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Kneser–Tits for a rank 1 form of E_6 (after Veldkamp)

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Abstract

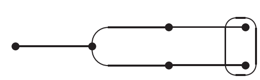
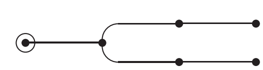
We prove the Kneser–Tits conjecture for groups of index ${}^2E_{6,1}^{29}$ using an argument inspired by a 1968 paper by Veldkamp. We also prove that these groups are stably rational varieties.

Introduction

The notion of *simple* for an algebraic group is different from the notion of simple for abstract groups. Recall that an abstract group Γ is *projectively simple* if $\Gamma/Z(\Gamma)$ is simple as an abstract group, where $Z(\Gamma)$ denotes the center of Γ. For a given field k, the Kneser–Tits conjecture asserts that *for every simply connected and absolutely quasi-simple k-isotropic algebraic group G, the abstract group $G(k)$ is projectively simple.*

A good survey of the conjecture is given in [PR94, § 7.2]. We give a few highlights. Many cases of the conjecture for classical groups are part of 'geometric algebra' as in the books by Artin [Art57] and Dieudonné [Die71]. The conjecture holds for k algebraically closed, for the real numbers (Cartan [Car27]), and for nonarchimedean locally compact fields (Platonov [PR94, p. 414]). It fails wildly if the simply connected hypothesis is dropped. Some groups of inner type A_n provide counterexamples to the conjecture; these amount to central division algebras with nontrivial SK_1. In order to prove the conjecture for a particular field k, Prasad and Raghunathan [PR85] showed that it suffices to consider the groups of k-rank 1.

For k a number field, no counterexamples are known. In order to prove the conjecture in that case, it remains only to prove it for groups with the following Tits indexes:

${}^2E_{6,1}^{29}$   ${}^2E_{6,1}^{35}$

(The *index* of a semisimple algebraic group is defined in [Tit66, 2.3], and the list of possible indexes is given in that paper. The conjecture has long been known for the classical groups, cf. [PR94, p. 410]. The trialitarian groups are treated in [Pra05].) We remark that when k is a totally imaginary number field or a global function field, the two indexes displayed above do not occur [Gil01, p. 315, Theorem 9b], hence the conjecture is proved in that case. The conjecture is still open for number fields with real embeddings, such as the rational numbers.

In fact, one of the two 'open' cases was (essentially) settled in 1968. The purpose of this paper is to give a proof of that case, i.e. to prove the following theorem.

Theorem. *For every field k of characteristic not equal to 2 or 3 and every simply connected k-group G of index ${}^2E_{6,1}^{29}$, the abstract group $G(k)$ is projectively simple.*

This theorem is 9.5(i) in Veldkamp's paper [Vel68], although his paper is missing an argument that his groups have index ${}^2E_{6,1}^{29}$ and that every simply connected group of index ${}^2E_{6,1}^{29}$ is one of his groups. We present a proof from a different viewpoint that is inspired by his and uses somewhat modernized language. We feel that this is worthwhile, partially because his result does not seem

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to have been incorporated into the literature. For example, it is not mentioned in Tits's excellent survey [Tit78]. Also, some delicate aspects of his proof can be avoided with modern techniques.

In [Vel69], Veldkamp modified his proof slightly in order to include also the cases where G is quasi-split or of index ${}^2E_{6,2}^{16'}$. However, these cases are already covered by Tits's survey, so we only consider groups as in the theorem.

We exclude characteristics 2 and 3, as Veldkamp did, in order to use convenient facts about quadratic forms and Jordan algebras. For the reader interested in global function fields, this hypothesis is harmless, as there are no groups of index ${}^2E_{6,1}^{29}$ over such a field.

We include also a proof of the following result which strengthens [CT00, Theorem 2.12b]. Recall that a variety X is *stably rational* if $X \times \mathbb{A}^n$ is birationally equivalent to an affine space for some n.

