right]_0 = \frac{1}{\lambda!}\sum_{\mu=0}^{\lambda}\binom{\lambda}{\mu}\left[\frac{d^{\mu}(f\circ\varphi)}{dt^{\lambda}}\right]_0\cdot\left[\frac{d^{\lambda-\mu}(g\circ\varphi)}{dt^{\lambda-\mu}}\right]_0 \\
&= \frac{1}{\lambda!}\sum_{\mu=0}^{\lambda}\frac{\lambda!}{\mu!(\lambda-\mu)!}\left[\frac{d^{\mu}(f\circ\varphi)}{dt^{\mu}}\right]_0\cdot\left[\frac{d^{\lambda-\mu}(g\circ\varphi)}{dt^{\lambda-\mu}}\right]_0 \\
&= \sum_{\mu=0}^{\lambda} f^{(\mu)}([\varphi]_r)\cdot g^{(\lambda-\mu)}([\varphi]_r) = \sum_{\mu=0}^{\lambda} f^{(\mu)}\cdot g^{(\lambda-\mu)}([\varphi]_r)
\end{aligned}
$$

for every $[\varphi]_r \in \overset{r}{T}M$. Q.E.D.

Let $\mathscr{T}(M) = \sum \mathscr{T}^p_q(M)$ be the tensor algebra of all tensor fields on M, where $\mathscr{T}^p_q(M)$ is the subspace of all tensor fields of type (p,q) on M (p-contravariant, q-covariant).

LEMMA 1.3. *Let $\tilde{X}, \tilde{Y} \in \mathscr{T}^1_0(TM)$. If $\tilde{X}(f^{(\nu)}) = \tilde{Y}(f^{(\nu)})$ for every $f \in C^\infty(M)$, $\nu = 0,1,\cdots,r$, then $\tilde{X} = \tilde{Y}$.*

Proof. Take an element $a = [\varphi]_r \in \overset{r}{T}M$ with $\overset{r}{\pi}(a) = x_0$. Let U be a coordinate neighborhood of x_0 with coordinate system $\{x_1,\cdots,x_n\}$ and let $\{\overset{(\nu)}{x_i} | 1,2,\cdots,n;\ \nu = 0,1,\cdots,r\}$ be the induced coordinate system (cf. [2]) on $(\overset{r}{\pi})^{-1}(U)$. By Definition 1.1 we have $\overset{(\nu)}{x_i} = (x_i)^{(\nu)}$. Therefore, we get $\tilde{X}_a(\overset{(\nu)}{x_i}) = \tilde{Y}(\overset{(\nu)}{x_i})$ for every $i = 1,2,\cdots,n$; $\nu = 0,1,\cdots,r$. Hence we have $\tilde{X}_a = \tilde{Y}_a$ for every $a \in \overset{r}{T}M$. Q.E.D.

For any vector field $X \in \mathscr{T}^1_0(M)$, we shall define the (λ)-lift $X^{(\lambda)}$ of X. For that purpose we shall prove the following

LEMMA 1.4. *For any $X \in \mathscr{T}^1_0(M)$ and any $\lambda = 0,1,\cdots,r$, there exists one and only one $X^{(\lambda)} \in \mathscr{T}^1_0(\overset{r}{T}M)$ satisfying the following equality*

$$X^{(\lambda)} f^{(\mu)} = (Xf)^{(\lambda+\mu-r)} \tag{1.3}$$

for every $f \in C^\infty(M)$ and $\mu = 0,1,\cdots,r$.

Proof. Take a coordinate neighborhood U in M with coordinate system $\{x_1,\cdots,x_n\}$. As in [2], let $\{\overset{(\nu)}{x_i}\}$ be the induced coordinate system on $(\overset{r}{\pi})^{-1}(U)$. The vector field X can be expressed on U as follows:

$$X|_U = \sum a_i \frac{\partial}{\partial x_i},$$

where $a_i \in C^\infty(U)$, $i = 1,2,\cdots,n$. Consider the vector field $\tilde{X} = \tilde{X}_U$ on $(\overset{r}{\pi})^{-1}(U)$

defined by

$$\tilde{X} = \sum_{\nu=r-\lambda}^{r} a_i^{(\nu+\lambda-r)} \frac{\partial}{\partial \overset{(\nu)}{x}_i}.$$

We shall prove

$$\tilde{X}(f^{(\nu)}) = (Xf)^{(\nu+\lambda-r)} \tag{1.4}$$

for every $f \in C^\infty(U)$ and $\nu = 0, 1, \cdots, r$.

First, we see that $\tilde{X}(\overset{(\nu)}{x}_i) = a_i^{(\nu+\lambda-r)} = (Xx_i)^{(\nu+\lambda-r)}$ and hence (1.4) holds for $f = x_i$ $(i = 1, 2, \cdots, n)$, since $\overset{(\nu)}{x}_i = (x_i)^{(\nu)}$ for every $i = 1, 2, \cdots, n;\ \nu = 0, 1, \cdots, r$. We denote by A the set of all $f \in C^\infty(U)$ such that (1.4) holds for every $\nu = 0, 1, \cdots, r$.

We assert that if $f, g \in A$, then $f \cdot g \in A$. For, we calculate, using Lemma 1.2, as follows:

$$\begin{aligned}
\tilde{X}((f \cdot g)^{(\nu)}) &= X \sum_{\mu=0}^{\nu} f^{(\nu)} g^{(\nu-\mu)} \\
&= \sum \{ \tilde{X} f^{(\mu)} \cdot g^{(\nu-\mu)} + f^{(\mu)} \cdot \tilde{X} g^{(\nu-\mu)} \} \\
&= \sum \{ (Xf)^{(\lambda+\mu-r)} \cdot g^{(\nu-\mu)} + f^{(\mu)} \cdot (Xg)^{(\nu+\lambda-r-\mu)} \} \\
&= \sum_{\mu'=0}^{\nu} (Xf)^{(\lambda+\nu-r-\mu')} g^{(\mu')} + \sum_{\mu=0}^{\nu+\lambda-r} f^{(\mu)} (Xg)^{(\nu+\lambda-r-\mu)} \\
&= \sum_{\mu=0}^{\lambda+\nu-r} (Xf)^{(\lambda+\nu-r-\mu)} g^{(\mu)} + (f \cdot Xg)^{(\nu+\lambda-r)} \\
&= (Xf \cdot g + f \cdot Xg)^{(\nu+\lambda-r)} = (X(f \cdot g))^{(\lambda+\nu-r)},
\end{aligned}$$

and hence $f \cdot g \in A$. On the other hand, it is clear that $f, g \in A$ implies $f + g \in A$ and that $c \in R$, $f \in A$ imply $c \cdot f \in A$. Therefore, every polynomial of $x_1, \cdots, x_n$ is contained in A.

Now, we shall prove (1.4) for every $f \in C^\infty(U)$. Take an element $[\varphi^0]_r \in \overset{r}{T}(U)) = (\overset{r}{\pi})^{-1}(U)$. We have to verify

$$(\tilde{X} f^{(\nu)})([\varphi^0]_r) = (Xf)^{(\lambda+\nu-r)}([\varphi^0]_r). \tag{1.5}$$

For that purpose, put $\overset{(\nu)}{x}_i([\varphi^0]_r) = \overset{(\nu)}{x}{}_i^*$ for $i = 1, \cdots, n;\ \nu = 0, 1, \cdots, r$, and $y_i = x_i - \overset{(0)}{x}{}_i^*$ for $i = 1, \cdots, n$. It is clear that $\overset{(\nu)}{x}_i = \overset{(\nu)}{y}_i$ for $\nu = 1, 2, \cdots, r$ and that $\overset{(0)}{y}_i([\varphi^0]_r) = 0$ for $i = 1, \cdots, n$. Since $f \in C^\infty(U)$, by considering Taylor expansion of f, we can find polynomials P_i, P_0 of $y_1, \cdots, y_n$ and a function $g_i \in C^\infty(U)$ $(i = 1, 2, \cdots, k)$ such that $\deg P_i > r$, $\deg P_0 \leq r$ and that

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$$f = \sum_{j=1}^{k} P_i \cdot g_i + P_0$$

on U. Put $f_i = P_i \cdot g_i$. Since $P_0 \in A$, it remains to prove (1.5) for $f = f_1$. Put $P = P_1$, $g = g_1$. Since $\deg P > r$, we have $P^{(\mu)}([\varphi^0]_r) = (1/\mu!) \cdot [d^\mu P \circ \varphi / dt^\mu]_0 = 0$. Now, by Lemma 1.2, we have $f_1^{(\nu)} = \sum P^{(\mu)} g^{(\nu-\mu)}$ and hence we get

$$\tilde{X}(f_1^{(\nu)}) = \sum_{\mu=0}^{\nu} \tilde{X} P^{(\mu)} \cdot g^{(\nu-\mu)} + \sum_{\mu=0}^{\nu} P^{(\mu)} \cdot \tilde{X} g^{(\nu-\mu)} \tag{1.6}$$

$$= \sum_{\mu=0}^{\nu} (XP)^{(\lambda+\mu-r)} \cdot g^{(\nu-\mu)} + \sum_{\mu=0}^{\nu} P^{(\mu)} \cdot \tilde{X} g^{(\nu-\mu)}.$$

