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CONFLUENT AND RELATED MAPPINGS DEFINED BY MEANS OF QUASI-COMPONENTS

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1. Introduction. In 1964, J. J. Charatonik in [**1**] introduced a new class of mappings, the so-called confluent mappings, which comprises the classes of open, monotone and quasi-interior mappings (see [**20**]). In 1966, A. Lelek started working on confluent mappings with applications to continua theory (see [**7**]). He introduced two other classes of mappings, the so-called weakly confluent and pseudo confluent mappings, he proved the invariance of rational continua under open, monotone and quasi-interior mappings and he asked about their invariance under confluent mappings. In 1976, E. D. Tymchatyn gave an example of a confluent mapping, which does not preserve the rationality of a curve (see [**18**]).

In this paper, we introduce some new classes of mappings defined by means of quasi-components, which comprise the classes of open, monotone and quasi-interior mappings but which are contained in the classes of confluent, weakly confluent and pseudo confluent mappings. We also give some characterizations of these mappings and some examples. Theorems 3.8 and 3.9 generalize some earlier results in [**10**], and Theorems 3.17 and 3.18 are analogous to Theorem 2 in [**16**]. Composition, product and union properties are proved also. Theorem 6.1 proves a union theorem for finite decompositions of the image space, while Example 6.4 resolves in the negative a question asked by A. Lelek about union theorems for h-confluent and H-confluent mappings for infinite decompositions of the image space. Theorem 8.1 generalizes earlier results about invariance of rational continua by proving that rational continua are preserved by H-pseudo confluent mappings. We also prove some invariance theorems of hereditarily σ-connected and hereditarily weakly σ-connected spaces, and Theorem 10.6 generalizes Theorem 4.6 of [**11**].

Finally, the author expresses his deep appreciation and his gratitude to Professor A. Lelek, who contributed to these investigations his kind advice and valuable improvements.

2. Definitions. By a *mapping* we always mean a continuous surjective function and by a *perfect mapping* we mean a mapping which is closed and has compact preimages of points. Let X be a topological space, A be a subset of X

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and x be a point of A. The *quasi-component of x in A*, denoted by $Q(A, x)$ is the intersection of all closed-open subsets of A containing x.

Let $f: X \to Y$ be a perfect mapping of a topological space X onto a topological space Y. The mapping f is said to be *H-confluent*, *H-weakly confluent*, or *H-pseudo confluent* provided, for each non-empty subset Z of Y, the following conditions are satisfied, respectively:

(H-c) For each point $z \in Z$ and each point $x \in f^{-1}(z)$, we have

$$f(Q(f^{-1}(Z), x)) = Q(Z, z).$$

(H-w.c.) For each point $z \in Z$, there exists a point $x \in f^{-1}(z)$ such that

$$f(Q(f^{-1}(Z), x)) = Q(Z, z).$$

(H-p.c.) For each point $z \in Z$, we have

$$\bigcup_{x \in f^{-1}(z)} f(Q(f^{-1}(Z), x)) = Q(Z, z).$$

Clearly, (H-c.) implies (H-w.c.), which implies (H-p.c.). The mapping f is said to be *h-confluent*, *h-weakly confluent*, or *h-pseudo confluent* provided, for each connected non-empty subset K of Y, the following conditions are satisfied, respectively:

(h-c.) For each quasi-component Q of $f^{-1}(K)$, we have $f(Q) = K$.

(h-w.c.) There exists a quasi-component Q of $f^{-1}(K)$ such that $f(Q) = K$.

(h-p.c.) For each pair of points $y, y' \in K$ there exists a quasi-component Q of $f^{-1}(K)$ such that $y, y' \in f(Q)$.

Clearly, (h-c.) implies (h-w.c.), which implies (h-p.c.).

We also mention four more classes of mappings that have been introduced by various authors since 1964:

The perfect mapping f is said to be *confluent*, *weakly confluent*, *pseudo confluent* provided, for each connected, closed non-empty subset K of Y, the following conditions are satisfied, respectively:

(c.) For each quasi-component Q of $f^{-1}(K)$, we have $f(Q) = K$.

(w.c.) There exists a quasi-component Q of $f^{-1}(K)$ such that $f(Q) = K$.

(p.c.) For each pair of points $y, y' \in K$, there exists a quasi-component Q of $f^{-1}(K)$ such that $y, y' \in f(Q)$.

The mapping f is said to be *strongly confluent* provided for each connected non-empty subset K of Y, and for each component C of $f^{-1}(K)$, we have $f(C) = K$.

Clearly, (c.) implies (w.c.), which implies (p.c.), and strongly confluent mappings satisfy condition (h-c.).

3. Preliminaries. The classes of open, monotone and quasi-interior mappings have been already established (see [**19**] and [**20**]). We say that a mapping

f of a topological space X onto a topological space Y is *quasi-interior* provided, f is perfect and for each point $y \in Y$, each component C of $f^{-1}(y)$ and each open neighborhood U of C, we have $y \in \operatorname{Int} f(U)$.

