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GENERALIZED SPECTRUM AND COMMUTING COMPACT PERTURBATIONS

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Let X be an infinite-dimensional complex Banach space and denote the set of bounded (compact) linear operators on X by $B(X)$ $(K(X))$. Let $N(A)$ and $R(A)$ denote, respectively, the null space and the range space of an element A of $B(X)$. Set $R(A^\infty)=\bigcap_n R(A^n)$ and $k(A)=\dim N(A)/(N(A)\cap R(A^\infty))$. Let $\sigma_g(A)=\mathbb{C}\backslash\{\lambda\in\mathbb{C}: R(A-\lambda)$ is closed and $k(A-\lambda)=0\}$ denote the generalized (regular) spectrum of A. In this paper we study the subset $\sigma_{gb}(A)$ of $\sigma_g(A)$ defined by $\sigma_{gb}(A)=\mathbb{C}\backslash\{\lambda\in\mathbb{C}: R(A-\lambda)$ is closed and $k(A-\lambda)<\infty\}$. Among other things, we prove that if f is a function analytic in a neighborhood of $\sigma(A)$, then $\sigma_{gb}(f(A))=f(\sigma_{gb}(A))$.

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1. Introduction and preliminaries

Let X be an infinite-dimensional complex Banach space and denote, respectively, the set of bounded, compact and finite dimensional operators on X by $B(X)$, $K(X)$ and $F(X)$. For A in $B(X)$ throughout this paper $N(A)$ and $R(A)$ will denote, respectively, the null space and the range space of A. Set $N(A^\infty)=\bigcup_n N(A^n)$, $R(A^\infty)=\bigcap_n R(A^n)$, $\alpha(A)=\dim N(A)$, $\beta(A)=\dim X/R(A)$ and $k(A)=\dim N(A)/(N(A)\cap R(A^\infty))$. Recall that an operator $A\in B(X)$ is semi-Fredholm if $R(A)$ is closed and at least one if $\alpha(A)$ and $\beta(A)$ is finite. For such an operator we define an index $i(A)$ by $i(A)=\alpha(A)-\beta(A)$. Let $\Phi_+(X)$ $(\Phi_-(X))$ denote the set of semi-Fredholm operators with $\alpha(A)<\infty(\beta(A)<\infty)$ and $\sigma_{ek}(A)$ Kato's essential spectrum of A, i.e., $\sigma_{ek}(A)=\{\lambda\in\mathbb{C}: A-\lambda\notin\Phi_+(X)\cup\Phi_-(X)\}$. Furthermore, let $\sigma(A)$, $\sigma_a(A)$ and $\sigma_{ab}(A)=\bigcap\{\sigma_a(A+K): K\in K(X)$ and $AK=KA\}$ denote, respectively, the spectrum, the approximate point spectrum and Browder's essential approximate point spectrum of A ([**17**]).

Set $V(X)=\{A\in B(X): R(A)$ is closed and $k(A)<\infty\}$ and $V_n(X)=\{A\in V(X): k(A)=n\}$, $n=0,1,2,\ldots$. Let us remark that $k(A)=n<\infty$ precisely when A has Kaashoek's property $P(I,n)$ ([**6**, pp. 452–453]), or when A has almost uniform descent ([**5**, Definition 1.3]). In particular $k(A)=0$ if and only if Kato's number $v(A:I)=\infty$ ([**9**, pp. 289–290]), i.e., if and only if $N(A^\infty)\subset R(A^\infty)$. Recall that $\Phi_+(X)\cup\Phi_-(X)\subset V(X)$ ([**5**, Theorem 3.7], [**10**, p. 197, Example 4]). Let $\sigma_g(A)=\{\lambda\in\mathbb{C}: A-\lambda\notin V_0(X)\}$ denote the generalized (regular) spectrum of A ([**1, 10, 13**]). $\sigma_g(A)$ is a non-empty compact subset of the set of complex numbers $\mathbb{C}$.

In this paper we study the subset $\sigma_{gb}(A)$ of $\sigma_g(A)$ defined by

$$\sigma_{gb}(A)=\{\lambda\in\mathbb{C}: A-\lambda\notin V(X)\}.$$

The relation between $\sigma_g(A)$ and $\sigma_{gb}(A)$ that is exhibited in this paper resembles the relation between the $\sigma_a(A)$ and the $\sigma_{ab}(A)$, and it is reasonable to call $\sigma_{gb}(A)$ Browder's essential generalized spectrum of A.

First in Section 2 we prove a Kato-type decomposition theorem for operators in $V(X)$ which is related to Kato's theorem for semi-Fredholm operators ([**9**, Theorem 4], [**19**, Proposition 2.5]).

In Section 3 we characterize $\sigma_{gb}(A)$ (Theorem 3.1) and derive several corollaries.

In Section 4 we prove that if f is a function analytic in a neighborhood of $\sigma(A)$, then $\sigma_{gb}(f(A))=f(\sigma_{gb}(A))$.

Finally, in Section 5 we investigate connected components of the set $\mathbb{C}\backslash\sigma_{gb}(A)$.