The first term of the right hand side in (1.6) is equal to

$$\sum_{\mu'=0}^{\nu} (XP)^{(\lambda+\nu-\mu'-r)} \cdot g^{(\mu')} = \sum_{\mu=0}^{\lambda+\nu-r} (XP)^{(\lambda+\nu-\mu-r)} \cdot g^{(\mu)}. \tag{1.7}$$

On the other hand, we calculate as follows:

$$\begin{aligned} (Xf_1)^{(\lambda+\nu-r)} &= (XP \cdot g + P \cdot Xg)^{(\lambda+\nu-r)} \\ &= (XP \cdot g)^{(\lambda+\nu-r)} + (P \cdot Xg)^{(\lambda+\nu-r)} \\ &= \sum_{\mu=0}^{\lambda+\nu-r} (XP)^{(\lambda+\nu-r-\mu)} \cdot g^{(\mu)} + \sum_{\mu=0}^{\lambda+\nu-r} P^{(\mu)} \cdot (Xg)^{(\lambda+\nu-r-\mu)}. \end{aligned} \tag{1.8}$$

From the equalities (1.6), (1.7), (1.8) and the fact that $P^{(\mu)}([\varphi^0]_r) = 0$ for $\mu = 0, 1, \cdots, r$, it follows that (1.5) holds for $f = f_1$. Thus (1.4) holds for every $f \in C^\infty(U)$ and $\nu = 0, 1, \cdots, r$.

Thus, for every coordinate neighborhood U in M, we have a vector field $\tilde{X}_U$ on $(\overset{r}{\pi})^{-1}(U)$ such that

$$\tilde{X}_U(f^{(\nu)}) = (Xf)^{(\nu+\lambda-r)}$$

for every $f \in C^\infty(U)$ and $\nu = 0, 1, \cdots, r$. If U and U' are both coordinate neighborhood in M such that $U \cap U' = U'' \neq \phi$, then we have $\tilde{X}_U | U'' = \tilde{X}_{U'} | U''$ since we can apply Lemma 1.3 for $\tilde{X} = \tilde{X}_U$, $\tilde{Y} = \tilde{X}_{U'}$ and $M = U''$. Therefore, we obtain a vector field $X^{(\lambda)}$ on M such that $X^{(\lambda)} | U = \tilde{X}_U$ for every coordinate neighborhood U in M. This vector field $X^{(\lambda)}$ clearly satisfies the condition (1.3) for every $f \in C^\infty(M)$ and $\mu = 0, 1, \cdots, r$.

The uniqueness of $X^{(\lambda)}$ is clear virtue of Lemma 1.3. Q.E.D.

COROLLARY 1.5. *We have the following*

$$\left(\frac{\partial}{\partial x_i}\right)^{(\lambda)} = \frac{\partial}{\partial x_i^{(r-\lambda)}}$$

for $i = 1, \cdots n$; $\lambda = 0, 1, \cdots, r$.

Proof. In the proof of Lemma 1.4 we have put $\tilde{X} = \sum a_i^{(\nu+\lambda-r)}\left(\frac{\partial}{\partial_{x_i}^{(\nu)}}\right)$ for $X = \sum a_i \frac{\partial}{\partial x_i}$. We have $a^{(0)} = a$ and $a^{(\nu)} = 0$ for $\nu = 1, \cdots, r$, if is a constant. Hence we get $(\partial/\partial x_i)^{(\lambda)} = \partial/\partial_{x_i}^{(r-\lambda)}$. Q.E.D.

COROLLARY 1.6. *Notations being as in the proof of Lemma 1.4, we have, for any $f \in C^\infty(U)$ and λ, $\mu = 0, 1, \cdots, r$ the following*

$$\frac{\partial f^{(\lambda)}}{\partial_{x_i}^{(\mu)}} = \left(\frac{\partial f}{\partial x_i}\right)^{(\lambda-\mu)}.$$

Proof. Making use of Lemma 1.4 and Corollary 1.5, we calculate as follows:

$$\begin{aligned}\frac{\partial f^{(\lambda)}}{\partial_{x_i}^{(\mu)}} &= \left(\frac{\partial}{\partial_{x_i}^{(\mu)}}\right) f^{(\lambda)} = \left(\frac{\partial}{\partial x_i}\right)^{(r-\mu)} f^{(\lambda)} \\ &= \left(\frac{\partial f}{\partial x_i}\right)^{(r-\mu+\lambda-r)} = \left(\frac{\partial f}{\partial x_i}\right)^{(\lambda-\mu)}\end{aligned}$$

Q.E.D.

DEFINITION 1.7. The vector field $X^{(\lambda)}$ in Lemma 1.4 will be called *the* (λ)-*lift of* X *to* $\overset{r}{T}M$.

LEMMA 1.8. *Let* $X, Y \in \mathscr{T}_0^1(M)$. *Then we have the following equality*

$$[X^{(\lambda)}, Y^{(\mu)}] = [X, Y]^{(\lambda+\mu-r)} \tag{1.9}$$

for every λ, $\mu = 0, 1, \cdots, r$.

Proof. Take a function $f \in C^\infty(M)$. For any $\nu = 0, 1, \cdots, r$, we calculate as follows:

$$\begin{aligned}[X^{(\lambda)}, Y^{(\mu)}] f^{(\nu)} &= X^{(\lambda)} Y^{(\mu)} f^{(\nu)} - Y^{(\mu)} X^{(\lambda)} f^{(\nu)} \\ &= X^{(\lambda)}(Yf)^{(\mu+\nu-r)} - Y^{(\mu+\nu-r)} \\ &= (XYf)^{(\lambda+\mu+\nu-2r)} - (XYf)^{(\lambda+\mu+\nu-2r)} \\ &= ([XY]f)^{(\lambda+\mu+\nu-2r)} = [X, Y]^{(\lambda+\mu-r)} f^{(\nu)}.\end{aligned}$$

Since f is arbitrary, we get (1.9) by Lemma 1.3. Q.E.D.

LEMMA 1.9. *The* (λ)-*lifting* $X \to X^{(\lambda)}$ *is a linear map of* $\mathscr{T}_0^1(M)$ *into* $\mathscr{T}_0^1(\overset{r}{T}M)$ *for every* $\lambda = 0, 1, \cdots, r$.

Proof. Clear from the uniqueness of $X^{(\lambda)}$ and the linearity of the (λ)-lifting of functions. Q.E.D.

LEMMA 1.10. *For any* $f \in C^\infty(M)$ *and* $X \in \mathscr{T}^1_0(M)$, *we have*

$$(f \cdot X)^{(\lambda)} = \sum_{\mu=0}^{\lambda} f^{(\mu)} X^{(\lambda-\mu)} \tag{1.10}$$

for every $\lambda = 0, 1, \cdots, r$.

Proof. Take a function $g \in C^\infty(M)$. By virtue of Lemma 1.3, it suffices to prove that the values of both hand sides of (1.9) are equal at $g^{(\nu)}$ for every $\nu = 0, 1, \cdots, r$. Now, we calculate as follows:

$$\begin{aligned}(fX)^{(\lambda)} g^{(\nu)} &= ((fX) \cdot g)^{(\nu+\lambda-r)} = (f \cdot Xg)^{(\nu+\lambda-r)} \\ &= \sum_{\mu=0}^{\lambda} f^{(\mu)} \cdot (Xg)^{(\nu+\lambda-r-\mu)} = \sum_{\mu=0}^{\lambda} f^{(\mu)} \cdot (X^{(\lambda-\mu)} g^{(\nu)}) \\ &= (\sum_{\mu=0}^{\lambda} f^{(\mu)} X^{(\lambda-\mu)}) g^{(\nu)},\end{aligned}$$

where we have used, in the third equality, the following fact: For every $\mu > \nu + \lambda - r$, we have $(Xg)^{(\nu+\lambda-r-\mu)} = 0$. Q.E.D.

Remark 1.11. If X is a vector field induced by a one-parameter group of transformations φ_t on M, then the r-tangent $\overset{r}{T}\varphi_t$ is also a one-parameter group of transformations on $\overset{r}{T}M$ and hence $\overset{r}{T}\varphi_t$ induces a vector field $\tilde{X}$ on $\overset{r}{T}M$. We can verify, by a tedious calculation, that $\tilde{X}$ is identical with $X^{(r)}$ in Definition 1.7.

In the case of usual tangent bundle $T(M)$, i.e. the case $r = 1$, we can verify, by a straightforward calculation, that the (0)-lift of X is identical with the vertical lift of X, while (1)-lift of X is identical with the complete lift of X (cf. [5]) as mentioned above.

§ 2. (λ)-lifts of 1-forms.