PROPOSITION 3.1. *Let $f: X \to Y$ be a quasi-interior mapping from a hereditarily normal space onto a topological space Y. Then f is H-confluent.*

Proof. Let Z be a subset of Y and $z \in Z$. To show that for each $x \in f^{-1}(z)$, $f(Q(f^{-1}(Z), x)) = Q(Z, z)$. On the contrary, suppose that there is some $x_0 \in f^{-1}(z)$ such that $f(Q(f^{-1}(Z), x_0)) \neq Q(Z, z)$. Let $y_0 \in Q(Z,z)\backslash f(Q(f^{-1}(Z), x_0))$. Then $f^{-1}(y_0) \cap Q_0 = \emptyset$, where $Q_0 = Q(f^{-1}(Z), x_0)$. Therefore, for each $x \in f^{-1}(y_0)$, there exists a closed-open subset A_x of $f^{-1}(Z)$ such that $Q_0 \subset A_x \subset f^{-1}(Z)\backslash\{x\}$. Then $f^{-1}(Z)\backslash A_x$ is open in $f^{-1}(Z)$ and $x \in f^{-1}(Z)\backslash A_x$. Then $\{f^{-1}(Z)\backslash A_x\}_{x \in f^{-1}(y_0)}$ is an open cover of the compact set $f^{-1}(y_0)$, so there exist points $x_1, \ldots, x_n \in f(y_0)$ such that

$$f^{-1}(y_0) \subset \bigcup_{i=1}^{n} (f^{-1}(Z)\backslash A_{xi}).$$

Put $A = \bigcap_{i=1}^{n} A_{xi}$. Then $Q_0 \subset A$ and $f^{-1}(y_0) \subset f^{-1}(Z)\backslash A$.

Since A is closed-open in $f^{-1}(Z)$, there exists an open set $U \subset X$ such that $A \subset U$, and

$$(1) \quad f^{-1}(Z) \cap (\bar{U}\backslash U) = \emptyset = \bar{U} \cap (f^{-1}(Z)\backslash A)$$

(see [**5**, page 145]). So we have $U \cap f^{-1}(y_0) = \emptyset$. Then $y_0 \in Z\backslash f(U)$. But $Q_0 \subset A \subset U$ and the non-empty set $f(Q_0)$ is contained in $Z \cap f(U)$. Since Z is connected between z and y_0, $f(U)$ is not closed-open in Z, so there exists a point $b_0 \in \overline{f(U)} \cap \overline{Y\backslash f(U)} \cap Z$. By (1) and the fact that $\overline{f(U)}\backslash f(U) \subset f(\bar{U}\backslash U)$, we get

$$Z \cap (\overline{f(U)}\backslash f(U)) \subset Z \cap f(\bar{U}\backslash U) = f(f^{-1}(Z) \cap (\bar{U}\backslash U)) = \emptyset,$$

so that $b_0 \in f(U)$. Let $a_0 \in U$ such that $f(a_0) = b_0$. Then $a_0 \in A$ by (1). Let C be the component of a_0 in $f^{-1}(b_0)$. The set A being closed-open in $f^{-1}(Z)$, we conclude that $C \subset A$, and $C \subset U$. Since f is quasi-interior, $b_0 \in \operatorname{Int} f(U)$ contrary to the fact that $b_0 \in \overline{Y\backslash f(U)}$. This completes the proof of 3.1.

Remark 3.2. It is easy to see that open mappings on topological spaces are quasi-interior. Also, monotone mappings on compact metric spaces are quasi-interior (see [**19**]).

The following diagram gives the relation between all the above-mentioned classes of mappings. By dashed arrows we denote those implications which are true under certain restrictions. In this diagram by (H-c.), (h-c.), (c.), etc. we denote the classes of perfect mappings which satisfy these conditions respectively. The implication (1) is true, provided that X and Y are compact metric spaces, and the implication (3) is true provided that X is hereditarily normal.

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Also, implications (2) and (5) are true provided that the mappings are perfect. It has been also proved that the implication (3) is invertible provided that X is a compact metric space and Y is a continuum (see [**13**]), that (5) is invertible provided that Y is a hereditarily locally connected continuum (see [**3**]), and that (6), (7), (11), (12), (16) and (17) are invertible provided that Y is a locally connected complete metric space and X is a hereditarily normal space (see [**11**, Propositions 3.1, 3.3 and 3.4]).

$$\begin{tikzcd}
\text{(open)} \arrow[dr, Rightarrow, dashed, "(2)"'] & & \text{(monotone)} \arrow[dl, Rightarrow, dashed, "(1)"] \arrow[rr, Rightarrow, "(4)"] & & \text{(strongly confluent)} \arrow[d, Rightarrow, dashed, "(5)"] & \\
& \text{(quasi-interior)} \arrow[rr, Rightarrow, dashed, "(3)"] & & \text{(H.c.)} \arrow[r, Rightarrow, "(6)"] \arrow[d, Rightarrow, "(8)"'] & \text{(h-c.)} \arrow[r, Rightarrow, "(7)"] \arrow[d, Rightarrow, "(9)"'] & \text{(c.)} \arrow[d, Rightarrow, "(10)"'] \\
& & & \text{(H-w.c.)} \arrow[r, Rightarrow, "(11)"] \arrow[d, Rightarrow, "(13)"'] & \text{(h-w.c.)} \arrow[r, Rightarrow, "(12)"] \arrow[d, Rightarrow, "(14)"'] & \text{(w.c.)} \arrow[d, Rightarrow, "(15)"'] \\
& & & \text{(H-p.c.)} \arrow[r, Rightarrow, "(16)"] & \text{(h-p.c.)} \arrow[r, Rightarrow, "(17)"] & \text{(p.c.)}
\end{tikzcd}$$

DIAGRAM 1.

The following examples show that the implications of Diagram 1 are not invertible unless otherwise indicated in Remark 3.2.

Example 3.3. There exists a confluent mapping $f: X \to Y$ of an arc-like continuum X onto an arc-like continuum Y, which is not h-confluent.

Proof. Let

$$X = \{(x, 1): |x| \leqq 1\} \cup \left\{\left(\sin \frac{\pi}{y-1}, y\right): 1 < y \leqq 2\right\} \cup \{(1, y): |y| \leqq 1\} \cup \left\{\left(x, \sin \frac{\pi}{x-1}\right): 1 < x \leqq 2\right\},$$

and let R be an equivalence relation in X, given by

$$R = \{((t, 1), (1, t)): |t| \leqq 1\} \cup \{((1, t), (t, 1)): |t| \leqq 1\} \cup \{(p, p): p \in X\}.$$

Then the natural projection f of X onto $Y = X/R$ is a confluent mapping which is not h-confluent (see [**11**, Example 3.7]).