2. A Kato-type decomposition theorem

Theorem 2.1. *Let $A\in B(X)$ be an operator with closed range. Then, $k(A)$ is finite if and only if the space X decomposes into the direct sum of two closed subspaces X_0 and X_1 which are A-invariant and have the following properties:*

(i) *if A_0 is the restriction of A to X_0 considered as an operator from X_0 to itself, then $N(A_0)\subset R(A_0^\infty)$,*

(ii) *the space X_1 is finite-dimensional and A is nilpotent on it.*

Proof. Suppose that the operator A satisfies conditions (i) and (ii). If A_1 is the restriction of A to X_1 considered as an operator from X_1 to itself, then there is an integer n such that $A_1^n=0$. Also, we have $N(A)=N(A_0)\oplus N(A_1)$ and $R(A^\infty)=R(A_0^\infty)\subset X_0$. By [**5**, Lemma 2.1(a)] $N(A)\cap R(A^\infty)=[N(A_0)\oplus N(A_1)]\cap R(A^\infty)=N(A_0)\oplus[N(A_1)\cap R(A^\infty)]=N(A_0)$. Hence $\dim[N(A)/(N(A)\cap R(A^\infty))]=\dim N(A_1)$ is finite, and $k(A)<\infty$.

Conversely, suppose that $k(A)=p$ is finite. Then, by [**5**, Theorem 3.8] $R(A^n)$ is closed for each positive integer n, and there are p vectors $x_{k,1}, k=1,\dots,p$, in $N(A)$ which are linearly independent modulo the subspace $N(A)\cap R(A^\infty)$. Now, as in [**8**] and [**12**] there are p finite chains associated with $x_{k,1}, k=1,\dots,p$, i.e., there are vectors

$$x_{k,1},\dots,x_{k,r_k},\quad (k=1,\dots,p) \tag{1}$$

such that $Ax_{k,m}=x_{k,m-1}$ $(m=2,\dots,r_k; k=1,\dots,p)$ and $Ax_{k,1}=0$ $(k=1,\dots,p)$. By [**8**] the adjoint operator A^* of A has exactly p elements in $N(A^*)$, say $x^*_{k,1}, k=1,\dots,p$, with finite chains. Moreover, the chain associated with $x^*_{k,1}$ has the same number of elements as the corresponding chain associated with $x_{k,1}$ for each $k=1,\dots,p$. Thus, there are elements

$$x^*_{k,1},\dots,x^*_{k,r_k},\quad (k=1,\dots,p) \tag{2}$$

in the dual space X^* of X such that $A^*x^*_{k,m}=x^*_{k,m-1}$ $(m=2,\dots,r_k; k=1,\dots,p)$ and

$A^*x^*_{k,1}=0$ $(k=1,\dots,p)$. Again, by [**8**] we can choose functionals in (2) such that the vectors in (2) and (1) are biorthogonal; i.e., $x^*_{k,r_k-j+1}(x_{m,i})=1$, if $k=m$ and $j=i$, $x^*_{k,r_k-j+1}(x_{m,i})=0$ in the other cases. Let X_1 be the subspace in X spanned by vectors in (1) and $X_0=\bigcap\{N(x^*_{k,m}): k=1,\dots,p; m=1,\dots,r_k\}$. X_0 and X_1 are closed subspaces in X, and by [**14**, pp. 150–151] we have $X=X_0\oplus X_1$. It is easy to see that X_1 is a finite dimensional space which is A-invariant and that A is nilpotent on it. Further, by [**12**, Remark] the subspace X_0 is invariant. Next, $R(A^n)$ is closed for each positive integer n [**5**, Theorem 3.8], by the proof of [**12**, Theorem 5] we have $N(A_0)\subset R(A_0^\infty)$. This completes the proof.

Remark 2.2. Since $R(A)$ is closed subspace in X, and $R(A)=R(A_0)\oplus R(A_1)$ by [**9**, Lemma 3.32] $R(A_0)$ is a closed subspace in X.

Remark 2.3. By [**19**, Lemma 1.3 and Corollary 1.4] we have $k(A_0)=0$. Thus, by Remark 2.2, $A_0\in V_0(X_0)$.