We shall now lift 1-forms on M to 1-forms on $\overset{r}{T}M$. For that purpose, we prove the following

LEMMA 2.1. *Let* $f_i, g_i \in C^\infty(M)$ $(i = 1, \cdots, k)$ *be functions on* M *such that* $\sum_{i=1}^{k} g_i df_i = 0$ *on* M. *Then the following equality*

$$\sum_{i=1}^{k}\sum_{\mu=0}^{\lambda} g_i^{(\mu)} d f_i^{(\lambda-\mu)} = 0$$

holds on $\overset{r}{T}M$ *for every* $\lambda = 0, 1, \cdots, r$.

Proof. Take a coordinate neighborhood U with coordinate system $\{x_1, \cdots, x_n\}$ and let $\{\overset{(\nu)}{x_i}\}$ be the induced coordinate system on $(\overset{r}{\pi})^{-1}(U)$. It is sufficient to verify the following

$$\sum_i \sum_\mu g_i^{(\mu)} d f_i^{(\lambda-\mu)} \Big(\frac{\partial}{\partial \overset{(\nu)}{x_j}}\Big) = 0$$

for $j = 1, \cdots, n$ and $\nu = 0, 1, \cdots, r$. Now, by the assumption, we have $\sum g_i \cdot (d f_i / d x_j) = 0$ for every $i = 1, \cdots, n$. Hence, by using Corrollay 1.6, we calculate as follows:

$$\begin{aligned}
\sum_i \sum_{\mu=0}^{\lambda} g_i^{(\mu)} d f_i^{(\lambda-\mu)} \Big(\frac{\partial}{\partial \overset{(\nu)}{x_j}}\Big) &= \sum_i \sum_{\mu=0}^{\lambda} g_i^{(\mu)} \frac{\partial f_i^{(\lambda-\mu)}}{\partial \overset{(\nu)}{x_j}} \\
&= \sum_i \sum_{\mu=0}^{\lambda} g_i^{(\mu)} \Big(\frac{\partial f_i}{\partial x_j}\Big)^{(\lambda-\mu-\nu)} = \sum \sum_{\mu=0}^{\lambda-\nu} g_i^{(\mu)} \Big(\frac{\partial f_i}{\partial x_j}\Big)^{(\lambda-\mu-\nu)} \\
&= \sum_i \Big(g_i \frac{\partial f_i}{\partial x_j}\Big)^{(\lambda-\nu)} = 0
\end{aligned}$$

Q.E.D.

LEMMA 2.2. *There exists one and only one lifting* $L_\lambda : \mathscr{T}_1^0(M) \to \mathscr{T}_1^0(\overset{r}{T}M)$ *for* $\lambda = 0, 1, \cdots, r$ *satisfying the following condition:*

$$L_\lambda(f \cdot dg) = \sum_{\mu=0}^{\lambda} f^{(\mu)} d g^{(\lambda-\mu)} \tag{2.1}$$

for every $f, g \in C^\infty(M)$.

Proof. Take a 1-form $\theta \in \mathscr{T}_1^0(M)$. Using a local coordinate system $\{x_1, \cdots, x_n\}$ on a neighborhood U, θ can be written as $\theta = \sum a_i d x_i$, $a_i \in C^\infty(U)$. Consider the 1-form

$$\tilde{\theta}_U = \sum_i \sum_{\mu=0}^{\lambda} a_i^{(\mu)} d x_i^{(\lambda-\mu)}$$

on $(\overset{r}{\pi})^{-1}(U)$. By Lemma 2.1, the form θ_U is independent of the choice of the coordinate system on U. Thus, we obtain a 1-form $\tilde{\theta}$ on $\overset{r}{T}M$ such that $\tilde{\theta} | (\overset{r}{\pi})^{-1}(U) = \tilde{\theta}_U$. Put $L_\lambda(\theta) = \tilde{\theta}$. We shall prove (2.1) on $(\overset{r}{\pi})^{-1}(U)$ as follows:

$$
\begin{aligned}
L_\lambda(f\cdot dg) &= L_\lambda\Big(f\cdot\sum_i \frac{\partial g}{\partial x_i}\,dx_i\Big)\\
&= \sum_i\sum_{\mu=0}^{\lambda}\Big(f\,\frac{\partial g}{\partial x_i}\Big)^{(\mu)} dx_i^{(\lambda-\mu)}\\
&= \sum_i\sum_{\mu=0}^{\lambda}\sum_{\nu=0}^{\mu} f^{(\nu)}\Big(\frac{\partial g}{\partial x_i}\Big)^{(\mu-\nu)} dx_i^{(\lambda-\mu)} = \sum_i\sum_{\mu'=0}^{\lambda}\sum_{\nu=0}^{\lambda-\mu'} f^{(\nu)}\Big(\frac{\partial g}{\partial x_i}\Big)^{(\lambda-\mu'-\nu)} dx_i^{(\mu')}\\
&= \sum_i\sum_{\nu=0}^{\lambda}\sum_{\mu=0}^{\nu} f^{(\lambda-\nu)}\Big(\frac{\partial g}{\partial x_i}\Big)^{(\nu-\mu)}(dx_i)^{(\mu)}\\
&= \sum_{\nu=0}^{\lambda} f^{(\lambda-\nu)}L_\nu(dg) = \sum_{\mu=0}^{\lambda} f^{(\mu)}L_{\lambda-\mu}(dg)\\
&= \sum_{\mu=0}^{\lambda} f^{(\mu)}(dg)^{(\lambda-\mu)}.
\end{aligned}
$$

The uniqueness of L_λ is now clear. Q.E.D.

DEFINITION 2.3. We denote $\theta^{(\lambda)} = L_\lambda(\theta)$ for $\theta\in\mathcal{T}^0_1(M)$ and call $\theta^{(\lambda)}$ *the* (λ)*-lift of* θ *for* $\lambda = 0, 1, \cdots, r$.

COROLLARY 2.4. *For* $f\in C^\infty(M)$ *and* $\theta\in\mathcal{T}^0_1(M)$ *we have*

$$(f\cdot\theta)^{(\lambda)} = \sum_{\mu=0}^{\lambda} f^{(\mu)}\cdot\theta^{(\lambda-\mu)} \tag{2.2}$$

for $\lambda = 0, 1, \cdots, r$.

Proof. Using a coordinate system $\{x_1, \cdots, x_n\}$ we can write $\theta = \sum a_i dx_i$ and hence we calculate as follows:

$$
\begin{aligned}
(f\theta)^{(\lambda)} &= \sum_i (f\cdot a_i dx_i)^{(\lambda)} = \sum_i\sum_{\mu=0}^{\lambda}(f\cdot a_i)^{(\mu)}dx_i^{(\lambda-\mu)}\\
&= \sum_i\sum_{\mu=0}^{\lambda}\sum_{\nu=0}^{\mu} f^{(\nu)}a_i^{(\lambda-\nu)}dx_i^{(\lambda-\mu)} = \sum_{\mu=0}^{\lambda} f^{(\mu)}\,(\sum_i a_i x_i)^{(\lambda-\mu)}\\
&= \sum_{\mu=0}^{\lambda} f^{(\mu)}\theta^{(\lambda-\mu)}.
\end{aligned}
$$

Q.E.D.

LEMMA 2.5. *For any* $\theta\in\mathcal{T}^0_1(M)$ *and* $X\in\mathcal{T}^1_0(M)$, *we have the following*

$$\theta^{(\lambda)}(X^{(\mu)}) = (\theta(X))^{(\lambda+\mu-r)} \tag{2.3}$$

for every λ, $\mu = 0, 1, \cdots, r$.

Proof. Let $\theta = \sum f_i dx_i$ be the local expression of θ. We calculate as follows:

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$$\begin{aligned}\theta^{(\lambda)}(X^{(\mu)}) &= (\sum f_i dx_i)^{(\lambda)}(X)^{(\mu)} \\ &= \sum_i \sum_{\nu=0}^{\lambda} f_i^{(\nu)}(dx_i)^{(\lambda-\nu)}(X^{(\mu)}) = \sum_i \sum_{\nu=0}^{\lambda} f_i^{(\nu)}(dx_i^{(\lambda-\nu)})\,(X^{(\mu)}) \\ &= \sum_i \sum_{\nu=0}^{\lambda} f_i^{(\nu)} d(X^{(\mu)} x_i^{(\lambda-\nu)}) = \sum_i \sum_{\nu=0}^{\lambda} f_i^{(\nu)} d(Xx_i)^{(\mu+\lambda-\nu-r)} \\ &= \sum_i \sum_{\nu=0}^{\lambda} f_i^{(\nu)} (dXx_i)^{(\mu+\lambda-\nu-r)} \\ &= \sum_i \sum_{\nu=0}^{\mu+\lambda-r} f_i^{(\nu)} (dXx_i)^{(\mu+\lambda-\nu-r)} \\ &= \sum_i (f_i \cdot dx_i(X))^{(\mu+\lambda-r)} = (\theta(X))^{(\mu+\lambda-r)}, \end{aligned}$$

Q.E.D.

§ 3. (λ)-lifts of tensor fields.

