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PRESENTATIONS OF THE TREFOIL GROUP

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Introduction. A presentation of a group G is an exact sequence of groups

$$1 \to R \subseteq F \to G \to 1$$

where F is a free group. Let $1 \to S \subseteq F \to G \to 1$ be another presentation of G involving the same free group F. The two presentations are said to be F-equivalent if there exist automorphisms α, β of F, G respectively making the diagram

$$\begin{array}{ccccccc} 1 \to R \subseteq & F & \to & G & \to 1 \\ & \downarrow\alpha & & \downarrow\beta & \\ 1 \to S \subseteq & F & \to & G & \to 1 \end{array}$$

commutative. If F has n free generators, then every ordered n-tuple of generators of G determines a presentation of G. Two such n-tuples of generators determine F-equivalent presentations if and only if they belong to the same T-system (see [4] or [2]). In this paper it is shown that if G is the trefoil group, i.e.

$$G = gp(a, b \mid a^2 = b^3),$$

then G has an infinite number of F-equivalence classes of presentations.

If F has n generators and r is the smallest number of elements whose normal closure in F is R, then the presentation

$$1 \to R \subseteq F \to G \to 1$$

is said to have deficiency $n-r$. It is asserted in [1] that the group

$$G = gp(a, b \mid a^{-1}b^2a = b^3)$$

has a presentation in which the deficiency is not 1. It is stated that Graham Higman has shown that the presentation determined by the pair of generators a, b^4 requires two relators.

It will be shown in this paper that if $G=gp(a, b \mid a^2=b^3)$, and i is a positive integer, then a^{2i+1}, b^{3i+1} is a pair of generators for G requiring more than one defining relator.

The results can easily be generalized to groups of the form $G=gp(a, b \mid a^r=b^s)$ where r, s are coprime.

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Nielsen transformations. Let Gp be the category of groups. Let Gp^n be the subcategory of Gp consisting of all groups

$$G \times G \times \cdots \times G \quad (n \text{ copies})$$

and maps

$$\theta \times \theta \times \cdots \times \theta : G \times G \times \cdots \times G \to H \times H \times \cdots \times H$$

where $\theta : G \to H$ is a homomorphism.

An n-transformation α is defined to be a natural transformation of the identity functor from Gp^n to Gp^n. Let F_n be the free group on $x_1, \ldots, x_n$. It is easy to see that there are fixed words $w_1(x_1, \ldots, x_n), \ldots, w_n(x_1, \ldots, x_n)$ such that

$$(g_1, \ldots, g_n)\alpha = (w_1(g_1, \ldots, g_n), \ldots, w_n(g_1, \ldots, g_n)).$$

A Nielsen transformation is an n-transformation α such that $(x_1, \ldots, x_n)\alpha$ is a set of generators for F_n. Thus there exists an automorphism $\gamma : F_n \to F_n$ such that

$$(x_1, \ldots, x_n)\alpha = (x_1\gamma, \ldots, x_n\gamma).$$

If $\theta : G \to H$ is a mapping, we write $(g_1\theta, \ldots, g_n\theta)$ as $(g_1, \ldots, g_n)\theta$.

THEOREM 1. *The set of all Nielsen transformations forms a group N under composition. The mapping*

$$\rho : N \to \operatorname{Aut} F_n$$

$$\alpha\rho = \gamma$$

is an anti-isomorphism.

Proof. Clearly ρ is bijective. If α_1, α_2 are Nielsen transformations

$$\begin{aligned}(x_1, \ldots, x_n)\alpha_1\alpha_2 &= (x_1, \ldots, x_n)\alpha_1\rho\alpha_2 \\ &= (x_1, \ldots, x_n)\alpha_2\alpha_1\rho\end{aligned}$$

since α_2 is a natural transformation. Hence

$$(x_1, \ldots, x_n)\alpha_1\alpha_2 = (x_1, \ldots, x_n)\alpha_2\rho\alpha_1\rho,$$

and so $(\alpha_1\alpha_2)\rho = \alpha_2\rho\alpha_1\rho$, which proves the theorem.

Every ordered n-tuple $\underline{\underline{g}} = (g_1, \ldots, g_n)$ of generators of a group G determines a presentation

$$1 \to R \subseteq F_n \xrightarrow{\theta} G \to 1$$

in which $x_i\theta = g_i$, $i = 1, \ldots, n$. Let $1 \to S \subseteq F_n \xrightarrow{\phi} G \to 1$ be another presentation of G. If these presentations are F_n-equivalent, $\phi = \gamma\theta\beta$ where γ, β are automorphisms of F_n and G respectively. Let $\alpha = \gamma\rho^{-1}$, then

$$\begin{aligned}(x_1, \ldots, x_n)\gamma\theta\beta &= (x_1, \ldots, x_n)\alpha\theta\beta \\ &= (x_1, \ldots, x_n)\theta\alpha\beta \\ &= (g_1, \ldots, g_n)\alpha\beta.\end{aligned}$$

Thus the n-tuple corresponding to ϕ can be obtained from $(g_1, \ldots, g_n)$ by a Nielsen transformation and an automorphism of G.