3. Characterization of $\sigma_{gb}(A)$

Theorem 3.1. *Let $A\in B(X)$. Then*

$$\sigma_{gb}(A)=\bigcap\{\sigma_g(A+K): K\in K(X) \text{ and } AK=KA\}.$$

Proof. If $\lambda\notin\bigcap\{\sigma_g(A+K): K\in K(X)$ and $AK=KA\}$, there is a $K\in K(X)$ such that $AK=KA$ and $\lambda\notin\sigma_g(A+K)$. Thus, $R(A+K-\lambda)$ is closed and $k(A+K-\lambda)=0$. Adding the operator $-K$ to $A+K-\lambda$, we see that $R(A-\lambda)$ is closed and $k(A-\lambda)<\infty$ ([**5**, Theorem 5.9]). Hence $A-\lambda\in V(X)$. To prove the converse suppose that $A-\lambda\in V(X)$. If $k(A-\lambda)=0$, then $\lambda\notin\sigma_g(A)$ and the proof is complete. If $0<k(A-\lambda)$, then by Theorem 2.1 we conclude that the space X decomposes into a direct sum of two closed subspaces X_0 and X_1. These subspaces are $(A-\lambda)$-invariant, hence A-invariant, and have the following properties: The space X_1 is finite dimensional (and $A-\lambda$ is nilpotent on it). If A_0 is the restriction of A to X_0 considered as an operator from X_0 into itself then $k(A_0-\lambda)=0$. Let F be the finite rank operator defined by $F=I$ on X_1, $F=0$ on X_0. Hence, $AF=FA$ and $R(A+F-\lambda)$ is closed. Since $A-\lambda$ is nilpotent on X_1 we have $N(A+F-\lambda)=N(A_0-\lambda)\subset R((A_0-\lambda)^\infty)\subset R((A_0-\lambda)^\infty)\oplus X_1=R((A+F-\lambda)^\infty$. Thus, $k(A+F-\lambda)=0$, and $\lambda\notin\sigma_g(A+K)$. This completes the proof.

Corollary 3.2 $\bigcap\{\sigma_g(A+K): K\in F(X)$ *and* $AK=KA\}=\sigma_{gb}(A)$.

Proof. Inclusion '$\supset$' is obvious. Suppose that $\lambda\notin\sigma_{gb}(A)$. From the proof of Theorem 3.1, there exists a finite rank operator F in $B(X)$ such that $AF=FA$ and $\lambda\notin\sigma_g(A+F)$, which proves the inclusion '$\subset$'. This completes the proof.

Let us point out that Theorem 3.1 and its corollary can be proved without using Theorem 2.1, but instead by using Kaashoek's [**6**, Theorem 3.2].

Corollary 3.3. *$\lambda \in \sigma_g(A) \setminus \sigma_{gb}(A)$ if and only if λ is an isolated point of $\sigma_g(A)$, $0 < k(A-\lambda) < \infty$ and $R(A-\lambda)$ is closed.*

Proof. This follows from Theorem 3.1, [**5**, Theorem 4.7] and [**6**, Theorem 4.1].

The polynomial hull $\hat{E}$ of a compact subset E of the complex plane $\mathbb{C}$ is the complement of the unbounded component of $\mathbb{C} \setminus E$. Given a compact subset E of the plane, a hole of E is a component of $\hat{E} \setminus E$. If F is another compact set such that $\partial E \subset F \subset E$, it follows that $\partial E \subset \partial F$, $\hat{E} = \hat{F}$ and E can be obtained from F by filling in some holes of F. (Here and in what follows ∂E denotes the boundary of the set E.)

Corollary 3.4. *Let $A \in B(X)$. Then*

(i) $\sigma_{gb}(A) \subset \sigma_{ek}(A)$,

(ii) *$\partial \sigma_{ek}(A) \subset \partial \sigma_{gb}(A)$ and $\sigma_{gb}(A)$ is nonempty,*

(iii) $\hat{\sigma}_{gb} = \hat{\sigma}_{ek}(A)$,

(iv) *$\sigma_{ek}(A)$ can be obtained from $\sigma_{gb}(A)$ by filling in some holes of $\sigma_{gb}(A)$,*

(v) *if $\sigma_{gb}(A)$ is connected, $\sigma_{ek}(A)$ is connected.*

Proof. It is sufficient to prove (ii). It is well known that $\sigma_{ek}(A)$ is nonempty and compact. Suppose $\lambda_0 \in \partial \sigma_{ek}(A)$ and $\lambda_0 \notin \sigma_{gb}(A)$. Hence, $k(A-\lambda_0) < \infty$ and $R(A-\lambda_0)$ is closed. Now, we know that there exists an $\varepsilon > 0$ such that $0 < |\lambda_0 - \lambda| < \varepsilon$ implies that $R(A-\lambda)$ is closed and $\alpha(A-\lambda)$ and $\beta(A-\lambda)$ are constant, i.e., $\alpha(A-\lambda) = \alpha(A-\lambda_0) - k(A-\lambda_0)$ and $\beta(A-\lambda) = \beta(A-\lambda_0) - k(A-\lambda_0)$ ([**6**, Theorem 4.1]). Thus $A - \lambda_0 \in \Phi_+(X) \cup \Phi_-(X)$, which is a contradiction. This completes the proof.

Corollary 3.5. *Let A^* be the adjoint operator of $A \in B(X)$. Then $\sigma_{gb}(A) = \sigma_{gb}(A^*)$.*

Proof. This follows from Theorem 3.1, [**15**, Theorem 2] and [**5**, Theorem 3.7].

Recall that $a(A)$, the ascent of A, is the smallest non-negative integer n such that $N(A^n) = N(A^{n+1})$. If no such n exists, then $a(A) = \infty$. Let $A_{|M}$ denotes the restriction of A to the subspace M of X.