Let $\mathscr{T}^{(r)}(M) = \sum \overset{(r)}{\mathscr{T}}{}^p_q(M)$ be the $(r+\,)$-times direct sum of the tensor algebra $\mathscr{T}(\overset{r}{T}M) = \sum \mathscr{T}^p_q(\overset{r}{T}M)$, where $\overset{(r)}{\mathscr{T}}{}^p_q(M) = \mathscr{T}^p_q(\overset{r}{T}M) \oplus \cdots \oplus \mathscr{T}^p_q(\overset{r}{T}M)$ $(r+1$ factors). Take two elements $T = (T_0, T_1, \cdots, T_r) \in \overset{(r)}{\mathscr{T}}{}^p_q(M)$ and $S = (S_0, S_1, \cdots, S_r) \in \overset{(r)}{\mathscr{T}}{}^{p'}_{q'}(M)$. Define the multiplication $U = T \otimes S = (U_0, U_1, \cdots, U_r) \in \overset{(r)}{\mathscr{T}}{}^{p+p'}_{q+q'}(M)$ as follows:

$$U_\lambda = \sum_{\mu=0}^{\lambda} T_\mu \otimes S_{\lambda-\mu} \tag{3.1}$$

for $\lambda = 0, 1, \cdots, r$. We can readily see that $\overset{(r)}{\mathscr{T}}(M)$ is a graded associative algebra with this multiplication.

In §1 and 2 we have define a linear map $L_\lambda : \mathscr{T}^p_q(M) \to \mathscr{T}^p_q(\overset{r}{T}M)$ for $(p, q) = (0, 0)$, $(p, q) = (1, 0)$ and $(p, q) = (0, 1)$. We denote by $L : \mathscr{T}^p_q(M) \to \overset{(r)}{\mathscr{T}}{}^p_q(M)$ the map

$$L = (L_0, L_1, \cdots, L_r)$$

for the above (p, q). Now, Lemma 1.2, Lemma 1.10 and Corollary 2.4 show that the following

$$\left\{\begin{aligned} L(f\cdot g) &= L(f) \otimes L(g), \\ L(f\cdot X) &= L(f) \otimes L(X), \\ L(f\cdot \theta) &= L(f) \otimes L(\theta). \end{aligned}\right. \tag{3.2}$$

hold for $f, g \in \mathscr{T}^0_0(M)$, $X \in \mathscr{T}^1_0(M)$ and $\theta \in \mathscr{T}^0_1(M)$.

In order to lift tensor fields on M to tensor fields on $\overset{r}{T}M$ we shall prove the following

THEOREM 3.1. *There exists one and only one linear map* $\tilde{L} : \mathscr{T}(M) \to \overset{(r)}{\mathscr{T}}(M)$ *such that*

$$\tilde{L}(S \otimes U) = \tilde{L}(S) \otimes \tilde{L}(U) \tag{3.3}$$

for $S, U \in \mathscr{T}(M)$, *i.e.* $\tilde{L}$ *is an algebra homomorphism and that* $\tilde{L} | \mathscr{T}^p_q(M) = L$ *for* $p + q \leq 1$.

Proof. Consider the map L^p_q of $\mathscr{T}^1_0(M) \times \cdots \times \mathscr{T}^1_0(M) \times \mathscr{T}^0_1(M) \times \cdots \times \mathscr{T}^0_1(M)$ (p factors of $\mathscr{T}^1_0(M)$ and q factors of $\mathscr{T}^0_1(M)$) into $\overset{(r)}{\mathscr{T}}{}^p_q(M)$ defined by

$$L^p_q(X_1, \cdots, X_p, \theta_1, \cdots, \theta_q) = L(X_1) \otimes \cdots \otimes L(X_p) \otimes L(\theta_1) \otimes L(\theta_1) \otimes \cdots \otimes L(\theta_q)$$

for $X_i \in \mathscr{T}^1_0(M)$, $i = 1, \cdots, p$ and $\theta_j \in \mathscr{T}^0_1(M)$, $j = 1, \cdots, q$. Clearly the map L^p_q is multilinear. Therefore, using (3.2), we see that there exists one and only one linear map $\tilde{L}$ of $\mathscr{T}^p_q(M)$ into $\overset{(r)}{\mathscr{T}}{}^p_q(M)$ such that

$$\tilde{L}(X_1 \otimes \cdots \otimes X_p \otimes \theta_1 \otimes \cdots \otimes \theta_q) = L(X_1) \otimes \cdots \otimes L(X_p) \otimes L(\theta_1) \otimes \cdots \otimes L(\theta_q),$$

from which it follows that $\tilde{L}(S \otimes U) = \tilde{L}(S) \otimes \tilde{L}(U)$ for $S, U \in \mathscr{T}^p_q(M)$ with $p > 0$ or $q > 0$, while $\tilde{L}$ satisfies (3.3) for all (p, q) with $p + q \leqq 1$. Therefore, $\tilde{L}$ satisfies (3.3) for every $S, U \in \mathscr{T}(M)$. On the other hand, the uniqueness of $\tilde{L}$ is clear. Q.E.D.

DEFINITION 3.2. Take a tensor field $K \in \mathscr{T}^p_q(M)$ on M. Then $\tilde{L}(K) = (K^{(0)}, K^{(1}, \cdots, K^{(r)})$ with $K^{(\lambda)} \in \mathscr{T}^p_q(\overset{r}{T}M)$ for $\lambda = 0, 1, \cdots, r$. We shall call $K^{(\lambda)}$ *the* (λ)*-lift of* K *to* $\overset{r}{T}M$. The equality (3.3) shows that for any $K, S \in \mathscr{T}(M)$ we have

$$(K \otimes S)^{(\lambda)} = \sum_{\mu=0}^{\lambda} K^{(\mu)} \otimes S^{(\lambda-\mu)}$$

for every $\lambda = 0, 1, \cdots, r$.

For the sake of convenience, we put $K^{(\lambda)} = 0$ for every negative integer λ.

Remark 3.3. In the case of usual tangent bundle, i.e. $r = 1$, $K^{(0)}$ is identical with the vertical lift of K, while $K^{(1)}$ is identical with the complete lift of K due to Yano-Kobayashi [5].

Let X be a vector field on M and let $\mathcal{L}_X : \mathcal{T}(M) \to \mathcal{T}(M)$ be the Lie derivation with respect to X. We shall prove the following

LEMMA 3.4. *Let $K \in \mathcal{T}(M)$ and $X \in \mathcal{T}^1_0(M)$. Then, we have*

$$\mathcal{L}_{X^{(\lambda)}} K^{(\mu)} = (\mathcal{L}_X K)^{(\lambda+\mu-r)} \tag{3.4}$$

for any λ, $\mu = 0, 1, \cdots, r$.

Proof. Since the Lie derivation is a derivation of the tensor algebra, it suffices to prove (3.4) for the special cases: $K = f \in \mathcal{T}^0_0(M)$, $K = df$ or $K = Y \in \mathcal{T}^1_0(M)$ and to prove that if $S \in \mathcal{T}^p_q(M)$ (resp $T \in \mathcal{T}^{p'}_{q'}(M)$) satisfies (3.4), then

$$\mathcal{L}_{X^{(\lambda)}}(S \otimes T)^{(\mu)} = \mathcal{L}_X(S \otimes T))^{(\lambda+\mu-r)} \tag{3.5}$$

holds for $\lambda, \mu = 0, 1, \cdots, r$. First, if $K = f$, we have

$$\mathcal{L}_{X^{(\lambda)}} f^{(\mu)} = X^{(\lambda)} f^{(\mu)} = (Xf)^{(\lambda+\mu-r)} = (\mathcal{L}_X f)^{(\lambda+\mu-r)}.$$

Second, if $K = df$, we have

$$\begin{aligned}\mathcal{L}_{X^{(\lambda)}}((df)^{(\mu)}) &= \mathcal{L}_{X^{(\lambda)}}(df^{(\mu)}) = d(X^{(\lambda)} f^{(\mu)}) \\ &= d(Xf)^{(\lambda+\mu-r)} = (d(Xf))^{(\lambda+\mu-r)} = (\mathcal{L}_X df)^{(\lambda+\mu-r)}.\end{aligned}$$

