A REMARK ABOUT A CERTAIN CLASS OF DISTRIBUTION SPACES

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1. Introduction

The object of this note is to exhibit a certain class of distribution spaces as being c-admissible in the sense of [1] and [2]. Throughout the terminology and notation are the same as in [1] and [2]. The only addition to this is that if $x \in R^n$ then τ_x will denote the translation operator which carries each distribution u onto the distribution u_x.

2. c-admissibility of certain spaces

We shall prove the following result.

PROPOSITION. *Let* $\boldsymbol{E}$ *be an admissible space which is barelled and* B_r*-complete, and which is a module over* $\boldsymbol{S}$ *with respect to convolution.*

Consider the following two hypotheses:

(i) $\boldsymbol{E}$ *is translation-invariant and for each* $u \in \boldsymbol{E}$ *the mapping* $x \to u_x$ *of* R^n *into* $\boldsymbol{E}$ *is bounded on compact subsets of* R^n.

(ii) $\boldsymbol{E}$ *is dilation-invariant and for each* $u \in \boldsymbol{E}$ *the mapping* $x \to u^x$ *of* $R^{\#}$ *into* $\boldsymbol{E}$ *is bounded on compact subsets of* $R^{\#}$

Then the conclusions are:

(a) *If* (i) *holds then* $\boldsymbol{E}$ *is c-admissible.*

(b) *If both* (i) *and* (ii) *hold then* $\boldsymbol{E}$ *is a dilation space.*

PROOF. We shall begin with the proof of assertion (a). Thus we assume that (i) holds.

Our first task is to show that for each $x \in R^n$ the mapping $u \to u_x$ of $\boldsymbol{E}$ into itself is continuous. Since $\boldsymbol{E}$ is both barrelled and B_r-complete, the closed graph theorem (Theorem 8.9.4 in Edwards [3] and the first Remark following it) tells us that it is sufficient to show that the linear operator τ_x (considered as a mapping of $\boldsymbol{E}$ into itself) has a closed graph. To this end

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we assume that $x \in R^n$ is fixed, and that (u_i) is a net in E such that $\lim_i u_i = u$ in $\boldsymbol{E}$ and $\lim \tau_x u_i = w$ in $\boldsymbol{E}$. Then for each $\phi \in \boldsymbol{S}$ it follows (because of relation (2.1) in [1]) that

$$\begin{aligned} w * \phi(0) &= \lim_i\, (\tau_x u_i) * \phi(0) \\ &= \lim_i u_i * \phi_x(0) \\ &= u * \phi_x(0) \\ &= (\tau_x u) * \phi(0). \end{aligned}$$

Thus $w = \tau_x u$ and the graph of τ_x is indeed closed.

Our second requirement is to show that for each $u \in \boldsymbol{E}$ and each $v \in \boldsymbol{E}'$ the mapping $x \to \langle u_x, v\rangle$ defines a continuous function on R^n. With this end in mind, let $b \in R^n$ be arbitrary but fixed. Let K be the set $\{x \in R^n : |x-b| \leqq 1\}$ and consider the set $\{\tau_x : x \in K\}$ of continuous linear mappings of $\boldsymbol{E}$ into itself. Write $\boldsymbol{E}_0$ for the set of all $u \in \boldsymbol{E}$ for which $\lim_{x\to b} \tau_x u$ exists in $\boldsymbol{E}$ and define the mapping T of $\boldsymbol{E}_0$ into $\boldsymbol{E}$ by $Tu = \lim_{x\to b} \tau_x u (u \in \boldsymbol{E}_0)$. We notice that for each fixed $\phi \in \boldsymbol{S}$, the mapping $x \in R^n \to \phi_x \in \boldsymbol{S} \to \phi_x \in \boldsymbol{E}$ is continuous; and hence that

$$\lim_{x\to b} \tau_x \phi = \tau_b \phi \text{ in } E. \tag{2.1}$$

It follows from this that $\boldsymbol{E}_0$ contains $\boldsymbol{S}$, which is dense in $\boldsymbol{E}$. Secondly, our assumption that (i) holds entails that the set $\{\tau_x : x \in K\}$ of continuous linear mappings is bounded at each point of $\boldsymbol{E}$. Thus, since $\boldsymbol{E}$ is B_r-complete and hence quasi-complete, we may refer to Corollary 7.1.4 in Edwards [3] and deduce that $\boldsymbol{E}_0 = \boldsymbol{E}$ and that T is a continuous linear mapping of $\boldsymbol{E}$ into itself. But relation (2.1) shows that T coincides with the continuous linear mapping τ_b on the dense vector subspace $\boldsymbol{S}$ of $\boldsymbol{E}$; whence it follows that the two mappings are identical. Thus $\lim_{x\to b} \tau_x u = \tau_b u$ for each $u \in \boldsymbol{E}$. Since $b \in R^n$ is arbitrary, we now infer that for each $u \in \boldsymbol{E}$, the mapping $x \to u_x$ is continuous from R^n into $\boldsymbol{E}$. It follows immediately that for each $u \in \boldsymbol{E}$ and each $v \in \boldsymbol{E}'$, the mapping $x \to \langle u_x, v\rangle$ defines a continuous function on R^n, which is what we wished to prove.