Corollary 3.6. *Let $A \in V(X)$. Then the following statements are equivalent:*

(i) *$A = V + F$, where $\alpha(V) = 0$, F is finite rank and $VF = FV$;*

(ii) *there exists a finite rank projection P commuting with A such that $\alpha(A_{|N(P)}) = 0$;*

(iii) *there exists $\varepsilon > 0$ such that $\alpha(A+\lambda) = 0$ for $0 < |\lambda| < \varepsilon$;*

(iv) $a(V) < \infty$.

Proof. If A satisfies any condition among (i)–(iv), then $A \in \Phi_+(X)$ and $i(A) \leqq 0$ ([**11**, Lemma 2.5], [**6**, Theorem 4.1]). Thus, the proof follows by [**17**, Corollary 2.7] or [**19**, Proposition 2.6].

Let $\mathscr{P}(X)$ denote the set of all bounded projections P in X such that $\operatorname{codim} P(X)$ is finite. The compression A_P is a bounded linear operator on the closed subspace PX defined by $A_P y = PAy$ for each y in PX. Consequently, $\sigma_g(A_P)$ is the generalized spectrum of this operator on the Banach space PX.

Theorem 3.7. *For every bounded linear operator on a Banach space X we have*

$$\sigma_{gb}(A) = \bigcap \{\sigma_g(A_P) : P \in \mathscr{P}(X) \text{ and } PA = AP\}.$$

Proof. Suppose that λ is not in $\sigma_{gb}(A)$. Then $R(A-\lambda)$ is closed and $k(A-\lambda) < \infty$, i.e., $A - \lambda \in V(X)$. Consequently, by Theorem 2.1 the space X is the direct sum of two closed subspaces X_0 and X_1 which are A-invariant and have the following properties: The space X_1 is finite dimensional (possibly zero) and $A-\lambda$ is nilpotent on it. If A_0 denotes the restriction of A to X_0 considered as an operator from X_0 into itself (and P the projection of X onto X_0 along X_1), then $N((A_0-\lambda)_P) \subset R((A_0-\lambda)_P^\infty)$. Let us remark that $PA = AP$, $P \in \mathscr{P}(X)$ and $R((A-\lambda)_P)$ is closed (Remark 2.2). Thus $\lambda \notin \sigma_g(A_P)$. This proves that $\sigma_{gb}(A) \supset \bigcap \{\sigma_g(A_P) : P \in \mathscr{P}(X) \text{ and } AP = PA\}$.

To prove the converse inclusion, suppose that λ is not in $\sigma_g(A_P)$ for some $P \in \mathscr{P}(X)$ such that $AP = PA$. Thus $R((A-\lambda)P)$ is closed and $k((A-\lambda)P) = 0$. Since $A - \lambda = (A-\lambda)P + (A-\lambda)(I-P)$ and $(A-\lambda)(I-P)$ is a finite rank operator, we conclude that $\lambda \notin \sigma_{gb}(A)$ ([**5**, Theorem 5.9]). The proof is complete.

Let us remark that it has been observed by Zemánek that for Browder's essential approximate point spectrum of A we have $\sigma_{ab}(A) = \bigcap \{\sigma_a(A_P) : P \in \mathscr{P}(X) \text{ and } AP = PA\}$ ([**21**, Theorem 3]).

4. Spectral mapping theorem for $\sigma_{gb}(A)$

Theorem 4.1. *If A is any operator and p is any polynomial, then*

$$\sigma_{gb}(p(A)) = p(\sigma_{gb}(A)).$$

Proof. Let $\lambda \notin p(\sigma_{gb}(A))$ and $p(t) - \lambda = c(t-\lambda_1)^{m_1} \dots (t-\lambda_k)^{m_k}$ with m_i integers, $c \neq 0$ and $\lambda_i \neq \lambda_j$ for $i \neq j$. Thus, $p(A) - \lambda = c(A-\lambda_1)^{m_1} \dots (A-\lambda_k)^{m_k}$ and $\lambda_i \notin \sigma_{gb}(A)$ for $i = 1, \dots, k$. Consequently, we have that $R(A-\lambda_i)$ is closed and $k(A-\lambda_i) < \infty$, for $i = 1, \dots, k$. From [**5**, Theorem 3.8], we know that $R((A-\lambda_i)^{m_i})$ is closed and by [**5**, Lemma 3.11] $k((A-\lambda_i)^{m_i}) < \infty$ for $i = 1, \dots, k$. Let us remark that by ([**4**, Corollary]) we have that

$$R(p(A)-\lambda) = R((A-\lambda_1)^{m_1}) \cap \dots \cap R((A-\lambda_k)^{m_k})$$

and

$$N(p(A)-\lambda) = N((A-\lambda_1)^{m_1}) \oplus \dots \oplus N((A-\lambda_k)^{m_k}).$$