Third, if $K = Y$, we have

$$\begin{aligned}\mathcal{L}_{X^{(\lambda)}}(Y^{(\mu)}) &= [X^{(\lambda)}, Y^{(\mu)}] = [X, Y]^{(\lambda+\mu-r)} \\ &= (\mathcal{L}_X Y)^{(\lambda+\mu-r)},\end{aligned}$$

where we have used Lemma 1.8. Finally, if $K = S \otimes T$, we calculate as follows:

$$\begin{aligned}\mathcal{L}_{X^{(\lambda)}}(S \otimes T)^{(\mu)} &= \mathcal{L}_{X^{(\lambda)}} \sum_{\nu=0}^{\mu} S^{(\nu)} \otimes T^{(\mu-\nu)} \\ &= \sum_{\nu=0}^{\mu} \{\mathcal{L}_{X^{(\lambda)}} S^{(\nu)} \otimes T^{(\mu-\nu)} + S^{(\nu)} \otimes \mathcal{L}_{X^{(\lambda)}} T^{(\mu-\nu)}\} \\ &= \sum_{\nu=0}^{\mu} \mathcal{L}_X S)^{(\lambda+\nu-r)} \otimes T^{(\mu-\nu)} + \sum_{\nu=0}^{\mu} S^{(\nu)} \otimes \mathcal{L}_X T)^{(\lambda+\mu-\nu-r)} \\ &= \sum_{\nu'=0}^{\mu} (\mathcal{L}_X S)^{(\lambda+\mu-\nu' - r)} \otimes T^{(\nu')} + \sum_{\nu=0}^{\lambda+\mu-r} S^{(\nu)} \otimes (\mathcal{L}_X T)^{(\lambda+\mu-\nu-r)} \\ &= \sum_{\nu=0}^{\lambda+\mu-r} (\mathcal{L}_X S)^{(\lambda+\mu-\nu-r)} \otimes T^{(\nu)} + \sum_{\nu=0}^{\lambda+\mu-r} S^{(\nu)} \otimes (\mathcal{L}_X T)^{(\lambda+\mu-\nu-r)} \\ &= ((\mathcal{L}_X S) \otimes T)^{(\lambda+\mu-r)} + (S \otimes \mathcal{L}_X T)^{(\lambda+\mu-r)}\end{aligned}$$

$$= (\mathcal{L}_X S \otimes T + S \otimes \mathcal{L}_X T)^{(\lambda+\mu-r)} = (\mathcal{L}_X (S \otimes T))^{(\lambda+\mu-r)}. \qquad \text{Q.E.D.}$$

COROLLARY 3.5. *Let* $X \in \mathcal{T}^1_0(M)$ *and* $K \in \mathcal{T}(M)$. *Then we have*

$$K_{X^{(r)}} K^{(r)} = (\mathcal{L}_X K)^{(r)} \tag{3.6}$$

$$\mathcal{L}_{X^{(r)}} K^{(0)} = \mathcal{L}_X K)^{(0)} \tag{3.7}$$

$$\mathcal{L}_{X^{(0)}} K^{(r)} = (\mathcal{L}_X K)^{(0)} \tag{3.8}$$

$$\mathcal{L}_{X^{(0)}} K^{(0)} = 0 . \tag{3.9}$$

Proof. Apply (3.4) for $\lambda, \eta = 0$ or r.

Remark 3.6. In Lemma 3.4 and Corollary 3.5 we have unified and generalized the formulas (1) of Prop. 4.1 and (1), (2), (3) of Prop. 5.1 [5] (cf. Remark 3.3.)

We now fix a positive integer k. Then, for $s \geq k$, every vector field X defines a linear map $\alpha_X = \alpha^k_X$ of $\mathcal{T}^p_s(M)$ into $\mathcal{T}^p_{s-1}(M)$ such that

$$\alpha^k_X (S \otimes \omega_1 \otimes \cdots \omega_k \otimes \cdots \otimes \omega_t) = S \otimes \omega_1 \otimes \cdots \otimes \omega_k(X) \otimes \cdots \otimes \omega_s,$$

where $S \in \mathcal{T}^p_0(M)$ and $\omega_i \in \mathcal{T}^0_1(M)$ for $i = 1, 2, \cdots, s$.

LEMMA 3.7. *For* $K \in \mathcal{T}^p_s(M)$ *and* $X \in \mathcal{T}^1_0(M)$, *we have the following*

$$\alpha^k_{X^{(\lambda)}} K^{(\mu)} = (\alpha^k_X K)^{(\lambda+\mu-r)} \tag{3.10}$$

Proof. It is sufficient to verify (3.10) for $K = T \otimes \omega_k \otimes U$, where $T \in \mathcal{T}^p_{k-1}(M)$ and $U \in \mathcal{T}^0_{s-k}(M)$. Making use of the equality (2.3), we calculate as follows:

$$\begin{aligned}
\alpha^k_{X^{(\lambda)}} (T \otimes \omega_k \otimes U)^{(\mu)} &= \alpha^k_{X^{(\lambda)}} \sum_{\alpha+\beta+\gamma=\mu} T^{(\alpha)} \otimes \omega_k^{(\beta)} \otimes U^{(\gamma)} \\
&= \sum_{\alpha+\beta+\gamma=\mu} T^{(\alpha)} \otimes \omega_k^{(\beta)}(X^{(\lambda)}) \otimes U^{(\gamma)} \\
&= \sum_{\alpha+\beta+\gamma=\mu} T^{(\alpha)} \otimes (\omega_k(X))^{(\beta+\lambda-r)} \otimes U^{(\gamma)} \\
&= \sum_{\alpha+\beta'+\gamma=\mu+\lambda-r} T^{(\alpha)} \otimes (\omega_k(X))^{(\beta')} \otimes U^{(\gamma)} \\
&= (T \otimes (\omega_k(X)) \otimes U)^{(\lambda+\mu-r)} = (\alpha^k_X (T \otimes \omega_k \otimes U))^{(\lambda+\mu-r)}.
\end{aligned}$$

Q.E.D.

COROLLARY 3.8. *For* $K \in \mathcal{T}^p_s(M)$ *and* $k \leq s$ *we have the following*

$$\alpha^k_{X^{(r)}} (K^{(r)}) = (\alpha^k_X K)^{(r)} \tag{3.11}$$

$$\alpha^k_{X^{(r)}}(K^{(0)} = (\alpha^k_X K)^{(0)} \tag{3.12}$$

$$\alpha^k_{X^{(0)}}(K^{(r)}) = (\alpha^k_X K)^{(0)} \tag{3.13}$$

$$\alpha^k_{X^{(0)}}(K^{(0)}) = 0. \tag{3.14}$$

Proof. Apply (3.10) for $\lambda, \mu = 0$ ro r.

Remark 3.9. In Corollary 3.8 and Lemma 3.7 we have unified and generalized the formulas (3) of Prop. 4.1 and (5), (6), (7) of Prop. 5.1 [5].

COROLLARY 3.10. *If* $K \in \mathscr{T}^p_s(M)$, *then*

$$K^{(\lambda)}(X_1^{(\mu_1)}, \cdots, X_s^{(\mu_s)}) = (K(X_1, \cdots, X_s))^{(\lambda+\mu-r\cdot s)} \tag{3.15}$$

for $X_i \in \mathscr{T}^1_0(M)$, *where* $\mu = \sum_{i=1}^{s} \mu_i$.

Proof. Apply the formula (3.10) s times. Q.E.D.

COROLLARY 3.11. *If* $K \in \mathscr{T}^p_s(M)$, $X_i \in \mathscr{T}^1_0(M)$, *then we have*

$$K^{(\lambda)}(X_1^{(r)}, \cdots, X_s^{(r)}) = (K(X_1, \cdots, X_s))^{(\lambda)} \tag{3.16}$$

for every $\lambda = 0, 1, \cdots, r$.

Proof. Put $\mu_i = r$ in (3.15), then we get (3.16). Q.E.D.

§ 4. Prolongations of almost complex structures.

LEMMA 4.1. *Let* $A, B \in \mathscr{T}^1_1(M)$ *and consider them as fields of linear endomorphisms of tangent spaces of* M. *Let* I_M *be the field of identity transformations of tangent spaces of* M. *Then,*

$$(A \circ B)^{(r)} = A^{(r)} \circ B^{(r)}, \tag{4.1}$$

$$(I_M)^{(r)} = I_{T^r_M}. \tag{4.2}$$

In particular, if P *is a polynomial of one variable, then we have*

$$P(A^{(r)}) = P(A)^{(r)}. \tag{4.3}$$

Proof. By Corollary 3.10, we have, for any $X \in \mathscr{T}^1_0(M)$ and $\lambda = 0, 1, \cdots, r$,

$$(A^{(r)} \circ B^{(r)})(X^{(\lambda)}) = A^{(r)}B^{(r)}(X^{(\lambda)})) = A^{(r)}(B(X)^{(\lambda)}$$
$$= (A(B(X)))^{(\lambda)} = ((A \circ B)(X))^{(\lambda)} = (A \circ B)^{(r)}(X^{(\lambda)}).$$

Therefore, we get $A^{(r)} \circ B^{(r)} = (A \circ B)^{(r)}$. Next, I_M can be written locally as follows:

$$I_M = \sum_i \frac{\partial}{\partial x_i} \otimes dx_i.$$

Making use of Corollary 1.5 and Lemma 2.2, we have

$$\begin{aligned}
(I_M)^{(r)} &= \sum_i \left(\frac{\partial}{\partial x_i} \otimes dx_i\right)^{(r)} = \sum_i \sum_{\lambda=0}^{r} \left(\frac{\partial}{\partial x_i}\right)^{(\lambda)} \otimes (dx_i)^{(r-\lambda)} \\
&= \sum_i \sum_{\lambda=0}^{r} \frac{\partial}{\partial \overset{(r-\lambda)}{x_i}} \otimes d\overset{(r-\lambda)}{x_i} \\
&= \sum_i \sum_{\lambda=0}^{r} \frac{\partial}{\partial \overset{(\lambda)}{x_i}} \otimes d\overset{(\lambda)}{x_i} = I_{\overset{r}{T}M}.
\end{aligned}$$

Q.E.D.