Example 3.4. There exists a strongly confluent mapping, hence also h-confluent, from a continuum onto a continuum, which is not H-pseudo confluent.

Proof. Let $X = \{(0, y): |y| \leqq 1\} \cup \{(x, \sin \pi/x): 0 < x \leqq 1\}$, and let R be an equivalence relation in X, given by $R = \{((0, y), (0, -y)): |y| \leqq 1\} \cup \{(p, p): p \in X\}$. Then the natural projection f of X onto $Y = X/R$ is strongly confluent but not H-pseudo confluent. To show that f is not H-pseudo confluent, let

$$A = \{(x, \sin \pi/2): 0 < x \leqq 1\} \cap \{(x, y): 1/2 \leqq y \leqq 1\} \quad \text{and}$$
$$B = \{(x, \sin \pi/2): 0 < x \leqq 1\} \cap \{(x, y): -1/2 \leqq y \leqq 0\}.$$

Put $Z = f(A \cup V) \cup \{f(0, 1), f(0, 1/2), f(0, 0)\}$. Then Z is a subset of Y and $Q(Z, f(0, 1)) = \{f(0, 1), f(0, 1/2), f(0, 0)\}$. Consider $f^{-1}(Z)$. Then it is easy to check that $Q(f^{-1}(Z), (0, 1)) = \{(0, 1), (0, 1/2)\} = Q(f^{-1}(Z), (0, 1/2))$, $Q(f^{-1}(Z), (0, 0)) = \{(0, 0), (0, -1/2)\} = Q(f^{-1}(Z), (0, -1/2))$ and finally, $Q(f^{-1}(Z), (0, -1)) = \{(0, -1)\}$. Consequently, there is no quasi-component of $f^{-1}(Z)$ whose image contains both points $f(0, 0)$ and $f(0, 1)$. Thus f is not H-pseudo confluent.

Example 3.5. There exists an H-weakly confluent mapping, hence also h-weakly confluent and weakly confluent, of a continuum X onto $[0, 1]$, which is not confluent.

Proof. Let $X = \{(x, 0): 0 \leqq x \leqq 1\} \cup \{(x, y): x = y \text{ and } 0 \leqq x \leqq 1/2\}$, and let $f((x, y)) = x$, for $(x, y) \in X$.

Example 3.6. There exists an H-pseudo confluent mapping from an arc-like continuum X onto a continuum Y, which is not weakly confluent.

Proof. Let $X = \{(0, y): 0 \leqq y \leqq 2\}$, and let $T = A_0 \cup A_1 \cup A_2$ be a triod, where A_i is an arc with end-points a_i, $b (i = 0, 1, 2)$ such that b is the only common point of any two of the arcs A_0, A_1 and A_2. Let f be a mapping such that $f(0) = a_0$ and f maps the intervals $\{(0, y): 0 \leqq y \leqq 1/3\}$, $\{(0, y): 1/3 \leqq y \leqq 2/3\}$ and $\{(0, y): 2/3 \leqq y \leqq 1\}$ homeomorphically onto the arcs $A_0 \cup A_1$, $A_1 \cup A_2$ and $A_2 \cup A_0$, respectively. Then f is H-pseudo confluent since condition (iii) of Proposition 3.13 is satisfied but is not weakly confluent (see [**11**, Example 3.6]).

Remarks 3.7. (i) It is trivial to check that the implication (4) in Diagram 1 is true but is not invertible.

(ii) Example 3.3 is an example of a weakly confluent mapping, hence also pseudo confluent, which is not h-pseudo confluent (see [**11**, Example 3.7]). Finally, Example 3.6 is a pseudo confluent but not weakly confluent mapping.

(iii) The example of the open mapping in [**3**] serves as an example of an h-confluent mapping, which is not strongly confluent.

The following two theorems are generalizations of Theorems 1.2 and 2.2 of [**10**].

THEOREM 3.8. *Let X and Y be metric spaces and f: $X \to Y$ a mapping from*

X onto Y. Then f is open if and only if

$$\lim_{n\to\infty} y_n = y \quad \textit{implies that} \quad \operatorname*{Ls}_{n\to\infty} f^{-1}(y_n) = f^{-1}(y).$$

Remark. The proof of Theorem 3.8, being an exact copy of the proof of Theorem 1.2 (ibidem), is omitted.

THEOREM 3.9. *Let X and Y be metric spaces and $f: X \to Y$ a perfect mapping from X onto Y. Then f is a quasi-interior if and only if* $\lim_{n\to\infty} y_n = y$ *implies that* $\mathrm{Ls}_{n\to\infty} f^{-1}(y_n)$ *meets each component of* $f^{-1}(y)$.

Proof. Let f be quasi-interior at y_0 and C be a component of $f^{-1}(y_0)$. Then if U is an open neighborhood of C in X, $\mathrm{Int} f(U)$ is open neighborhood of y_0 in D, so $f(U)$ contains some points $y_{n1}, y_{n2}, \ldots$ such that $\lim_{i\to\infty} y_{n_i} = y_0$. Therefore, $U \cup f^{-1}(y_{n_i}) \neq \emptyset$, $(i = 1, 2, \ldots)$. Hence, $\mathrm{Ls}_{n\to\infty} f^{-1}(y_n) \cap C \neq \emptyset$, since C is compact.