Thus $R(p(A)-\lambda)$ is closed. Further, by ([5, Lemma 2.1(a)]) and the elementary fact that if $\lambda\neq 0$, then $N((A+\lambda)^\infty)\subset R(A^\infty)$, for each integer n we have

$$\frac{N(p(A)-\lambda)}{N(p(A)-\lambda)\cap R((p(A)-\lambda)^n)}$$

$$=\frac{N((A-\lambda_1)^{m_1})\oplus\ldots\oplus N((A-\lambda_k)^{m_k})}{(N((A-\lambda_1)^{m_1})\oplus\ldots\oplus N((A-\lambda_k)^{m_k}))\cap R((A-\lambda_1)^{m_1 n})\cap\ldots\cap R((A-\lambda_k)^{m_k n})}$$

$$=\frac{N((A-\lambda_1)^{m_1})\oplus\ldots\oplus N((A-\lambda_k)^{m_k})}{N((A-\lambda_1)^{m_1})\cap R((A-\lambda_1)^{m_1 n})\oplus\ldots\oplus N((A-\lambda_k)^{m_k n})\cap R((A-\lambda_k)^{m_k n})}.$$

Thus,

$$\dim\frac{N(p(A)-\lambda)}{N(p(A)-\lambda)\cap R((p(A)-\lambda)^n)}\leqq\sum_{i=1}^{n}\dim\frac{N((A-\lambda_i)^{m_i})}{N((A-\lambda_i)^{m_i})\cap R((A-\lambda_i)^{m_i n})}$$

and by [5, Theorem 3.7] it follows that $k(p(A)-\lambda)\leqq\Sigma k((A-\lambda_i)^{m_i})$. Hence, $\lambda\notin\sigma_{gb}(p(A))$.

We now turn to the proof of the opposite inclusion. Suppose that $\lambda\in p(\sigma_{gb}(A))$ and $\lambda\notin\sigma_{gb}(p(A))$. By the definition of $\sigma_{gb}(A)$, we have that $R(p(A)-\lambda)$ is closed and $k(p(A)-\lambda)<\infty$. By ([4, Corollary (iii)]) we know that $R((A-\lambda_i)^{m_i})$ is closed for $i=1,\ldots,k$. Since $N((A-\lambda_i)^{m_i})\subset N(p(A)-\lambda)$, and for each positive integer m and $n, N((A-\lambda_i)^m)\subset R((A-\lambda_j)^n)$, $(i\neq j)$, then by [7, Lemma 2.3] we have

$$\dim\frac{N(A-\lambda_i)}{N(A-\lambda_i)\cap R((A-\lambda_i)^{m_i n})}\leqq\dim\frac{N(p(A)-\lambda)}{N(p(A)-\lambda)\cap R((p(A)-\lambda)^n)}.$$

This shows that $k(A-\lambda_i)<\infty$ ([5, Theorem 3.7]), and by [5, Theorem 3.8] $R(A-\lambda_i)$ is closed. According to this, we have that $\lambda_i\notin\sigma_{gb}(A)$, $(i=1,\ldots,k)$, which provides a contradiction. The proof is complete.

Theorem 4.2. *Let $A\in B(X)$, and let D be an open neighbourhood of $\sigma(A)$. If f is a rational function on D with no poles in D, then*

$$\sigma_{gb}(f(A))=f(\sigma_{gb}(A)).$$

Proof. We can write $f=p/q$, where p and q are polynomials and q has no zeros in D. Hence, $0\notin q(\sigma(A))$, $q(A)$ is invertible and $f(A)=p(A)q(A)^{-1}=q(A)^{-1}p(A)$. For each $\lambda\in\mathbb{C}$ we now write, assuming that p/q is not constant,

$$\frac{p}{q}-\lambda=\frac{p-\lambda q}{q}=\frac{1}{q}c(z-\lambda_1)(z-\lambda_2)\ldots(z-\lambda_n).$$

Hence

$$f(A)-\lambda=q(A)^{-1}c(A-\lambda_1)(A-\lambda_2)\dots(A-\lambda_n),$$

and the proof of Theorem 4.2 follows by Theorem 4.1.

Let (G_n) be a sequence of compact subsets of $\mathbb{C}$. The limit superior, $\limsup G_n$, is the set of all λ in $\mathbb{C}$ such that every neighbourhood of λ intersects infinitely many G_n. To show that if f is an analytic function defined on a neighbourhood of $\sigma(A)$, then $f(\sigma_{gb}(A))=\sigma_{gb}(f(A))$ we shall prove the following statement.

Theorem 4.3. *Let A, $A_n\in B(X)$, $A_n\to A$ and $AA_n=A_nA$ for each positive integer n. Then*

(i) $\limsup \sigma_g(A_n)\subset\sigma_g(A)$,

(ii) $\limsup \sigma_{gb}(A_n)\subset\sigma_{gb}(A)$.