Using the same arguments as in the proof of Proposition 6.7 and 6.12 [5], we obtain the following theorems:

THEOREM 4.2. *If $J \in \mathscr{T}^1_1(M)$ defines an almost complex structure on M, so does $J^{(r)}$ on $\overset{r}{T}M$. If N_J denotes the Nijenhuis tensor of J, $(N_J)^{(r)}$ is the Nijenhuis tensor of $J^{(r)}$.*

THEOREM 4.3. *If $\overset{r}{M}$ is a homogeneous (almost) complex manifold with almost complex structure J, so is $\overset{r}{T}M$ with (almost) complex structure $J^{(r)}$.*

§ 5. Prolongations of affine connections.

Let ∇ be the covariant differentiation defined by an affine connection of M. We shall prove the following

THEOREM 5.1. *There exists one and only one affine connection of $\overset{r}{T}M$ whose covariant differentiation $\tilde{\nabla}$ satisfies the following condition:*

$$\tilde{\nabla}_{X^{(\lambda)}} Y^{(\mu)} = (\nabla_X Y)^{(\lambda+\mu-r)} \tag{5.1}$$

for every $X, Y \in \mathscr{T}^1_0(M)$ and $\lambda, \mu = 0, \cdots, r$.

Proof. Take a coordinate neighborhood U with coordinate system $\{x_1, \cdots, x_n\}$ and let Γ^k_{ij} be the connection components of ∇ with respect to $\{x_1, \cdots, x_n\}$, i.e.

$$\nabla_{\frac{\partial}{\partial x_i}} \left(\frac{\partial}{\partial x_j}\right) = \sum_{k=1}^{n} \Gamma^k_{ij} \frac{\partial}{\partial x_k} \tag{5.2}$$

for $i, j = 1, \cdots, n$. Let $\{y_1, \cdots, y_n\}$ be another coordinate system on U and let Γ'^{k}_{ij} be the connection components of ∇ with respect to $\{y_1, \cdots, y_n\}$. Then, we have the following equalities:

$$\Gamma'^{k}_{ij} = \sum_{a,b,c} \frac{\partial x_b}{\partial y_i} \frac{\partial x_c}{\partial y_j} \frac{\partial y_k}{\partial x_a} \Gamma^{a}_{bc} + \sum_{a} \frac{\partial^2 x_a}{\partial y_i \partial y_j} \frac{\partial y_k}{\partial x_a} \tag{5.3}$$

for $i, j, k = 1, 2, \cdots, n$. Let $\{\overset{(\nu)}{x}_i\}$ (resp. $\{\overset{(\nu)}{y}_i\}$) be the induced coordinate system on $(\pi_r)^{-1}(U)$. Define

$$\tilde{\Gamma}^{(k,\lambda)}_{(i,\nu)(j,\mu)} = (\Gamma^{k}_{ij})^{(\lambda-\nu-\mu)} \tag{5.4}$$

for $i, j, k = 1, 2, \cdots, n$; $\lambda, \mu, \nu = 0, 1, \cdots, r$. We shall prove that there exists a connection $\tilde{\nabla}$ whose connection components with respect to the coordinate system $\{\overset{(\nu)}{x}_i\}$ are given by (5.4). To prove this we have to prove the following equalities similar to (5.3):

$$\tilde{\Gamma}'^{(k,\lambda)}_{(i,\nu)(i,\mu)} = \sum_{\alpha,\beta,\gamma} \sum_{a,b,c} \frac{\partial \overset{(\beta)}{x}_b}{\partial \overset{(\nu)}{y}_i} \frac{\partial \overset{(\gamma)}{x}_c}{\partial \overset{(\mu)}{y}_j} \frac{\partial \overset{(\lambda)}{y}_k}{\partial \overset{(\alpha)}{x}_a} \tilde{\Gamma}^{(a,\alpha)}_{(b,\beta)(c,\gamma)} + \sum_{\alpha} \sum_{a} \frac{\partial^{2} \overset{(\alpha)}{x}_a}{\partial \overset{(\nu)}{y}_i \partial \overset{(\mu)}{y}_j} \frac{\partial \overset{(\lambda)}{y}_k}{\partial \overset{(\alpha)}{x}_a} \tag{5.5}$$

for $i, j, k = 1, 2, \cdots, n$; $\lambda, \mu, \nu = 0, 1, \cdots, r$, where $\tilde{\Gamma}'^{(k,\lambda)}_{(i,\nu)(j,\mu)}$ denote the connection components of $\tilde{\nabla}$ with respect to the coordinate system $\{\overset{(\nu)}{y}_i\}$. To prove (5.5), we consider the $(\lambda-\mu-\nu)$-lift of Γ'^{k}_{ij} in the equality (5.3). Then by putting $\rho = \lambda - \mu - \nu$ and by using Corollary 1.6, we calculate as follows:

$$\begin{aligned}
(\Gamma'^{k}_{ij})^{(\rho)} &= \sum_{a,b,c} \sum_{\alpha+\beta+\gamma+\delta=\rho} \left(\frac{\partial x_b}{\partial y_i}\right)^{(\beta)} \left(\frac{\partial x_c}{\partial y_j}\right)^{(\gamma)} \left(\frac{\partial y_k}{\partial x_a}\right)^{(\alpha)} (\Gamma^{a}_{bc})^{(\delta)} \\
&\quad + \sum_{a} \sum_{\alpha+\beta=\rho} \left(\frac{\partial^2 x_a}{\partial y_i \partial y_j}\right)^{(\alpha)} \left(\frac{\partial y_k}{\partial x_a}\right)^{(\beta)} \\
&= \sum_{a,b,c=1}^{n} \sum_{\alpha'-\beta'-\gamma'=0}^{\rho} \left(\frac{\partial x_b}{\partial y_i}\right)^{(\beta'-\nu)} \left(\frac{\partial x_c}{\partial y_j}\right)^{(\gamma'-\mu)} \left(\frac{\partial y_k}{\partial x_a}\right)^{(\lambda-\alpha')} (\Gamma^{a}_{bc})^{(\alpha'-\beta'-\gamma')} \\
&\quad + \sum_{a}^{n} \sum_{\alpha=\mu+\nu}^{r} \left(\frac{\partial^2 x_a}{\partial y_i \partial y_j}\right)^{(\alpha-\mu-\nu)} \left(\frac{\partial y_k}{\partial x_a}\right)^{(\lambda-\alpha)} \\
&= \sum_{a,b,c=1}^{n} \sum_{\alpha,\beta,\gamma=0}^{r} \left(\frac{\partial x_b}{\partial y_i}\right)^{(\beta-\nu)} \left(\frac{\partial x_c}{\partial y_j}\right)^{(\gamma-\mu)} \left(\frac{\partial y_k}{\partial x_a}\right)^{(\lambda-\alpha)} (\Gamma^{a}_{bc})^{(\alpha-\beta-\gamma)}
\end{aligned}$$

$$+\sum_a \sum_{\alpha=0}^{r} \left(\frac{\partial^2 x_a}{\partial y_i \partial y_j} \right)^{(\alpha-\mu-\nu)} \left(\frac{\partial y_k}{\partial x_a} \right)^{(\lambda-\alpha)}$$

$$= \sum_{a,b,c} \sum_{\alpha,\beta,\gamma} \left(\frac{\overset{(\beta)}{\partial x_b}}{\underset{(\nu)}{\partial y_i}} \right) \left(\frac{\overset{(\gamma)}{\partial x_c}}{\underset{(\mu)}{\partial y_j}} \right) \left(\frac{\partial y_k^{(\lambda)}}{\underset{(\alpha)}{\partial x_a}} \right) \tilde{\Gamma}^{(a,\alpha)}_{(b,\beta)(c,\gamma)}$$

$$+ \sum_a \sum_\alpha \frac{\overset{2(\alpha)}{\partial x_a}}{\overset{(\nu)}{\partial y_i} \overset{(\mu)}{\partial y_i}} \frac{\overset{(\lambda)}{\partial y_k}}{\overset{(\alpha)}{\partial x_a}},$$

where we have used, in the third equality, the following fact: if $\alpha-\beta-\gamma>\rho=\lambda-\mu-\nu$, then at least one of the three numbers $\beta-\nu$, $\gamma-\mu$, $\lambda-\alpha$ is negative. Thus, we have proved that (5.5) holds.