Let A be the free abelian group (written multiplicatively) on free generators a_1, a_2. Let α be a Nielsen transformation for which $(a_1, 1)\alpha=(a_1, 1)$. Clearly then

$$(a_1, a_2)\alpha = (a_1a_2^i, a_2^j)$$

where $j=\pm 1$. It follows that if τ, μ are the Nielsen transformations such that

$$(x_1, x_2)\tau = (x_1x_2, x_2)$$
$$(x_1, x_2)\mu = (x_1, x_2^{-1}),$$

then multiplying α by a suitable power of τ and also by μ if $j=-1$ we obtain a Nielsen transformation α' for which $(a_1, a_2)\alpha'=(a_1, a_2)$. Thus $(x_1, x_2)\alpha'=(x_1\gamma, x_2\gamma)$ where γ is an automorphism of F_2 which induces the identity automorphism on F_2/F_2'. It is proved in [5] that γ is an inner automorphism. Let G be an arbitrary group and let $g_1, g_2 \in G$, then the second component of $(g_1, g_2)\alpha'$ is a conjugate of g_2. But the second component of $(g_1, g_2)\alpha'$ is either the same as or the inverse of the second component of $(g_1, g_2)\alpha$. Thus we have proved the following lemma.

LEMMA 1. *Let G be an arbitary group and let $g_1, g_2 \in G$. Let $C=gp(c)$ be the infinite cyclic group. If α is a Nielsen transformation such that $(c, 1)\alpha=(c, 1)$, then the second component of $(g_1, g_2)\alpha$ is a conjugate of g_2 or g_2^{-1}.*

Presentations of the trefoil group. Let $G=gp(a, b \mid a^2=b^3)$. The automorphism group of G is generated by inner automorphisms and the automorphism

$$\nu: G \to G$$
$$a\nu = a^{-1}, \qquad b\nu = b^{-1}.$$

Suppose (g, h) is an ordered pair of generators for G. By an extension of Gruschko's theorem (see [3]), it follows that there is a Nielsen transformation α_1 for which $(g, h)\alpha_1=(a^m, b^n)$ for some integers m, n. If σ is the Nielsen transformation such that $(x_1, x_2)=(x_1^{-1}, x_2^{-1})$, then $(g, h)\nu\alpha_1=(g, h)\alpha_1\nu=(a^m, b^n)\nu=(a^{-m}, b^{-n})=(g, h)\alpha_1\sigma$. Hence $(g, h)\nu=(g, h)\alpha_1\sigma\alpha_1^{-1}$. If β is an inner automorphism of G, then clearly there is a Nielsen transformation α for which $(g, h)\beta=(g, h)\alpha$. Hence for any pair of generators (g, h) of G and any automorphism β of G, there is a Nielsen transformation α for which $(g, h)\beta=(g, h)\alpha$. It follows from the previous section therefore that two ordered pairs of generators $\underline{\underline{g}}, \underline{\underline{g}}'$ of G determine F_2-equivalent presentations if and only if there is a Nielsen transformation α for which $\underline{\underline{g}}\alpha=\underline{\underline{g}}'$.

Let $\underline{\underline{g}}_i=(a^{2i+1}, b^{3i+1})$. It follows from [3] that $\underline{\underline{g}}_i$ is an ordered pair of generators for G. Now

$$\underline{\underline{g}}_i\tau^{-1} = (ab^{-1}, b^{3i+1})$$

and so clearly there is a Nielsen transformation α_i for which

$$\underline{\underline{g}}_i\alpha_i = (ab^{-1}, b^{3i+1}(ab^{-1})^{-6i-2}).$$

Let C be the infinite cyclic group generated by c, then there is a homomorphism $\theta: G \to C$ such that $(ab^{-1})\theta = c$ and $(b^{3i+1}(ab^{-1})^{-6i-2})\theta = 1$. Suppose that there were a Nielsen transformation α for which $\underline{\underline{g}}_i\alpha_i\alpha = \underline{\underline{g}}_j\alpha_j$. Then since α is a natural transformation $(c, 1)\alpha = (c, 1)$, and so by Lemma 1, $b^{3i+1}(ab^{-1})^{-6i-2}$ is a conjugate of $b^{3j+1}(ab^{-1})^{-6j-2}$ or its inverse. It is easy to verify that this is true if and only if $i=j$.

In her paper [6], E. S. Rapaport proves that any two presentations of the trefoil group involving F_2 and one relator are F_2-equivalent. It follows that the presentation of G determined by $\underline{\underline{g}}_i = (a^{2i+1}, b^{3i+1})$, $i \neq 0$, requires more than one relator. Thus we have proved the following theorem.

THEOREM 2. *If* $G = gp(a, b \mid a^2 = b^3)$ *and* $\underline{\underline{g}}_i = (a^{2i+1}, b^{3i+1})$, *then the presentations of* G *determined by* $\underline{\underline{g}}_i$ *and* $\underline{\underline{g}}_j$ *are* F_2*-equivalent only if* $i=j$. *For* $i \neq 0$ *the presentation determined by* $\underline{\underline{g}}_i$ *requires more than one relator.*

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