Next consider a fixed $\phi \in \boldsymbol{S}$. We claim that the mapping $u \to u * \phi$ of $\boldsymbol{E}$ into itself is continuous. To verify this assertion, it is sufficient to show that the graph of this mapping is closed; the desired conclusion will then follow from the closed graph theorem. Thus let (u_i) be a net in $\boldsymbol{E}$ such that $\lim_i u_i = w$ in $\boldsymbol{E}$ and $\lim_i u_i * \phi = w$ in E. Then for each $\psi \in \boldsymbol{S}$ we have $w * \psi(0) = \lim_i u_i * \phi * \psi(0) = u * \phi * \psi(0)$. Hence $w = u * \phi$ and the mapping $u \in \boldsymbol{E} \to u * \phi \in \boldsymbol{E}$ is closed, as required.

If we recall that $\boldsymbol{S}$ is barrelled, then a similar argument shows that for each $u \in \boldsymbol{E}$, the mapping $\phi \to u * \phi$ of $\boldsymbol{S}$ into $\boldsymbol{E}$ is continuous.

We shall now complete the proof of part (a) of the Proposition. Let $u \in \boldsymbol{E}$ and $v \in \boldsymbol{E}'$ be arbitrary but fixed. We must show that the continuous function $x \to \langle u_x, v\rangle$ $(x \in R^n)$ generates a temperate distribution on R^n. In view of the last paragraph, the mapping $\phi \to \langle u * \phi, v\rangle$ $(\phi \in \boldsymbol{S})$ defines a temperate distribution, which we denote by s. We shall show that the function $x \to \langle u_x, v\rangle$ $(x \in R^n)$ generates precisely this distribution s. To do this it is sufficient to show that for each $\psi \in \boldsymbol{D}$

$$\int_{R^n} \langle u_x, v\rangle \psi(-x)\, dx = s * \psi(0). \tag{2.2}$$

Let $\psi \in \boldsymbol{D}$ be arbitrary. Choose a net (ϕ_i) in $\boldsymbol{S}$ such that $\lim_i \phi_i = u$ in $\boldsymbol{E}$. We notice that, because of Theorem 2.2(a) in [1] and the continuity of the functions $x \to \langle u_x, v\rangle$ on R^n, the mapping $x \to v_x$ is continuous from R^n into $\boldsymbol{E}'$ for the weak topology on $\boldsymbol{E}'$. Therefore the set $\{v_x : x \in \operatorname{supp} \psi\}$ is a weakly compact, hence weakly bounded, hence equicontinuous (because $\boldsymbol{E}$ is barrelled) subset of $\boldsymbol{E}'$. In view of this we conclude that

$$\lim_i \phi_i * v(x) = \lim_i \langle \phi_i, v_x\rangle = \langle u, v_x\rangle = \langle u_x, v\rangle$$

uniformly for $x \in \operatorname{supp} \psi$. It follows that

$$\begin{aligned} \int_{R^n} \langle u_x, v\rangle \psi(-x) dx &= \lim_i \int_{R^n} \phi_i * v(x)\psi(-x) dx \\ &= \lim_i \phi_i * \psi * v(0) \\ &= \lim_i \langle \phi_i * \psi, v\rangle. \end{aligned} \tag{2.3}$$

Now we have shown above that the mapping $w \in \boldsymbol{E} \to w * \psi \in \boldsymbol{E}$ is continuous. Therefore

$$\lim_i \langle \phi_i * \psi, v\rangle = \langle u * \psi, v\rangle = s * \psi(0). \tag{2.4}$$

Relations (2.3) and (2.4) together ensure that (2.2) holds; whence we infer that the function $x \to \langle u_x, v\rangle$ can indeed be identified with a temperate distribution on R^n. Since $u \in \boldsymbol{E}$ and $v \in \boldsymbol{E}'$ were arbitrary, this completes the proof of (a).

The validity of part (b) of the Proposition will be established if we can show that (ii) entails that the mapping $x \to u^x$ of $R^{\#}$ into $\boldsymbol{E}$ and the mapping $u \to u^x$ of $\boldsymbol{E}$ into itself are both continuous (for the given topology on $\boldsymbol{E}$); and the truth of this may be verified by using arguments analogous to those which we employed above to establish the continuity of the mappings $x \to u_x$ of R^n into $\boldsymbol{E}$ and $u \to u_x$ of $\boldsymbol{E}$ into itself.

References

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[2] S. R. Harasymiv, 'A note on approximation of distributions by quasi-analytic functions', *J. Aust. Math. Soc.* 10 (1969), 95—105.
[3] R. E. Edwards, *Functional Analysis: Theory and Applications* (Holt, Rinehart and Winston, New York, 1965).

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