Next, to prove (5.1), we first prove the following

$$\tilde{\nabla}_{\left(\frac{\partial}{\partial x_i}\right)} \left(\frac{\partial}{\partial x_j} \right)^{(\mu)} = \left(\nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_j} \right)^{(\lambda+\mu-r)} \tag{5.6}$$

The left hand side of (5.6) is equal to

$$\tilde{\nabla}_{\frac{\partial}{\overset{(r-\lambda)}{\partial x_i}}} \frac{\partial}{\overset{(r-\mu)}{\partial x_j}} = \sum \tilde{\Gamma}^{(k,\nu)}_{(i,r-\lambda)(j,r-\mu)} \frac{\partial}{\overset{(\nu)}{\partial x_k}}$$

$$= \sum_k \sum_{\nu=0}^{r} (\Gamma^k_{ij})^{(\nu+\mu+\lambda-2r)} \left(\frac{\partial}{\partial x_k} \right)^{(r-\nu)}$$

$$= \sum_k \sum_{\nu'=0}^{r} (\Gamma^k_{ij})^{(\mu+\lambda-r-\nu')} \left(\frac{\partial}{\partial x_k} \right)^{(\nu')}$$

$$= \sum_k \sum_{\nu=0}^{\mu+\lambda-r} (\Gamma^k_{ij})^{(\mu+\lambda-r-\nu)} \left(\frac{\partial}{\partial x_k} \right)^{(\nu)}$$

$$= \sum_k \left(\Gamma^k_{ij} \frac{\partial}{\partial x_k} \right)^{(\mu+\lambda-r)} = \left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \right)^{(\mu+\lambda-r)},$$

which proves (5.6).

Next, we shall prove the following

$$\tilde{\nabla}_{\left(f \frac{\partial}{\partial x_i}\right)^{(\lambda)}} \left(\frac{\partial}{\partial x_j} \right)^{(\mu)} = \left(\nabla_{f \frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \right)^{(\lambda+\mu-r)} \tag{5.7}$$

for any $f \in C^\infty(U)$, $i, j = 1, 2, \cdots, n$; $\lambda, \mu = 0, 1, \cdots, r$. The left hand side of (5.7) is equal to

$$\tilde{\nabla}_{\sum_{\nu=0}^{\lambda} f^{(\nu)} \left(\frac{\partial}{\partial x_i} \right)^{(\lambda-\nu)}} \left(\frac{\partial}{\partial x_j} \right)^{(\mu)} = \sum f^{(\nu)} \tilde{\nabla}_{\left(\frac{\partial}{\partial x_i}\right)^{(\lambda-\nu)}} \left(\frac{\partial}{\partial x_j} \right)^{(\mu)}$$

$$= \sum_{\nu=0}^{\lambda} f^{(\nu)}\left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}\right)^{(\lambda+\mu-\nu-)} = \sum_{\nu=0}^{\lambda+\mu-r} f^{(\nu)}\left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}\right)^{(\lambda+\mu-\nu-r)}$$

$$= \left(f\cdot\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}\right)^{(\lambda+\mu-r)} = \left(\nabla_{f\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_j}\right)^{(\lambda+\mu-r)},$$

which proves (5.7).

From (5.7), it follows that

$$\tilde{\nabla}_{X^{(\lambda)}}\left(\frac{\partial}{\partial x_j}\right)^{(\mu)} = \left(\nabla_X \frac{\partial}{\partial x_j}\right)^{(\lambda+\mu-r)} \tag{5.8}$$

holds for every $X\in\mathcal{T}_0^1(U)$, $j=1,\cdots,n$; r, $\mu=0,1,\cdots,r$.

Finally, take $Y\in\mathcal{T}_0^1(U)$ such that $Y=\sum f_j \frac{\partial}{\partial x_j}$. Then we can calculate as follows:

$$\tilde{\nabla}_{X^{(\lambda)}}Y^{(\mu)} = \sum_j \tilde{\nabla}_{X^{(\lambda)}}\left(f_j\frac{\partial}{\partial x_j}\right)^{(\mu)} = \sum_j \tilde{\nabla}_{X^{(\lambda)}}\left(\sum_{\nu=0}^{\mu} f_j^{(\mu-\nu)} \frac{\partial}{\underset{(r-\nu)}{\partial x_j}}\right)$$

$$= \sum_j \sum_{\nu=0}^{\mu}\left\{f_j^{(\mu-\nu)}\tilde{\nabla}_{X^{(\lambda)}}\left(\frac{\partial}{\partial x_j}\right)^{(\nu)} + X^{(\lambda)}f_j^{(\mu-\nu)}\cdot\left(\frac{\partial}{\partial x_j}\right)^{(\nu)}\right\}$$

$$= \sum_j \sum_{\nu=0}^{\mu}\left\{f_j^{(\mu-\nu)}\left(\nabla_X\frac{\partial}{\partial x_j}\right)^{(\lambda+\nu-r)} + (Xf_j)^{(\lambda+\mu-\nu-r)}\left(\frac{\partial}{\partial x_j}\right)^{(\nu)}\right\}$$

$$= \sum_j\left\{\sum_{\nu'=0}^{\mu} f_j^{(\nu')}\left(\nabla_X\frac{\partial}{\partial x_j}\right)^{(\lambda+\mu-r-\nu')} + \sum_{\nu=0}^{\mu}(Xf_j)^{(\lambda+\mu-r-\nu)}\left(\frac{\partial}{\partial x_j}\right)^{(\nu)}\right\}$$

$$= \sum_j\left\{\sum_{\nu=0}^{\lambda+\mu-r} f_j^{(\nu)}\left(\nabla_X\frac{\partial}{\partial x_j}\right)^{(\lambda+\mu-r-\nu)} + \sum_{\nu=0}^{\lambda+\mu-r}(Xf_j)^{(\lambda+\mu-r-\nu)}\left(\frac{\partial}{\partial x_j}\right)^{(\nu)}\right\}$$

$$= \sum_j\left\{\left(f_j\cdot\nabla_X\frac{\partial}{\partial x_j}\right)^{(\lambda+\mu-r)} + \left(Xf_j\cdot\frac{\partial}{\partial x_j}\right)^{(\lambda+\mu-r)}\right\}$$

$$= \sum_j\left(\nabla_X\left(f_j\frac{\partial}{\partial x_j}\right)\right)^{(\lambda+\mu-r)} = (\nabla_X Y)^{(\lambda+\mu-r)}.$$

Thus, we have proved the existence of a connection $\tilde{\nabla}$ on $\overset{r}{T}M$ satisfying the condition (5.1). The uniqueness of such $\tilde{\nabla}$ is clear since $\left\{\left(\frac{\partial}{\partial x_i}\right)^{(\lambda)} \mid i=1,2,\cdots, n;\ \lambda=0,1,\cdots,r\right\}$ is a basis of $\mathcal{T}_0^1(\overset{r}{\pi}{}^{-1}(U))$. Q.E.D.

DEFINITION 5.2. The unique connection $\tilde{\nabla}$ in Theorem 5.1 will be called *the prolongation* (or *complete lift*) *of* ∇ *to* $\overset{r}{T}M$ and will be denoted by $\tilde{\nabla}=\overset{(r)}{\nabla}$.

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We note that, in the case $r=1$, $\overset{(1)}{\nabla}$ is identical with the complete lift ∇^c due to Yano-Kobayashi [5].

PROPOSITION 5.3. *If T and R are the torsion and the curvature tensor fields for ∇, then $T^{(r)}$ and $R^{(r)}$ are the torsion and the curvature tensor fields for $\overset{(r)}{\nabla}$.*

Proof. Let $\tilde{T}$ and $\tilde{R}$ be the torsion and the curvature tensor fields for $\overset{(r)}{\nabla}$. Making use of Corollary 3.10, we have

$$\begin{aligned} T^{(r)}(X^{(\lambda)}, Y^{(\mu)}) &= (T(X,Y))^{(\lambda+\mu-r)} \\ &= (\nabla_X Y - \nabla_Y X - [X,Y])^{(\lambda+\mu-r)} \\ &= \overset{(r)}{\nabla}_{X^{(\lambda)}} Y^{(\mu)} - \overset{(r)}{\nabla}_{Y^{(\mu)}} X^{(\lambda)} - [X^{(\lambda)}, Y^{(\mu)}] \\ &= \tilde{T}(X^{(\lambda)}, Y^{(\mu)}) \end{aligned}$$

for every $X, Y \in \mathcal{T}^1_0(M)$ and $\lambda, \mu = 0, 1, \cdots, r$. Hence we get $T^{(r)} = \tilde{T}$. Similarly, we have

$$\begin{aligned} R^{(r)}(X^{(\lambda)}, Y^{(\mu)})Z^{(\nu)} &= (R(X, Y)Z)^{(\lambda+\mu+\nu-2r)} \\ &= ([\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z)^{(\lambda+\mu+\nu-2r)} \\ &= \overset{(r)}{\nabla}_{X^{(\lambda)}}(\nabla_Y Z)^{(\mu+\nu-r)} - \overset{(r)}{\nabla}_{Y^{(\mu)}}(\nabla_X Z)^{(\lambda+\nu-r)} - \overset{(r)}{\nabla}_{[X,Y]^{(\lambda+\mu-r)}} Z^{(\nu)} \\ &= [\overset{(r)}{\nabla}_{X^{(\lambda)}}, \overset{(r)}{\nabla}_{Y^{(\mu)}}]Z^{(\nu)} - \overset{(r)}{\nabla}_{[X^{(\lambda)}, Y^{(\mu)}]} Z^{(\nu)} \\ &= \tilde{R}(X^{(\lambda)}, Y^{(\mu)})Z^{(\nu)}. \end{aligned}$$

Therefore, we obtain $R^{(r)} = \tilde{R}$. Q.E.D.