Proposition 0.1. *For every field k of characteristic not equal to 2, every k-group G of index ${}^2E_{6,1}^{29}$, simply connected or adjoint, is stably rational as a variety.*

The theorem and Proposition 0.1 are connected by the notion of R-equivalence of k-points of an algebraic group due to Manin, Colliot-Thélène, and Sansuc, see e.g. [Vos98, ch. 6] for definitions and basic properties. Fix G as in the theorem. One writes $RG(k)$ for the subset of $G(k)$ of elements that are R-equivalent to the identity. Proposition 0.1 implies that $RG(k)$ is all of $G(k)$, see e.g. [Mer96, Proposition 1]. On the other hand, $RG(k)$ is a noncentral normal subgroup of $G(k)$, so the theorem also implies that $RG(k)$ is all of $G(k)$.

Rationality results for isotropic groups of type E_6 with other indexes can be found in [CP98, § 9].

Notation and conventions

Throughout this paper, C denotes an octonion k-algebra and K is a quadratic field extension of k. We occasionally write C also for the quadratic norm form on C. We write C^K for the 'K-associate' of the norm $N_{C/k}$ on C as defined in [KMRT98, p. 499]: if the norm form on C is $\langle 1\rangle \oplus q$ and $K = k(\sqrt{\alpha})$, then C^K is the quadratic form $\langle 1\rangle \oplus \langle\alpha\rangle q$. We record that

$$C^K \text{ is Witt-equivalent to } \langle\alpha\rangle C \oplus \langle 1, -\alpha\rangle. \tag{0.2}$$

We write $H^1(k, G)$ for the Galois cohomology group $H^1(\mathrm{Gal}(k_{\mathrm{sep}}/k), G(k_{\mathrm{sep}}))$, where k_{sep} denotes a separable closure of k.

1. Outline of proof of the theorem

For a semisimple group G, write $G(k)^+$ for the subgroup generated by the k-points of the unipotent radicals of the parabolic k-subgroups of G. For G quasi-simple, $G(k)^+$ is projectively simple [Tit64].

Fix G as in the theorem and a rank 1 k-split torus S in G. Write H for the centralizer of S in G; it is reductive of type ${}^2D_4$. As in [Tit64, 3.2(18)], we have

$$G(k) = H(k) \cdot G(k)^+. \tag{1.1}$$

In § 3 below, we prove that

$$D(k) \subseteq G(k)^+, \tag{1.2}$$

where D is the stabilizer of a particular vector in the irreducible 54-dimensional representation of G. In §§ 4 and 5, we observe that

$$H(k) \subseteq D(k) \cdot G(k)^+. \tag{1.3}$$

Combining (1.1)–(1.3), we find that $G(k)^+ = G(k)$, which proves the theorem.

2. Explicit description of G

In this section, we assume that we are given a group G_0 of index ${}^2E_{6,1}^{29}$ as in the theorem, and we produce an explicit description of G_0 using Galois descent.

Write K for the quadratic extension of k over which G_0 is of inner type. The semisimple anisotropic kernel of G_0 is isomorphic to $\mathrm{Spin}(C^K)$ for some octonion k-algebra C that is not split by K (see [GP05, 2.10]). By Tits's Witt-type theorem, this sets up a bijection between the set of isomorphism classes of simply connected k-groups of index ${}^2E_{6,1}^{29}$ and the set of isomorphism classes of pairs (K, C), where K is a quadratic extension of k and C is an octonion k-algebra not split by K; cf. [GP05, Proposition 11.1]. We remark that one can detect C using the Rost invariant of G_0, see [GP05, § 2].

Write A for the Albert k-algebra of hermitian 3-by-3 matrices with entries in C. (See [SV00] or [KMRT98] for background about Albert algebras.) It has a nondegenerate symmetric bilinear form 'tr' defined by

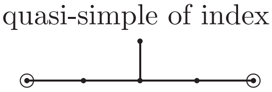
$$\mathrm{tr}\left(\begin{pmatrix}\varepsilon_1 & c_3 & \cdot \\ \cdot & \varepsilon_2 & c_1 \\ c_2 & \cdot & \varepsilon_3\end{pmatrix}, \begin{pmatrix}\nu_1 & d_3 & \cdot \\ \cdot & \nu_2 & d_1 \\ d_2 & \cdot & \nu_3\end{pmatrix}\right) = \sum_{i=1}^{3}[\varepsilon_i\nu_i + C(c_i, d_i)],$$

where C denotes the symmetric bilinear form deduced from the norm on C. (When writing elements of A, we replace some entries with a '$\cdot$'. No information is lost, because these entries are determined by the condition that the matrix is hermitian.)