Conversely, let the condition be satisfied and let U be an open neighborhood of a component C of $f^{-1}(y_0)$. If $y_0 \notin \mathrm{Int}\, f(U)$, then there exist points $y_n \in Y \backslash f(U)$ $(n = 1, 2, \ldots)$ such that $\lim_{n\to\infty} y_n = y_0$. Hence, $f^{-1}(y_n) \subset X \backslash U$ $(n = 1, 2, \ldots)$ and since $X\backslash U$ is closed in X we have that $\mathrm{Ls}_{n\to\infty} f^{-1}(y_n) \cap U = \emptyset$, contradicting the fact that $\mathrm{Ls}_{n\to\infty} f^{-1}(y_n)$ has to meet $C \subset U$. Therefore, $y \in \mathrm{Int} f(U)$.

Let A be any one of the classes of H-confluent, H-weakly confluent, H-pseudo confluent, h-confluent, h-weakly confluent and h-pseudo confluent mappings. Then the following is true:

PROPOSITION 3.10. *If $f: X \to Y$ is a mapping belonging to the class A and B is any subset of Y, then the restriction of f on $f^{-1}(B)$ onto B belongs to the class A.*

The following three propositions can be obtained by using 1.3, 2.1, 2.2 and 2.3 in [**11**].

PROPOSITION 3.11. *Let $f: X \to Y$ be a perfect mapping from a hereditarily normal space X onto a topological space Y. Then the following are equivalent:*

(i) *f is H-confluent;*

(ii) *for each open set $Z \subset Y$, each point $y \in Z$ and each $x \in f^{-1}(y)$, we have $f(Q(f^{-1}(Z), x)) = Q(Z, y)$;*

(iii) *for each closed set $C \subset X$ and each $y \in Y\backslash f(C)$, we have $Q(Y\backslash f(C), y) \subseteq f(Q(X\backslash C, x))$, $(x \in f^{-1}(y))$.*

PROPOSITION 3.12. *Let $f: X \to Y$ be a perfect mapping from a hereditarily normal space X onto a topological space Y. Then the following are equivalent:*

(i) *f is H-weakly confluent;*

(ii) *for each open set $Z \subset Y$ and each point $y \in Z$, there exists a point $x \in f^{-1}(y)$ such that $f(Q(f^{-1}(Z), x)) = Q(Z, y)$;*

(iii) *for each closed set $C \subset X$ and each $y \in Y\backslash f(C)$, there exists a point $x \in f^{-1}(y)$ such that $Q(Y\backslash f(C), y) \subseteq f(Q(X\backslash C, x))$.*

Consider the following condition: Let $f: X \to Y$ be a mapping from a topological space X onto a topological space Y. We say that *f satisfies condition* (S), provided that for any pair of points $y, z \in Y$ and any closed subset C of X, which separates $f^{-1}(y)$ and $f^{-1}(z)$ in X, we have that $f(C)$ separates Y between y and z.

PROPOSITION 3.13. *Let $f: X \to Y$ be a perfect mapping from a hereditarily normal space X onto a topological space Y. Then the following are equivalent:*

(i) *f is H-pseudo confluent;*

(ii) *for each open set $Z \subset Y$ and each point $y \in Z$, we have*

$$Q(Z, y) = \bigcup_{x \in f^{-1}(y)} f(Q(f^{-1}(Z), x));$$

(iii) *f satisfies condition (S);*

(iv) *for each closed set $C \subset X$ and each $y \in Y \backslash f(C)$, we have*

$$Q(Y \backslash f(C), y) \subseteq \bigcup_{x \in f^{-1}(y)} f(Q(X \backslash C, x)).$$

PROPOSITION 3.14. *Let $f: X \to Y$ be a perfect mapping from a hereditarily normal space X onto a topological space Y. Then the following are equivalent:*

(i) *f is h-confluent;*

(ii) *for each connected subset B of Y such that $B = L \cap K$ where L is open and K is closed in Y, and for each quasi-component Q of $f^{-1}(B)$, we have $f(Q) = B$.*

Proof. (We wish to thank the referee for the simplification of the proof of Proposition 3.14). (i) implies (ii): Obvious.

(ii) implies (i): Let $B \subset Y$ be a connected set such that for some $y \in B$ and some $x \in f^{-1}(B)$, $f^{-1}(B)$ is not connected between x and $f^{-1}(y)$. Then $f^{-1}(B) = M \cup N$ such that $x \in M$, $f^{-1}(y) \subset N$ and $\bar{M} \cap N = \emptyset = M \cap \bar{N}$. But then there exists an open subset G of X such that $M \subset G$, $\bar{G} \cap N = \emptyset$; hence $(\bar{G} \backslash G) \cap f^{-1}(B) = \emptyset$ or $f(\bar{G} \backslash G) \cap B = \emptyset$ (see [**4**, page 130]). Let $V = Y \backslash f(\bar{G} \backslash G)$. This is an open subset of Y such that $B \subset V$. Since B is connected there exists a component C_0 of V such that $B \subset C_0$. Since C_0 is closed in V, there exists a closed subset K of Y such that $C_0 = V \cap K$. So C_0 meets the requirements of (ii). We also have $(\bar{G} \backslash G) \cap f^{-1}(C_0) = \emptyset$, so $f^{-1}(C_0) \subset X \backslash (\bar{G} \backslash G)$. Let Q be the quasi-component of $f^{-1}(C_0)$ containing x. Then $Q \subset G \cup (X \backslash \bar{G})$ and since $x \in M \subset G$ it follows that $Q \subset G$, so $Q \cap f^{-1}(y) = \emptyset$, contradicting the fact that $y \in C_0 = f(Q)$.