Proof. (i) It is enough to show that if $0\notin\sigma_g(A)$, then $0\notin\limsup\sigma_g(A_n)$. Suppose that $0\notin\sigma_g(A)$. Then $R(A)$ is closed and $k(A)=0$. Then, by [5, Lemma 4.2] we know that there exists an $\varepsilon>0$ and an integer n_0 such that $R(A_n-\lambda)$ are closed for $n\geqq n_0$ and $k(A_n-\lambda)=0$ for $|\lambda|<\varepsilon$. Therefore, for $n\geqq n_0$ we see that $\sigma_g(A_n)\cap\{\lambda\in\mathbb{C}:|\lambda|<\varepsilon\}$ is empty. Thus, we have that $0\notin\limsup\sigma_g(A_n)$.

(ii) To prove (ii), it is enough to show that if $0\notin\sigma_{gb}(A)$, then $0\notin\limsup\sigma_{gb}(A_n)$. If $0\notin\sigma_g(A)$, then by (i) we know that $0\notin\limsup\sigma_g(A_n)$, and $0\notin\limsup\sigma_{gb}(A_n)$. If $0\in\sigma_g(A)\setminus\sigma_{gb}(A)$ then $R(A)$ is closed and $0<k(A)<\infty$. Consequently, by [5, Theorem 4.10(a)] there exists an $\varepsilon>0$ and an integer n_0 such that $R(A_n-\lambda)$ are closed and $k(A_n-\lambda)<\infty$ for $|\lambda|<\varepsilon$ and $n\geqq n_0$. Therefore, for $n\geqq n_0$ we see that $\sigma_{gb}(A_n)\cap\{\lambda\in\mathbb{C}:|\lambda|<\varepsilon\}$ is empty. Thus we have $0\notin\limsup\sigma_{gb}(A_n)$, and the proof is complete.

Remark 4.4. Let us remark that the commutativity conditions in Theorem 4.3 are necessary. Examples in which σ_g and σ_{gb} are not upper semi-continuous can be constructed using the result of Goldman [3, Theorem 1]. In fact, if $A\in V(X)$ ($V_0(X)$), $\alpha(A)=\infty$ and $\beta(A)=\infty$, by [3, Theorem 1] there exists a sequence A_n of linear bounded operators on X, with non-closed ranges, such that $A_n\to A$. Thus, we have that $0\notin\sigma_{gb}(A)$ ($\sigma_g(A)$) and $0\in\sigma_{gb}(A_n)$ for each n (which implies $0\in\limsup\sigma_{gb}(A_n)$).

Theorem 4.5. *Let $A\in B(X)$ and let f be an analytic function defined on a neighbourhood of $\sigma(A)$. Then*

$$f(\sigma_{gb}(A))=\sigma_{gb}(f(A)).$$

Proof. Let D be a neighbourhood of $\sigma(A)$, and let $(f_n(t))$ be a sequence of rational functions, with no poles in D, converging to $f(t)$ on D. We have

$$f(\sigma_{gb}(A))=\lim f_n(\sigma_{gb}(A))$$

$$= \limsup \sigma_{gb}(f_n(A)) \quad \text{(by Theorem 4.2)}$$

$$\subset \sigma_{gb}(f(A)) \quad \text{(by Theorem 4.3(ii))}.$$

To prove the converse suppose that $\mu \notin f(\sigma_{gb}(A))$. Thus, for each $\lambda \in \sigma_{gb}(A)$ we have that $f(\lambda)-\mu \neq 0$. Set $g(\lambda)=f(\lambda)-\mu$. If $g(\lambda)\neq 0$ for each $\lambda \in \sigma(A)$, then $g(A)$ is invertible, and $\mu \notin \sigma(f(A))$. Thus $\mu \notin \sigma_{gb}(f(A))$. Now suppose that $g(\lambda)$ has zeros of order n_i at $\lambda_i \in \sigma(A)$, $i=1,\dots,k$. Then

$$g(\lambda)=\prod_{i=1}^{k}(\lambda-\lambda_i)^{n_i}h(\lambda) \text{ and } h(\lambda)\neq 0 \text{ for each } \lambda \in \sigma(A).$$

Set

$$p(\lambda)=\prod_{i=1}^{k}(\lambda-\lambda_i)^{n_i}.$$

By Theorem 4.1, we know that $0 \notin \sigma_{gb}(p(A))$. Then $R(p(A))$ is closed and $k(p(A))<\infty$. Consequently, since $h(A)$ is an invertible operator commuting with $p(A)$, it is easy to see that $g(A)=p(A)h(A)$ has closed range and $k(g(A))<\infty$. Thus, we have that $\mu \notin \sigma_{gb}(f(A))$, i.e., $\sigma_{gb}(f(A)) \subset f(\sigma_{gb}(A))$. This completes the proof of the theorem.

5. Connected components of $\mathbb{C}\backslash\sigma_{gb}(A)$

If $A \in B(X)$, then $\mathbb{C}\backslash\sigma_{gb}(A)$ is an open set in the complex plane $\mathbb{C}$. Let U be a connected component of $\mathbb{C}\backslash\sigma_{gb}(A)$ and $G=\{\lambda \in \mathbb{C}\backslash\sigma_{gb}(A): k(A-\lambda)\neq 0\}$. By [**6**, Theorem 4.1] we know that G has no accumulation point in $\mathbb{C}\backslash\sigma_{gb}(A)$. A complex number $\lambda \in G \cap U$ is called a jumping point in U.