We write ${}^1G$ for the group of isometries of the cubic form

$$\begin{pmatrix}\varepsilon_1 & c_3 & \cdot \\ \cdot & \varepsilon_2 & c_1 \\ c_2 & \cdot & \varepsilon_3\end{pmatrix} \mapsto \varepsilon_1\varepsilon_2\varepsilon_3 + \mathrm{trace}_C(c_1c_2c_3) - \sum_{i=1}^{3}\varepsilon_i C(c_i)$$

on A. This group is simply connected quasi-simple of index

${}^1E_{6,2}^{28}$ 

It acts on the 54-dimensional vector space $A \oplus A$ via the homomorphism ρ defined by

$$\rho(g)(a_1, a_2) = (ga_1, g^\dagger a_2),$$

where $g^\dagger$ is defined by the equation $\mathrm{tr}(gx, g^\dagger y) = \mathrm{tr}(x, y)$ for all $x, y \in A$. The map $g \mapsto g^\dagger$ is an automorphism of ${}^1G$ of order 2, see [Jac61, p. 76] or [SV00, 7.3.1]; it is outer because it is not the identity on the center of ${}^1G$.

Abbreviate $A \otimes K$ as A_K and write ι for the nonidentity k-automorphism of K. Consider the k-space V of elements of $A_K \oplus A_K$ fixed by

$$(a_1, a_2) \mapsto (\tau\iota a_2, \tau\iota a_1),$$

where $\tau \in {}^1G(k)$ is defined by

$$\tau\begin{pmatrix}\varepsilon_1 & c_3 & \cdot \\ \cdot & \varepsilon_2 & c_1 \\ c_2 & \cdot & \varepsilon_3\end{pmatrix} = \begin{pmatrix}\varepsilon_1 & \pi(c_2) & \cdot \\ \cdot & \varepsilon_3 & \pi(c_1) \\ \pi(c_3) & \cdot & \varepsilon_2\end{pmatrix},$$

where π denotes the canonical involution on C.

Define G to be the group ${}^1G$ with a twisted $\mathrm{Gal}(K/k)$-action given by

$$\iota * g = \tau\iota(g)^\dagger\tau^{-1} \quad \text{for } g \in G(K), \tag{2.1}$$

where the action on the right is the usual action on ${}^1G(K)$. Note that ρ is a k-homomorphism $G \to GL(V)$.

Proposition 2.2. *The group G constructed above is simply connected quasi-simple of index ${}^2E_{6,1}^{29}$. It is k-isomorphic to the given group G_0.*

Proof. The group G is simply connected quasi-simple of type E_6 because it is so over K. It is of type ${}^2E_6$ because it is obtained by twisting the group ${}^1G$ of inner type by the outer automorphism $\dagger$.

Since C is not K-split, G has index ${}^1E_{6,2}^{28}$ over K. Every circled vertex in the k-index of G must also be circled in the K-index, so G is anisotropic or is of index ${}^2E_{6,1}^{29}$. Although it is easy to show that G is isotropic, we must do some more work in order to identify the semisimple anisotropic kernel of G.

Let Rel be the group of related triples of proper similarities of C as defined in [Gar01, § 7]; it is a reductive group of type ${}^1D_4$ with a two-dimensional center. A k-point of Rel is a triple (t_1, t_2, t_3), where t_i is a proper similarity of the norm on C and the t_i satisfy the identities

$$\mu(t_i)^{-1} t_i(\pi(x)\pi(y)) = \pi(t_{i+2}(x))\pi(t_{i+1}(y)) \quad (x, y \in C)$$

for $i = 1, 2, 3$, where $\mu(t_i) \in k^\times$ satisfies $C(t_i c_i) = \mu(t_i) C(c_i)$ for all $c_i \in C$. There is an injection $\psi\colon \text{Rel} \to {}^1G$ defined by

$$\psi_{(t_1,t_2,t_3)} \begin{pmatrix} \varepsilon_1 & c_3 & \cdot \\ \cdot & \varepsilon_2 & c_1 \\ c_2 & \cdot & \varepsilon_3 \end{pmatrix} = \begin{pmatrix} \mu(t_1)^{-1}\varepsilon_1 & t_3 c_3 & \cdot \\ \cdot & \mu(t_2)^{-1}\varepsilon_2 & t_1 c_1 \\ t_2 c_2 & \cdot & \mu(t_3)^{-1}\varepsilon_3 \end{pmatrix}. \tag{2.3}$$