PROPOSITION 3.15. *Let $f: X \to Y$ be a perfect mapping from a hereditarily normal space X onto a topological space Y. Then the following are equivalent:*

(i) *f is h-weakly confluent;*

(ii) *for each connected subset B of Y such that $B = L \cap K$, where L is open and K is closed in Y, there exists a quasi-component Q of $f^{-1}(B)$ such that $f(Q) = B$.*

PROPOSITION 3.16. *Let $f\colon X \to Y$ be a perfect mapping from a hereditarily normal space X onto a topological space Y. Then the following are equivalent:*

(i) *f is h-pseudo confluent;*

(ii) *for each connected subset B of Y such that $B = L \cap K$, where L is open and K is closed in Y, and for any pair of points $y, z \in B$, there exists a quasi-component Q of $f^{-1}(B)$ such that $y, z \in f(Q)$.*

The proofs of Propositions 3.15 and 3.16 are similar to the proof of 3.14 and as such they are omitted.

D. Read (see [**16**, Theorem 2],) proved that Y is an hereditarily indecomposable continuum if and only if each continuous mapping of any continuum onto Y is confluent. The following propositions are analogous to Read's theorem.

PROPOSITION 3.17. *Let Y be a T_1 space. Then the following are equivalent:*

(i) *Y is hereditarily disconnected, i.e., each component of Y is degenerate;*

(ii) *for each topological space X and each perfect mapping f from X onto Y, f is strongly confluent;*

(iii) *for each topological space X and each perfect mapping f from X onto Y, f is h-confluent.*

Proof. (i) implies (ii) and (ii) implies (iii): Obvious.

(iii) implies (i): Suppose that Y is not hereditarily disconnected. Then there exists a non-degenerate connected subset L of Y. Let $q \in L$. Put $X = Y \cup \{p\}$, $(p \notin Y)$ endowed with the sum topology. Define a mapping $f\colon X \to Y$ by $f(x) = x$, if $x \in Y$ and $f(p) = q$. Then f is a perfect mapping and $f^{-1}(L)$ has $\{p\}$ as its quasi-component which is not mapped onto L, which contradicts (iii).

PROPOSITION 3.18. *Let Y be a T_1-space. Then the following are equivalent:*

(i) *Y is totally disconnected, i.e., each quasi-component of Y is degenerate;*

(ii) *for each topological space X and each perfect mapping f from X onto Y, f is H-confluent.*

Proof. (i) implies (ii): Obvious.

(ii) implies (i): Suppose Y is not totally disconnected. Then Y has a non-degenerate quasi-component Q. Let $q \in Q$. Put $X = Y \cup \{p\}$ $(p \notin Y)$, endowed with the sum topology. Define a mapping $f\colon X \to Y$ by $f(x) = x$, if $x \in Y$ and $f(p) = q$. Then f is a perfect mapping and $\{p\}$ is a quasi-component of X not mapped onto Q, which contradicts (ii).

4. Composition properties. We say that a class A of mappings has the *composition property*, provided that for any two mappings $f\colon X \to Y$ and $g\colon Y \to Z$ belonging to A, their composition $gf; X \to Z$ belongs to A. We also say that a class A of mappings has the *composition factor property*, provided that for any two mappings $f\colon X \to Y$ and $g\colon Y \to Z$ such that $gf\colon X \to Z$ belongs to A, g belongs to A.

PROPOSITION 4.1. *The classes of H-confluent, H-weakly confluent and H-pseudo confluent mappings have the composition property.*

Proof. We only prove it for the class of H-confluent mappings. The proofs for the other two classes being similar are omitted.

Let f: $X \to Y$ and g: $Y \to Z$ be two H-confluent mappings and let A be a a subset of Z, $z \in A$ and $x \in f^{-1}g^{-1}(z)$. Then since f is H-confluent and $f(x) \in g^{-1}(z) \subset g^{-1}(A)$, we have that

(1) $f(Q(f^{-1}g^{-1}(A), x)) = Q(g^{-1}(A), f(x))$.

But g being H-confluent we have

(2) $g(Q(g^{-1}(A), f(x))) = Q(A, z)$.

From (1) and (2) we take $gf(Q(f^{-1}g^{-1}(A), x)) = Q(A, z)$. Finally, gf is perfect and onto, since f and g are perfect and onto. Hence, gf is H-confluent.

PROPOSITION 4.2. *The classes of H-confluent, H-weakly confluent and H-pseudo confluent mappings have the composition factor property.*

Proof. We prove it for the class of H-confluent mappings, the proofs for the other two classes being similar.

Let f: $X \to Y$ and g: $Y \to Z$ be two mappings such that gf is H-confluent. It is easily seen that g is perfect. Let A be a subset of Z, $z \in A$ and $y \in g^{-1}(z)$. Then if $x \in f^{-1}(y)$, since gf is H-confluent, we obtain that

(1) $gf(Q(f^{-1}g^{-1}(A), x)) = Q(A, z)$.

Since $f(Q(f^{-1}g^{-1}(A), x)) \subset Q(g^{-1}(A), y)$ for any mapping f, (1) gives us

(2) $Q(A, z) = gf(Q(f^{-1}g^{-1}(A), x)) \subset g(Q(g^{-1}(A), y))$.

On the other hand $g(Q(g^{-1}(A), y)) \subset Q(A, z)$ for any continuous mapping g, which in accordance with (2) completes the proof.