Remark 5.1. If λ is a jumping point in U, then by Theorem 2.1(ii), there is an A-invariant finite dimensional subspace N_λ in X such that $A-\lambda$ is nilpotent on it. Consistent with the matrix case we define the (algebraic) multiplicity of the jumping point λ to be $\dim N_\lambda$. If U is a connected component of the semi-Fredholm region of A, then our definition of the multiplicity of the jumping point λ in U is consistent with the definition in [**18**, p. 232] and [**22**, p. 449].

Theorem 5.2. *Let $A \in B(X)$ and let U and G be as above. Then the functions*

$$\lambda \to N((A-\lambda)^\infty)+R((A-\lambda)^\infty) \text{ and } \lambda \to N((A-\lambda)^\infty)\cap R((A-\lambda)^\infty)$$

are constant on U, while the functions

$$\lambda \to R((A-\lambda)^\infty) \text{ and } \lambda \to N((A-\lambda)^\infty)$$

are constant on $U \backslash G$.

Proof. The proof follows from [**9**, Theorem 3] and ([**5**, Theorem 4.7(d), (e); Lemma 4.2(d), (e); Lemma 3.6]).

Remark 5.3. Let us remark that by [**5**, Lemma 3.6(a)] and Theorem 5.2 we have that

$$R((A-\lambda)^{\infty})+N(A-\lambda)^{\infty})=R((A-\lambda)^{\infty})+cl(N((A-\lambda)^{\infty}))$$

$$=R((A-\lambda)^{\infty})\oplus N_{\lambda}=W$$

for each $\lambda\in U$, where N_λ is a finite dimensional subspace, N_λ is A-invariant and $(A-\lambda)_{|N_\lambda}$ is nilpotent on it. Thus, W is closed, hence a Banach subspace in X ([**5**, Theorem 3.8]). The restriction of A to the subspace W has been studied in [**16**] and [**19**].

Theorem 5.4. *If* $A\in V(X)$, *set* $v_0(A)=\sup\{\varepsilon>0: A-\lambda\in V_0(X)$ *for* $0<|\lambda|<\varepsilon\}$ *and* $v(A)=\sup\{\varepsilon>0: A-\lambda\in V(X)$ *for* $|\lambda|<\varepsilon\}$. *Then*

$$v(A)=\sup\{v_0(A+F): F\in F(X) \text{ and } AF=FA\}.$$

Proof. Let $F\in F(X)$ and $AF=FA$. Then $A+F\in V(X)$ ([**5**, Theorem 5.9]). If $|\lambda|< v_0(A+F)$, again by ([**5**, Theorem 5.9]) we have that $A-\lambda=(A+F-\lambda)-F\in V(X)$. Hence, $v_0(A+F)\leqq v(A)$, i.e., $\sup\{v_0(A+F): F\in F(X)$ and $AF=FA\}\leqq v(A)$.

To prove the other inequality suppose that $\varepsilon>0$, and let p denote the total multiplicity of the jumps having absolute value less than $v(A)-\varepsilon$. As in the proof of [**18**, Theorem 1.1(II)] (using Theorem 2.1 instead of Kato's decomposition theorem [**9**, Theorem 4]) we conclude that the space X decomposes into the direct sum of two closed subspaces Z and Y which are A-invariant, $\dim Z=p$ and Z is the direct sum of the finite dimensional summands at the jumping points $\lambda_1(A),\dots,\lambda_p(A)$ (where each jump appears consecutively according to its multiplicity). Let $P^2=P\in B(X)$ be the idempotent with $R(P)=Z$ and $N(P)=Y$. It is clear that $P\in F(X)$ and $AP=PA$. Set $F=\alpha P$, with $|\alpha|>\|A\|+v(T)$. Now, as in the proof of [**20**, Theorem 7.1], for each λ with $|\lambda|<v(A)-\varepsilon$ we have that $R(A+F-\lambda)$ is closed and $N(A+F-\lambda)\subset R((A+F-\lambda)^{\infty})$. Thus, $v_0(A+F)\geqq v(A)-\varepsilon$, and the proof is complete.

Lemma 5.5. *Let* $A\in B(X)$ *and let* U, G *and* W *be as above. Then:*

(i) $(A-\lambda)_{|W}\in\Phi_-(W)$ *for each* $\lambda\in U$;

(ii) *if* $\lambda\in U$, *then* $\lambda\in U\cap G$ *if and only if* λ *is a jumping point in the semi-Fredholm region of* $A_{|W}$.

Proof. Let $\lambda\in U$. Then $W=R((A-\lambda)^{\infty})\oplus N_\lambda$ (Remark 5.3). By [**5**, Theorem 3.4] we have that $(A-\lambda)W=(A-\lambda)R((A-\lambda)^{\infty})\oplus(A-\lambda)N_{\lambda}=R((A-\lambda)^{\infty})\oplus(A-\lambda)N_{\lambda}$. Thus, $(A-\lambda)_{|W}\in\Phi_{-}(W)$, which proves (i). (ii) follows by Remark 5.1 and (i).