We identify Rel with its image in ${}^1G$. We remark that

$$\psi^\dagger_{(t_1,t_2,t_3)} = \psi_{(\mu(t_1)^{-1}t_1, \mu(t_2)^{-1}t_2, \mu(t_3)^{-1}t_3)}.$$

The center of Rel has k-points the triples $(\lambda_1, \lambda_2, \lambda_3)$ of elements of $k^\times$ such that $\lambda_1\lambda_2\lambda_3 = 1$. The automorphism $\dagger$ acts on the center by sending such a triple to $(\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1})$.

The image of $s\colon \mathbb{G}_m \to {}^1G$ defined by

$$s(\lambda) = \psi_{(1,\lambda,\lambda^{-1})}$$

is a rank 1 torus S in the center of Rel. We claim that Rel is the centralizer in G of S. To see this, consider an element $g \in {}^1G$ that centralizes S. Write e_i for the element of A whose only nonzero entry is a 1 in the (i, i) place. The weight spaces of S in A (e.g., ke_2 and ke_3) are invariant under g. Since $s(\lambda)^\dagger = s(\lambda^{-1})$, the element $g^\dagger$ also commutes with S, hence

$$g(e_3 \times A) = (g^\dagger e_3) \times A = e_3 \times A,$$

where $\times$ denotes the Freudenthal cross product as in [KMRT98, p. 519] or [SV00, p. 122]. The space $e_3 \times A$ is the direct sum of the S-weight spaces

$$ke_1, \quad ke_2, \quad \text{and} \quad \begin{pmatrix} 0 & C & \cdot \\ \cdot & 0 & 0 \\ 0 & \cdot & 0 \end{pmatrix},$$

with weights 0, -2, and -1. Therefore, g leaves the subspace ke_i invariant for all i, hence g is in Rel by [All67, p. 254, Corollary].

The map ψ defines a map from a twisted form H of Rel into G, where the twisted ι-action on H sends $(t_1, t_2, t_3) \in H(K)$ to a triple

$$(\pi\iota(\mu(t_1)^{-1}t_1)\pi, \pi\iota(\mu(t_3)^{-1}t_3)\pi, \pi\iota(\mu(t_2)^{-1}t_2)\pi). \tag{2.4}$$

Since

$$\iota * s(\lambda) = \tau s(\iota(\lambda)^{-1})\tau = s(\iota(\lambda)),$$

S is a rank 1 k-split torus in G. By the preceding paragraph, H is the centralizer in G of S.

Finally, we claim that the semisimple anisotropic kernel of G is isomorphic to the group $\mathrm{Spin}(C^K)$. To see this, note that ψ restricts to an inclusion $\mathrm{Spin}(C) \hookrightarrow {}^1G$, where $\mathrm{Spin}(C)$ consists of the triples (t_1, t_2, t_3) such that $\mu(t_i) = 1$ for all i. This gives rise to an inclusion of a twisted form T of $\mathrm{Spin}(C)$ in G, where ι acts on a related triple (t_1, t_2, t_3) in $T(K)$ via

$$\iota * (t_1, t_2, t_3) = (\pi\iota(t_1)\pi, \pi\iota(t_3)\pi, \pi\iota(t_2)\pi).$$

That is, T is the spin group of the quadratic form obtained by restricting the norm on $C \otimes K$ to the elements fixed by the map $v \mapsto \pi\iota v$, which is C^K by [KMRT98, 36.21(1)].

Since $\mathrm{Spin}(C^K)$ is also the semisimple anisotropic kernel of G_0, the last sentence of the proposition follows from Tits's Witt-type theorem [Spr98, 16.4.2]. □

Remark 2.5. Our explicit construction of a group of index ${}^2E_{6,1}^{29}$ is different from that in Veldkamp's paper. The groups arising as in his 3.3(3) are indeed of index ${}^2E_{6,1}^{29}$ by [GP05, 9.6], and all groups of index ${}^2E_{6,1}^{29}$ are obtained as in his 3.3(3) by [GP05, 11.1].