PROPOSITION 4.3. *The classes of h-confluent, h-weakly confluent and h-pseudo confluent mappings have the composition factor property.*

Proof. Let f: $X \to Y$ and g: $Y \to Z$ be two mappings such that gf: $X \to Z$ is h-confluent. It is easily seen that g is perfect. Let K be a connected subset of Z and Q a quasi-component of $g^{-1}(K)$. Then $g(Q) \subseteq K$. On the other hand, since gf is h-confluent we have that for each $x \in f^{-1}(Q)$,

$$gf(Q(f^{-1}g^{-1}(K), x)) = K$$

and since $f(Q(f^{-1}g^{-1}(K), x)) \subseteq Q$, we get

$$K = gf(Q(f^{-1}g^{-1}(K), x)) \subseteq g(Q).$$

Thus, $g(Q) = K$.

The proofs for the other two classes being similar are omitted.

Remark 4.4. It is an open problem whether or not the classes of h-confluent, h-weakly confluent and h-pseudo confluent mappings possess the composition property. It is true though that the composition of two mappings is any one of the above classes is in the same class, provided that the space Y is locally connected (see [**11**, 3.1, 3.3 and 3.4]).

PROPOSITION 4.5. *Let $f: X \to Y$ be an H-confluent and $g: Y \to Z$ an h-confluent mapping. Then gf is an h-confluent mapping.*

Proof. Clearly gf is perfect. Let $K \subset Z$ be connected and $Q = Q(f^{-1}g^{-1}(K), x)$ a quasi-component of $f^{-1}g^{-1}(K)$ at x. Since f is H-confluent we have that $f(Q) = f(Q(f^{-1}g^{-1}(K), x)) = Q(g^{-1}(K), f(x))$ and since g is h-confluent, we obtain $gf(Q) = g(Q(g^{-1}(K), f(x))) = K$.

Similarly, we can prove the following two propositions:

PROPOSITION 4.6. *If $f: X \to Y$ is an H-weakly confluent and $g: Y \to Z$ an h-weakly confluent mapping, then $gf: X \to Z$ is an h-weakly confluent mapping.*

PROPOSITION 4.7. *If $f: X \to Y$ is an H-pseudo confluent and $g: Y \to Z$ an h-pseudo confluent mapping, then $gf: X \to Z$ is an h-pseudo confluent mapping.*

The following result is also known (see [**14**, 5.4 and 5.16]):

PROPOSITION 4.8. *The classes of confluent, weakly confluent and pseudo confluent mappings have the composition property as well as the composition factor property.*

5. Product properties. We say that a class A of mappings has the *product property*, provided that for any two mappings $f_i: X_i \to Y_i$ ($i = 1, 2$) belonging to A, their product $f_1 \times f_2$ from $X_1 \times X_2$ onto $Y_1 \times Y_2$ belongs to A. We also say that a class A of mappings has the *product factor property*, provided that for any two mappings $f_i: X_i \to Y_i$ ($i = 1, 2$) such that $f_1 \times f_2$ belongs to A, then f_1 and f_2 belong to A.

In what follows in this paragraph we consider all spaces to be continua, i.e., connected metric compacta.

T. Maćkowiak has constructed an example of a confluent mapping, the product of which with the identity on the unit interval I is not pseudo confluent (see [**14**, Example 5.37]). The following example also serves the same purpose and is due to Professor A. Lelek.

Example 5.1. Let

$$X = \{z \in \mathbf{C}: |z| = 1\} \cup \left\{z \in \mathbf{C}: z = r \exp i\left(\frac{\pi}{2}\sin\frac{\pi}{r-1} + \frac{\pi}{2}\right): 1 < r \leqq 2\right\}$$
$$\cup \left\{z \in \mathbf{C}: z = r \exp i\left(\frac{\pi}{2}\sin\frac{\pi}{r-1} + \frac{3\pi}{2}\right): 1 < r \leqq 2\right\},$$

a subset of the plane. Then X is a continuum, and let R be an equivalence relation in X given by

$$R = \{(z_1, z_2): |z_1| = |z_2| = 1, z_1^2 = z_2^2\} \cup \{(z, z): z \in X\}.$$

Put $Y = X/R$, and let f be the natural projection of X onto Y. Then it is easy to see that f is a confluent mapping.

Let $I = [0, 1]$ be the unit interval and h the identity on I. Consider the mapping $f \times h$: $X \times I \to Y \times I$. Both $X \times I$ and $Y \times I$ can be considered to be subsets of R^3. To show that $f \times h$ is not pseudo confluent, put

$$X_1 = \{z \in X: |z| = 1\},$$

$$X_2 = \left\{z \in X: z = r \exp i\left(\frac{\pi}{2} \sin \frac{\pi}{r-1} + \frac{\pi}{2}\right): 1 < r \leqq 2\right\}$$

and

$$X_2 = \left\{z \in X: z = r \exp i\left(\frac{\pi}{2} \sin \frac{\pi}{r-1} + \frac{3\pi}{2}\right): 1 < r \leqq 2\right\}.$$

Let A be the helix $x = \cos(2\pi t)$, $y = \sin(2\pi t)$, $z = t$, $t \in [0, 1]$, subset of $Y \times I$, and S_1, S_2 two sinusoid curves on $X_1 \times I$ and $X_2 \times I$ respectively, such that $\overline{(f \times h)(S_1)} = \overline{(f \times h)(S_2)} = A$. Then the set $K = (f \times h)(S_1) \cup (f \times h)(S_2) \cup A$ is a subcontinuum of $Y \times I$, such that $(f \times h)^{-1}(K)$ has two components, namely, the sets $C_1 = S_1 \cup A_1$ and $C_2 = S_2 \cup A_2$, where A_1 and A_2 are the helices $x = \cos(\pi t)$, $y = \sin(\pi t)$, $z = t$, $t \in [0, 1]$ and $x = -\cos(\pi t)$, $y = -\sin(\pi t)$, $z = t$, $t \in [0, 1]$, respectively. Let now $a \in (f \times h)(S_1)$ and $b \in (f \times h)(S_2)$. Then there is no component of $(f \times h)^{-1}(K)$ the image of which contains both a and b. Thus $f \times h$ is not pseudo confluent.