PROPOSITION 5.4. *For any tensor field K on M and any vector field X on M, we have*

$$\overset{(r)}{\nabla}_{X^{(\lambda)}}(K^{(\mu)}) = (\nabla_X K)^{(\lambda+\mu-r)} \tag{5.9}$$

$$\overset{(r)}{\nabla} K^{(\mu)} = (\nabla K)^{(\mu)} \tag{5.10}$$

for every $\lambda, \mu = 0, 1, \cdots, r$.

Proof. It is sufficient to prove these formulas in the special cases, where $K = f \in \mathcal{T}^0_0(M)$, $K = \theta \in \mathcal{T}^0_1(M)$ and $K = Y \in \mathcal{T}^1_0(M)$.

If $K = f$, then $\overset{(r)}{\nabla}_{X^{(\lambda)}} f^{(\mu)} = K^{(\lambda)} = f^{(\mu)} = (Xf)^{(\lambda+\mu-r)} = (\nabla_X f)^{(\lambda+\mu-r)}$. If $K = Y$, then (5.9) is nothing but the formula (5.1).

If $K=\theta$, then, using Lemma 2.5, we have

$$(\overset{(r)}{\nabla}_{X^{(\lambda)}}\theta^{(\mu)})(Y^{(\nu)})=\overset{(r)}{\nabla}_{X^{(\lambda)}}(\theta^{(\mu)}(Y^{(\nu)}))-\theta^{(\mu)}(\overset{(r)}{\nabla}_{X^{(\lambda)}}Y^{(\nu)})$$
$$=\overset{(r)}{\nabla}_{X^{(\lambda)}}((\theta(Y))^{(\mu+\nu-r)})-\theta^{(\mu)}(\nabla_X Y)^{(\lambda+\nu-r)}$$
$$=(\nabla_X(\theta(Y)))^{(\lambda+\mu+\nu-2r)}-(\theta(\nabla_X Y))^{(\lambda+\mu+\nu-2r)}$$
$$=((\nabla_X\theta)Y)^{(\lambda+\mu+\nu-2r)}=(\nabla_X\theta)^{(\lambda+\mu-r)}(Y^{(\nu)})$$

for every $Y\in\mathscr{T}^1_0(M)$ and $\nu=0,1,\cdots,r$, and hence we get $\overset{(r)}{\nabla}_{X^{(\lambda)}}\theta^{(\mu)}=(\nabla_X\theta)^{(\lambda+\mu-r)}$.

Next, to prove (5.10), using Lemma 3.7, we have

$$\alpha_{X^{(\lambda)}}\overset{(r)}{\nabla}K^{(\mu)}=\overset{(r)}{\nabla}_{X^{(\lambda)}}K^{(\mu)}=(\nabla_X K)^{(\lambda+\mu-r)}$$
$$=(\alpha_X\nabla K)^{(\lambda+\mu-r)}=\alpha_{X^{(\lambda)}}(\nabla K)^{(\mu)}$$

fore very $X\in\mathscr{T}^1_0(M)$ and $\lambda=0,1,\cdots,r$, and hence we get (5.10).

Q.E.D.

COROLLARY 5.5. *For any tensor field K on M and any vector field X on M, we have*

(5.11) $$\overset{(r)}{\nabla}_{X^{(r)}}K^{(r)}=(\nabla_X K)^{(r)}$$

(5.12) $$\overset{(r)}{\nabla}_{X^{(r)}}K^{(0)}=(\nabla_X K)^{(0)}$$

(5.13) $$\overset{(r)}{\nabla}_{X^{(0)}}K^{(r)}=(\nabla_X K)^{(0)}$$

(5.14) $$\overset{(r)}{\nabla}_{X^{(0)}}K^{(0)}=0$$

(5.15) $$\overset{(r)}{\nabla}K^{(r)}=(\nabla K)^{(r)}$$

$$\overset{(r)}{\nabla}K^{(0)}=(\nabla K)^{(0)}.$$

Proof. Apply Lemma 5.4 for λ, $\mu=0$ or r.

Remark 5.6. In Proposition 5.4 and Corollary 5.5 we have unified and generalized the formulas (1) ~ (6) of Prop. 7.2 [5], where we should correct (6) as $\nabla^c_{X^V}(K^V)=0$.

PROPOSITION 5.7. *Let X be an infinitesimal affine transformation of an affine connection ∇ on M. Then $X^{(\lambda)}$ is an infinitesimal affine transformation of $\overset{(r)}{\nabla}$ for every $\lambda=0,1,\cdots,r$.*

Proof. A necessary and sufficient condition for X to be an infinitesimal affine transformation of M is that

$$\mathscr{L}_X \circ \nabla_Y - \nabla_Y \circ \mathscr{L}_X = \nabla_{[X,Y]}$$

for every $Y \in \mathscr{T}_0^1(M)$. Take $K \in \mathscr{T}(M)$. Using Lemma 3.4 and Proposition 5.4, we calculate as follows:

$$\begin{aligned}
&\mathscr{L}_{X^{(\lambda)}} \circ \overset{(r)}{\nabla}_{Y^{(\mu)}} K^{(\nu)} - \overset{(r)}{\nabla}_{Y^{(\lambda)}} \circ \mathscr{L}_{X^{(\lambda)}} K^{(\nu)} \\
&= \mathscr{L}_{X^{(\lambda)}} (\nabla_Y K)^{(\mu+\nu-r)} - \overset{(r)}{\nabla}_{Y^{(\mu)}} (\mathscr{L}_X K)^{(\lambda+\nu-r)} \\
&= \mathscr{L}_X (\nabla_Y K))^{(\lambda+\mu+\nu-2r)} - (\nabla_Y \mathscr{L}_X K)^{(\lambda+\mu+\nu-2r)} \\
&= (\nabla_{[X,Y]} K)^{(\lambda+\mu+\nu-2r)}, = \overset{(r)}{\nabla}_{[X,Y]^{(\lambda+\mu-r)}} K^{(\nu)} \\
&= \overset{(r)}{\nabla}_{[X^{(\lambda)},Y^{(\mu)}]} K^{(\nu)}
\end{aligned}$$

for every $\nu = 0, 1, \cdots, r$, and hence we get

$$\mathscr{L}_{X^{(\lambda)}} \circ \overset{(r)}{\nabla}_{\tilde{Y}} - \overset{(r)}{\nabla}_{\tilde{Y}} \circ \mathscr{L}_{X^{(\lambda)}} = \overset{(r)}{\nabla}_{[X^{(\lambda)}, \tilde{Y}]}$$

for every $\tilde{Y} \in \mathscr{T}_0^1(\overset{r}{T}M)$, which proves that $X^{(\lambda)}$ is an infinitesimal affine transformation of $\overset{(r)}{\nabla}$. Q.E.D.

COROLLARY 5.8. *If the group of affine transformations of M with ∇ is transitive on M, then the group of affine transformations of $\overset{r}{T}M$ with $\overset{(r)}{\nabla}$ is transitive on $\overset{r}{T}M$.*

From Proposition 5.3 and 5.4 we obtain

THEOREM 5.9. *Let T and R be the torsion and the curvature tensor fields of an affine connection ∇ of M. According as $T = 0$, $\nabla T = 0$, $R = 0$ or $\nabla R = 0$, we have $T^{(r)} = 0$, $\overset{(r)}{\nabla} T^{(r)} = 0$, $R^{(r)} = 0$ or $\overset{(r)}{\nabla} R^{(r)} = 0$. In particular, if M is locally symmetric with respect to ∇, so is $\overset{r}{T}M$ with respect to $\overset{(r)}{\nabla}$.*

§ 6. Final Remarks.

In [5], Yano and Kobayashi considered the complete lifts of special tensor fields such as pseudo-Riemannian metrics, almost symplectic structures and others. We can also prove the similar results to those in [5] for our tangent bundles of higher order. We shall enumerate some of them, without proof, as follows:

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PROPOSITION 6.1. *If M is a homogeneous pseudo-Riemannian manifold with metric g, so is $\overset{r}{T}M$ with metric $g^{(r)}$.*

PROPOSITION 6.2. *If M is a pseudo-Riemannian symmetric space with metric g, then $\overset{r}{T}M$ is also a pseudo-Riemannian symmetric space with metric $g^{(r)}$.*

On the other hand, by the same arguments as in [5] we have the following

PROPOSITION 6.3. *If M is an affine symmetric space with connection ∇, then $\overset{r}{T}M$ is also an affine symmetric space with connection $\overset{(r)}{\nabla}$.*

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