3. The subgroup D

Write e for $(e_1, e_1) \in V$, and let D be the subgroup of G that fixes the vector $e \in V$. Over K, it is isomorphic to $\mathrm{Spin}(\langle 1, -1\rangle \oplus C)$ by [Spr62, Proposition 4].

LEMMA 3.1. *The subgroup D is k-isomorphic to the spin group of a* 10*-dimensional quadratic form of Witt index* 1.

Proof. From the description of D over K, we can conclude that D is, as a k-group, quasi-simple simply connected of type D_5 and of k-rank at most 1. The k-rank of D is exactly 1 because D contains the rank 1 k-split torus S from the proof of Proposition 2.2.

To complete the proof, it suffices to show that the vector representation of D is k-defined. Suppose not. Then D is k-isomorphic to the spin group of a five-dimensional, isotropic skew-hermitian form over a quaternion division k-algebra that is split by K. It follows that the K-rank of D is at least 2, which is a contradiction. □

Remark 3.2. We can realize D concretely in the following way. (We omit details as this will not be used below.) The 10-dimensional subspace

$$e_1 \times A_K = \begin{pmatrix} 0 & 0 & \cdot \\ \cdot & K & C \otimes K \\ 0 & \cdot & K \end{pmatrix}$$

of A_K is D-invariant and the equation $a \times a = q(a)e_1$ defines a quadratic form q given by

$$q\begin{pmatrix} 0 & 0 & \cdot \\ \cdot & \varepsilon_2 & x \\ 0 & \cdot & \varepsilon_3 \end{pmatrix} = \varepsilon_2\varepsilon_3 - N_{C/k}(x).$$

The action of D on $e_1 \times A_K$ gives a K-homomorphism $D \to \mathrm{SO}(q)$. A descent computation using (2.1) shows that D is k-isomorphic to the spin group of a k-form of q, namely $\langle 1, -\alpha\rangle \oplus \langle -1\rangle C^K$. That is, D is isomorphic to $\mathrm{Spin}(\langle 1, -1\rangle \oplus C)$ also over the base field k.

A classical result from geometric algebra [Die71, § II.9(C)] implies that $D(k)^+$ is all of $D(k)$, see [PR94, pp. 409–410]. The proof of (1.2) is completed by the following lemma, pointed out to me by Prasad.

LEMMA 3.3. *Let G' and G be isotropic reductive k-groups such that G' is a subgroup of G. Then $G'(k)^+$ is contained in $G(k)^+$.*

We remark that the lemma is obvious when k is perfect. In that case, $G(k)^+$ is the subgroup of $G(k)$ generated by the unipotent elements, and the lemma follows by the definition of $G'(k)^+$.

Proof. Let u be an element of $G'(k)$ contained in the unipotent radical of a parabolic subgroup. Then there is a k-homomorphism $s\colon \mathbb{G}_m \to G'$ such that u lies in the k-subgroup U' of G' that is, over an algebraic closure of k, directly spanned by the one-dimensional root subgroups U_α as α varies over the roots of G' such that $\langle \alpha, s\rangle$ is positive, cf. [Bor91, § 21] or [Spr98, 15.1.2]. (The group U' is even the radical of the parabolic subgroup $Z_{G'}(\operatorname{im} s)\cdot U'$ of G'.)

The homomorphism s defines a rank 1 k-split torus in G, and we define a subgroup U of G analogously to U' above. Note that u belongs to the unipotent radical U of the parabolic subgroup $Z_G(\operatorname{im} s)\cdot U$ of G, hence u is in $G(k)^+$.

Since elements such as u generate $G'(k)^+$, the lemma follows. □

4. Multipliers

Consider the k-subspace $\{(\mu e_1, \iota(\mu)e_1) \mid \mu \in K\}$ of V. It is an H-invariant subspace of V, and for $h \in H$, we define $\gamma(h) \in K^\times$ by

$$\rho(h)e = (\gamma(h)e_1, \iota(\gamma(h))e_1).$$

We have

$$1 = \operatorname{tr}(e_1, e_1) = \operatorname{tr}(\rho(h)e) = \gamma(h)\iota(\gamma(h)),$$

so γ defines a k-homomorphism from H to the rank 1 torus T whose k-points are the norm 1 elements of $K^\times$. The purpose of this section is to prove the following.