A recent result of A. Lelek shows the following (see [**13**, Theorem 1]):

THEOREM 5.2. *If f is an H-confluent mapping from a compactum onto a continuum, then f is quasi-interior.*

It is known that the classes of open, monotone and quasi-interior mappings have the product property (see [**12**, Theorems 1.9 and 1.10]). Thus Theorem 5.2 together with Theorem 1.10 in [**12**] give the following:

COROLLARY 5.3. *The class of H-confluent mappings of continua has the product property.*

Example 5.4. There exists an h-confluent and H-weakly confluent mapping of an arc-like continuum onto a planar continuum, the product of which with the identity on the interval $[-2, 2]$ is not h-pseudo confluent.

Proof. Let $X = \{(y, \sin \pi/y): y \in [-1, 0)\} \cup \{(x, 0): -1 \leqq x \leqq 1\}$, a subset of the xy-plane, and let R be an equivalence relation in X given by $R = \{(x, 0), (-x, 0)): -1 \leqq x \leqq 1\} \cup \{(p, p): p \in X\}$. Let f be the natural projection of X onto $Y = X/R$. Then f is h-confluent and it is easy to check that

f is H-weakly confluent. Let h be the identity on $I = [-2, 2]$. Consider the product $f \times h$: $X \times I \to Y \times I$. We will show that $f \times h$ is not h-pseudo confluent. For this, consider the points $a(1, 0, 0)$ and $b(-1, 0, 0)$ of $X \times I$. Then $(f \times h)(a) = (f \times h)(b)$.

Let P_1 be the plane $-x + z = -1$, i.e., the plane passing through the point $a(1, 0, 0)$ and normal to the vector $N = -i + k$, and let P_2 be the plane $-x + z = 1$, i.e., the plane passing through the point $b(-1, 0, 0)$ and normal to the same vector N.

Put

$$A = [\{(y, \sin \pi/y): y \in [-1, 0)\} \times I] \cap P_1, \quad \text{and}$$
$$B = [\{(y, \sin \pi/y): y \in [-1, 0)\} \times I] \cap P_2.$$

Then $K = f(A \cup B \cup \{a, b\})$ is a connected subset of $Y \times I$, the preimage of which consists of connected quasi-components $Q_1 = A \cup \{a\}$ and $Q_2 = B \cup \{b\}$ none of which is mapped onto K. Moreover. if c is a point of A and d is a point of B, then there is no quasi-component of $(f \times h)^{-1}(K)$ such that $(f \times h)(c)$ and $(f \times h)(d)$ belong to the image of it. Thus, $f \times h$ is not h-pseudo confluent.

An immediate consequence of this example is the following:

COROLLARY 5.5. *The classes of H-weakly confluent, H-pseudo confluent, h-confluent, h-weakly confluent and h-pseudo confluent mappings do not have the product property.*

Z. Rudy investigated in [**17**] the product factor property, where the following is proved:

THEOREM 5.6. *Let a class of mappings A satisfy the following conditions:*
(i) *if $f \in A$, then $f|f^{-1}(B) \in A$ for each closed set $B \subset Y$, and*
(ii) *if $gf \in A$ and f is open, then $g \in A$.*
Then the class A has the product factor property.

Remark 5.7. It is easy to observe that the classes of confluent, weakly confluent and pseudo confluent mappings have the product factor property (see [**14**, 5.39 and 5.40]).

By Propositions 3.10, 4.2 and 4.3 we obtain the following:

COROLLARY 5.8. *The classes of H-confluent, H-weakly confluent, H-pseudo confluent, h-confluent, h-weakly confluent and h-pseudo confluent mappings have the product factor property.*

6. Union properties.

THEOREM 6.1. *Let Y be a topological space with the property that the intersection of any two connected subsets is connected. Suppose X is a topological space,*

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$f: X \to Y$ is a closed mapping of X onto Y, and $Y = Y_1 \cup Y_2 \cup \ldots \cup Y_n$ is a decomposition of Y into connected subsets with the following properties:

(i) *either $Y_i \cap Y_j \neq \emptyset$ or Y_i, Y_j are separated, for $i \neq j$ $i, j = 1, 2, \ldots, n$;*

(ii) *$f|f^{-1}(Y_i)$ is h-confluent for $i = 1, \ldots, n$.*

Then f is h-confluent.

Proof. Let K be a connected subset of Y. Assume that $n = 2$, so that $Y = Y_1 \cup Y_2$. If $Y_1 \cap \bar{Y}_2 = \emptyset = \bar{Y}_1 \cap Y_2$, then either $K \subset Y_1$ or $K \subset Y_2$ and by (ii) we infer that each quasi-component of $f^{-1}(K)$ is mapped onto K. So assume that $Y_1 \cap Y_2 \neq \emptyset$ and that $K \setminus Y_1 \neq \emptyset \neq K \setminus Y_2$. Let Q be a quasi-component of $f^{-1}(K)$ and $x_1 \in Q$. We may assume that $f(x_1) \in Y_1$. Since $K \cap Y_1$ is a connected subset of Y_1, if Q_1 is the quasi-component of x_1 in $f^{-1}(K \cap Y_1)$, by (ii) we have

(1) $f(Q_1) = K \cap Y_1$ and $Q_1 \subset Q$.

Let now $y \in K \cap Y_1 \cap Y_2$ and $x_2 \in Q_1 \cap f^{-1}(y)$. Since $K \cap Y_2$ is connected subset of Y_2, if Q_2 is the quasi-component of x_2 in $f^{-1}(K \cap Y_2)$, by (ii) we have

(2) $f(Q_2) = K \cap Y_2$ and $Q_2 \subset Q$.