For a technical reason we suppose that the connected component U contains zero. Then the points in $G\cap U$ can be ordered in such a way that

$$|\lambda_1(A)|\leqq|\lambda_2(A)|\leqq\dots<v(A),$$

where each jump appears consecutively according to its multiplicity. If there are only p $(=0,1,2,\dots)$ such jumps, we put $|\lambda_{p+1}(A)|=|\lambda_{p+2}(A)|=v(A)$.

Let S denote the closed unit ball of X. Let

$$q(A)=\sup\{\varepsilon\geqq 0: AS\supset\varepsilon S\}$$

be the surjection modulus of A. For each $r=1,2,\dots$ we define

$$q_r(A)=\sup\{q(A+F):\operatorname{rank}F<r\}.$$

Theorem 5.6. *Let $A\in V(X)$, $0\in U$, and let U, G and W be as above. Then for each jumping point $\lambda_r(A)$, $r=1,2,\dots$ we have*

$$|\lambda_r(A)|=\lim_k q_r((A_{|W})^k)^{1/k}.$$

Proof. By Lemma 5.5 we know that $(A-\lambda)_{|W}\in\Phi_{-}(W)$ for each $\lambda\in U$, and that $\lambda_r(A)$, $r=1,2,\dots$ are jumps (with the same multiplicity) in the semi-Fredholm region of $A_{|W}$ (Remark 5.1). Thus, the proof of the theorem follows by [**18**, Theorem 1.1, pp. 232–233] (since the stability index of the semi-Fredholm operator $A_{|W}$ is 0).

If T is a linear operator from a Banach space X to another Banach space Y, then the reduced minimum modulus of T is defined by

$$\gamma(T)=\inf\{\|Tx\|:\operatorname{dist}(x,N(T))=1\}.$$

For each $r=1,2,\dots$ we put

$$\gamma_r^{-}(T)=\sup\{\gamma(Q_VT):\dim V<r\},$$

where Q_V is the canonical map of X onto the quotient space X/V. Now, we have:

Corollary 5.7. *Let $A\in B(X)$ and let $\lambda_r(A)$, $r=1,2,\dots,U$ and W be as above. Then for each jumping point $\lambda_r(A),r=1,2,\dots$ we have*

$$|\lambda_{\rho+r}(A)| = \lim_k \gamma_r^-((A_{|W})^k)^{1/k},$$

where ρ is the multiplicity of the jump at zero.

Proof. The proof follows by [**22**, Theorem 1, p. 451], Lemma 5.5 and Theorem 5.6.

Corollary 5.8. *If $A \in V(X)$, then*

$$v_0(A) = \lim_k \gamma((A_{|W})^k)^{1/k}.$$

Proof. This follows from Corollary 5.7.

Let $A \in B(X)$ be a semi-Fredholm operator. Then the semi-Fredholm radius $s(A)$ of A is the supremum of all $\varepsilon \geqq 0$ such that the operator $A - \lambda$ is semi-Fredholm for $|\lambda| < \varepsilon$.

Corollary 5.9. *Let $A \in V(X)$ and let $\lambda_r(A)$, $r = 1, 2, \ldots, U$ and W be as above. Then:*

(i) *if there is a finite number of jumps, then $v(A) \leqq s(A_{|W})$.*

(ii) *if there is an infinite number of jumps, then $v(A) = s(A_{|W})$.*

Proof. This follows by Lemma 5.5 and [**6**, Theorem 4.1].

We would like to finish this paper with the following questions:

Question 1. If $A \in V(X)$, must $\lim_k \gamma(A^k)^{1/k} = v_0(0)$?

(Let us remark that the limit exists (by Theorem 2.1 and the proof of [**2**, Theorem 2]). If X is a Hilbert space, then the answer to the Question 1 is positive (see [**1**, Theorem 3.2, Corollary 3.4] or [**13**, Théorème 3.1, Corollaire 3.9]).)

Question 2. If A, $B \in B(X)$ and $AB = BA \in V(X)$, must A, $B \in V(X)$?

(Let us remark that if A, $B \in B(X)$ and $AB = BA \in V_0(X)$, then A, $B \in V_0(X)$ ([**13**, Lemma 4.15]).)

Question 3. If A, $B \in B(X)$, $AB = BA$ and B is a quasinilpotent operator, must

$$\sigma_{gb}(A + B) = \sigma_{gb}(A)?$$

(Recall that if X is a Hilbert space, A, $B \in B(X)$, $AB = BA$ and B is a quasinilpotent operator, then $\sigma_g(A + B) = \sigma_g(A)$ ([**13**, Théorème 4.8].)

Question 4. If A, $B \in V(X)$ (or $V_0(X)$) and $AB = BA$, must $AB \in V(X)$ (or $V_0(X)$), and possibly $k(AB) \leqq k(A) + k(B)$?

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