Lemma 4.1. *The image of $\gamma\colon H(k) \to T(k)$ consists of elements $\lambda\iota(\lambda)^{-1}$ for $\lambda \in K^\times$ such that $\lambda\iota(\lambda) \in k^\times$ is a norm from C.*

Proof. From the explicit description of H as a twist of Rel, we find that the kernel of γ is generated by $\operatorname{Spin}(C^K)$ and S. Projection on the first entry defines a surjection $\ker\gamma \to \operatorname{SO}(C^K)$ whose restriction to $\operatorname{Spin}(C^K)$ is the vector representation. We compute the image of $\lambda\iota(\lambda)^{-1} \in T(k)$ under the composition

$$T(k) \longrightarrow H^1(k, \ker\gamma) \longrightarrow H^1(k, \operatorname{SO}(C^K)), \tag{4.2}$$

where the first map is induced by the short exact sequence

$$1 \longrightarrow \ker\gamma \longrightarrow H \xrightarrow{\gamma} T \longrightarrow 1.$$

Equation (2.3) shows that the element $\lambda\iota(\lambda)^{-1}$, viewed as a point of T over the separable closure k_{sep} of k, is the image of

$$t := (\sqrt{N_\lambda}\lambda^{-1}, \sqrt{\lambda}, \sqrt{\lambda N_\lambda^{-1}}) \in H(k_{\text{sep}})$$

for some fixed square roots of λ and $N_\lambda := \lambda\iota(\lambda)$ in k_{sep}. For $\sigma \in \operatorname{Gal}(k_{\text{sep}}/k)$, we claim that the first entry of $t^{-1}(\sigma * t)$ is $\sqrt{N_\lambda}^{-1}\sigma(\sqrt{N_\lambda})$, where $\sigma*$ denotes the action on k_{sep} twisted as in (2.1) and the σ in the latter expression acts in the usual manner on k_{sep}. If σ is the identity on K, then the two actions agree, σ fixes λ, and the claim is obvious. If σ is not the identity on K, then

$$t^{-1}(\sigma * t) = \frac{\lambda}{\sqrt{N_\lambda}}\frac{\sigma(\lambda)}{\sigma(\sqrt{N_\lambda})} = \frac{\sqrt{N_\lambda}}{\sigma(\sqrt{N_\lambda})}.$$

As $\sqrt{N_\lambda}\,\sigma(\sqrt{N_\lambda})^{-1}$ equals ± 1, the claim is proved. It follows that the image of $\lambda\iota(\lambda)^{-1}$ under the composition (4.2) is the image of N_λ under the map

$$k^\times/k^{\times 2} \cong H^1(k, Z(\operatorname{SO}(C^K))) \to H^1(k, \operatorname{SO}(C^K)).$$

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If $\lambda\iota(\lambda)^{-1}$ is in the image of $H(k)$, it has trivial image in $H^1(k, \ker\gamma)$ and consequently in $H^1(k, \mathrm{SO}(C^K))$. It follows that N_λ is a similarity factor of C^K. Combining (0.2) and the following proposition shows that N_λ is represented by the norm of C. □

PROPOSITION 4.3. *Let q be a quadratic form that is Witt-equivalent to $\sum_{i=1}^n u_i\phi_i$, where $u_1, u_2, \dots, u_n$ are odd-dimensional quadratic forms and $\phi_1, \phi_2, \dots, \phi_n$ are Pfister forms of different dimensions. An element $\mu \in k^\times$ is a similarity factor of q if and only if μ is represented by ϕ_i for every i.*

We allow the possibility that one of the ϕ_i is the '0-fold' Pfister form $\langle 1\rangle$. In that case, q is odd-dimensional and the proposition is the standard fact that the group of similarity factors of an odd-dimensional quadratic form is the group of squares in $k^\times$.

Proof. As q and $\sum u_i\phi_i$ are Witt-equivalent, they have the same similarity factors, so we may assume that q actually is $\sum u_i\phi_i$. The case where q is a Pfister form (i.e. $n = 1$ and $u_1 = \langle 1\rangle$) is a standard result, see [Lam05, X.1.8]; we call this the *base case*.

We now prove the general case. The 'if' direction follows directly from the base case. To prove the 'only if' direction, we assume that μ is a similarity factor of q, i.e. $\langle\mu\rangle q$ is isomorphic to q. In the Witt ring, $\sum u_i \cdot \langle 1, -\mu\rangle\phi_i$ is zero. Equivalently, we have

$$\sum_{i=1}^n u_i \cdot \langle 1, -\mu\rangle\phi_i = \sum_{i=1}^n u_i \cdot h_i,$$

where h_i is a hyperbolic form of the same dimension as $\langle 1, -\mu\rangle\phi_i$. As h_i and $\langle 1, -\mu\rangle\phi_i$ are Pfister forms and the u_i are odd-dimensional, [Ser03, Lemma 22.2] gives that $\langle 1, -\mu\rangle\phi_i$ and h_i are isomorphic for all i. Consequently, μ is represented by ϕ_i by the base case. □

Quadratic forms as in the proposition are common: for example, when k is a global field (or, more generally, a linked field), every quadratic form can be written as in the proposition by [Lam05, X.6.27].

5. Conclusion of the proof of the theorem

This section contains a proof of (1.3), i.e. we prove the following lemma.

LEMMA 5.1. *We have $H(k) \subseteq D(k) \cdot G(k)^+$.*

Proof. Fix $h \in H(k)$. We will produce an element g of $G(k)^+$ such that $\rho(g)e = (\gamma(h)e_1, \iota(\gamma(h))e_1)$. Then $g^{-1}h$ will belong to $D(k)$ and the lemma will follow.

By Lemma 4.1, $\gamma(h)$ is of the form $\lambda\iota(\lambda)^{-1}$ for some $\lambda \in K^\times$ such that $\lambda\iota(\lambda)$ is a norm from C. Fix a quadratic subfield ℓ of C such that $\lambda\iota(\lambda)$ is a norm from ℓ. Since C is not split by K, the tensor product $K \otimes \ell$ is a biquadratic extension field of k which we denote simply by $K\ell$. As described in [Jac61, § 5], there is an injective k-homomorphism

$$\phi\colon R_{\ell/k}(\mathrm{SL}_3) \to {}^1G \quad \text{via } \phi_g(a) = ga\pi(g)^{\mathrm{t}},$$

for $g \in \mathrm{SL}_3(\ell)$ and $a \in A$, where juxtaposition denotes usual matrix multiplication, t denotes the transpose, and π means to apply the nontrivial ℓ/k-automorphism to the entries of g. Moreover,

$$\phi_g^\dagger = \phi_{\pi(g)^{-t}} \quad \text{and} \quad \tau\phi_g\tau^{-1} = \phi_{(2\,3)g(2\,3)},$$

where $(2\quad 3)$ denotes the matrix $\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}\right)$. (The first equation is from [Jac61, p. 77]. The second equation is verified in the same manner, i.e. by checking it for elementary matrices g.)

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Let SU denote the group $R_{\ell/k}(\mathrm{SL}_3)$ with the twisted $\mathrm{Gal}(K/k)$-action given by

$$\iota * g = (2\,3)\iota\pi(g)^{-t}(2\,3)$$

for $g \in \mathrm{SL}_3(K\ell)$, i.e. a K-point of $R_{\ell/k}(\mathrm{SL}_3)$. By the preceding paragraph and (2.1), ϕ is a k-injection $\mathrm{SU} \to G$.

Write M for the subfield of $K\ell$ fixed by $\iota\pi$. The k-points of SU are elements of the special unitary group of the three-dimensional $K\ell/M$-hermitian form h such that

$$h(x, y) = x_1\pi\iota(y_1) + x_2\pi\iota(y_3) + x_3\pi\iota(y_2), \tag{5.2}$$

cf. [KMRT98, pp. 23ff and 42ff]. That is, SU is the transfer (the Weil restriction) from M to k of a group of outer type A_2. As the hermitian form (5.2) is isotropic, SU is quasi-split.

As $\lambda\iota(\lambda) \in k^\times$ is a norm from ℓ and K, the biquadratic lemma (see, e.g., [Wad86, 2.14]) gives an element $\gamma \in K\ell$ such that $\gamma\pi(\gamma) = \alpha\lambda$ for some $\alpha \in k^\times$. Consider the element

$$g := \begin{pmatrix} \gamma\iota\pi(\gamma)^{-1} & & \\ & \gamma^{-1} & \\ & & \iota\pi(\gamma) \end{pmatrix} \in \mathrm{SL}_3(K\ell).$$

Note that g is in $\mathrm{SU}(k)$. On the other hand,

$$\phi_g(e_1) = \gamma\iota\pi(\gamma)^{-1}\pi(\gamma)\iota(\gamma)^{-1}e_1 = (\alpha\lambda)\iota(\alpha\lambda)^{-1}e = \lambda\iota(\lambda)^{-1}e_1.$$

Since SU is k-quasi-split, $\mathrm{SU}(k)^+$ is all of $\mathrm{SU}(k)$ (see [Ste59, § 8]). By Lemma 3.3, ϕ_g is in $G(k)^+$. This proves Lemma 5.1, which in turn completes the proof of the theorem. □

6. Proof of Proposition 0.1

This section consists of a proof of Proposition 0.1, i.e. we prove that every group G of index ${}^2E_{6,1}^{29}$ is stably rational as a variety. We assume throughout this section that the characteristic of k_0 is not equal to 2, and we explicitly allow characteristic 3.

The crux of proving Proposition 0.1 is the following proposition.

Proposition 6.1. *Let q be a quadratic form that is Witt-equivalent to $u_1\phi_1 + u_2\phi_2$ where u_1 and u_2 are odd-dimensional quadratic forms and ϕ_1 and ϕ_2 are Pfister forms of different dimensions. Then the variety* $\mathrm{PSO}(q)$ *is stably rational.*

Merkurjev handled the case where q is of the form $u_1\phi_1$ in [Mer96, p. 204, Proposition 7]. The proof of Proposition 6.1 is a small extension of his arguments.

Proof. Write V_i for the vector space underlying ϕ_i. Let Y be the rational quadric in $V_1 \oplus V_2$ defined by the form $\phi_1 - \phi_2$, and let X be the open subvariety of Y consisting of vectors $v_1 + v_2$ such that $\phi_1(v_1)$ is not zero.

Define $\psi\colon X \to \mathbb{G}_m$ via $\phi(v_1 + v_2) = \phi_1(v_1)$. For every extension E of k, the image of ψ consists of those elements of $E^\times$ represented by both ϕ_1 and ϕ_2, which by Proposition 4.3 are the similarity factors of q. Moreover, the fiber over an $x \in E^\times$ in the image of α is the product of the rational varieties defined by the equations $\phi_1 = x$ and $\phi_2 = x$. By [Mer96, p. 198, Corollary 1], it follows that $\mathrm{PSO}(q)$ is stably rational. □

With Proposition 6.1 in hand, the proof of Proposition 0.1 follows by standard arguments as in [CP98, p. 5] or [Tha99].

Proof of Proposition 0.1. Fix a k-group G of index ${}^2E_{6,1}^{29}$. Write M for the centralizer of a maximal k-split torus in G. The generalized Bruhat decomposition implies that G is birationally equivalent

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to $U \times M \times U$ where U is the unipotent radical of a minimal parabolic k-subgroup of G. As U is k-rational, it suffices to prove that M is stably rational.

Set S' to be the connected center of M. If G is simply connected, then M is the group H from § 2, hence S' is isomorphic to $R_{K/k}(\mathbb{G}_m)$ and $H^1(E, S')$ is zero for every extension E/k. If G is adjoint, then S' is quasi-trivial (see [Tit88, p. 89, Lemme] or [Spr98, p. 279]) and again we find that $H^1(E, S')$ is zero for every extension E/k. It follows that M is birationally equivalent to $S' \times M/S'$. The first term, S', is a rank 2 torus, so it is rational [Vos98, § 4.9]. The second term, M/S', is isomorphic to $\mathrm{PSO}(C^K)$ for C and K as in § 2. Combining (0.2) and Proposition 6.1 gives that M/S' is stably rational, hence G is stably rational. □

It is important that the eight-dimensional quadratic form C^K is of the special type to which Proposition 6.1 applies: Merkurjev [Mer96, p. 212] and Gille [Gil97] give explicit eight-dimensional quadratic forms q such that $\mathrm{PSO}(q)$ is *not* stably rational.

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