By (1) and (2) we obtain $K = (K \cap Y_1) \cup (K \cap Y_2) = f(Q_1) \cup f(Q_2) \subset f(Q)$. But we always have $f(Q) \subset K$. Hence, f is h-confluent. Induction now completes the proof.

Example 6.2. This example shows that the condition (i) in theorem 5.1 is essential.

Let

$$X = \{(x, 1): -2 \leqq x \leqq 0\} \cup \{(0, y): -1 \leqq y \leqq 1\} \cup \{(x, \sin \pi/x): 0 < x \leqq 1\}$$

and let R be an equivalence relation in X given by

$$R = \{((-2 + t, 1), (0, -1 + t)): 0 \leqq t \leqq 2\} \cup \{(p, p): p \in X\}.$$

Put $Y = X/R$ and let f be the natural projection of X onto Y. Let $Y = Y_1 \cup Y_2$ be the following decomposition of Y into connected subsets $Y_1 = \{(0, y): -1 \leqq y \leqq 1\}$ and $Y_2 = \{(x, \sin \pi/x): 0 < x \leqq 1\}$. One can easily see that Y has the property that the intersection of any two connected subsets is connected and that condition (ii) of theorem 6.1 is satisfied, but f is not h-confluent and Y_1 and Y_2 are not separated.

A. Lelek has proved the following union theorem for confluent mappings (see [**9**, Thoerem 1]):

THEOREM 6.3. *Suppose X, Y are compact metric spaces, f is a continuous mapping of X onto Y, and $Y = Y_0 \cup Y_1 \cup Y_2 \cup \ldots$ is a decomposition of Y into closed subsets Y_i such that the following three conditions are satisfied:*

(i) $f|f^{-1}(Y_i)$ *is a confluent mapping of* $f^{-1}(Y_i)$ *onto* Y_i, $(i = 0, 1, 2, \ldots)$;

(ii) $Y_i \cap Y_j \subset Y_0$ *for* $i \neq j$, $i, j = 1, 2, \ldots$; *and*

(iii) $K \cap Y_0$ *has only a finite number of components for each subcontinuum* K *of* Y.

Then f *is confluent.*

In a discussion with the author, Professor A. Lelek asked if there is an analogous "infinite union theorem" for h-confluent and H-confluent mappings. The answer to both questions is in the negative and this can be shown by the following example:

Example 6.4. There exists a mapping f of a continuum X onto a continuum Y and a decomposition of Y into subcontinua $Y_0, Y_1, Y_2, \ldots$ such that the following conditions are satisfied:

(i) $f|f^{-1}(Y_i)$ is an open mapping, hence also H-confluent and h-confluent, of $f^{-1}(Y_i)$ onto Y_i for $i = 0, 1, 2, \ldots$;

(ii) $Y_i \cap Y_j \subset Y_0$ for $i \neq j$ and $i, j = 1, 2, \ldots$;

(iii) $K \cap Y_0$ is connected for each connected subset K of Y, and f is not h-confluent.

Proof. Let C be the Cantor ternary set on the interval $\{(x, 0): 0 \leqq x \leqq 1\}$ and $D = \{(0, 1)\}$. Consider the set $C \cup D$ and let $X = [(C \cup D) \times I]/R$, where I is the unit interval $[0, 1]$ and R is an equivalence relation in $(C \cup D) \times I$ given by: $R = \{(x, 1), (x', 1)): x, x' \in C \cup D\} \cup \{(p, p): p \in (C \cup D) \times I\}$. We can describe X as the cone over the set $C \cup D$. Then we can take the vertex of the cone to be the point $a(0, 0, 1)$.

Let Y be the cone over the set C and $f: X \rightarrow Y$ be the mapping such that $f((0, y, z)) = (0, 0, z)$, for each $(0, y, z) \in X$ and $f((x, 0, z)) = (x, 0, z)$, for each $(x, 0, z) \in X$. Consider the following decomposition of $C = A_0 \cup A_1 \cup A_2 \cup \ldots$ where $A_0 = C \cap [2/3, 1]$, $A_1 = \{0\}$, $A_2 = C \cap [2/9, 1/3]$, $\ldots$, $A_n = C \cap [2/3^n, 1/3^{n-1}]$, for $n = 2, 3, \ldots$, and denote by Y_i the cone over A_i in Y, for $i = 0, 1, 2, \ldots$ Then it is easy to check that the decomposition $Y = Y_0 \cup Y_1 \cup Y_2 \cup \ldots$ satisfies all the three conditions. To show that f is not h-confluent, let K be the Knaster-Kuratowski biconnected subset of Y (see [**5**, p. 135]). Then $f^{-1}(K)$ has one quasi-component Q, a subset of the cone over C in X, which is mapped onto K, and countably many degenerate quasi-components lying on the cone over D in X, none of which is mapped onto K. Thus f is not h-confluent.

7. Limit properties. We say that a class A of mappings has *the general limit property* (respectively, *the weak limit property*) if for any sequence $f_n: X \rightarrow Y$ in A, where Y is a compactum (respectively, locally connected compactum), the uniform limit of the sequence is in A.

It has been proved (see [**14**, 5.48 and 5.50]) that the classes of confluent, weakly confluent and pseudo confluent mappings have the weak limit property. These results together with remark 3.2 give the following:

COROLLARY 7.1. *The classes of H-confluent, H-weakly confluent, H-pseudo confluent, h-confluent, h-weakly confluent and h-pseudo confluent mappings have the weak limit property.*

Remark 7.2. T. Maćkowiak proved (see [**14**, 5.49 and 5.54]) that the classes of weakly confluent and pseudo confluent mappings of compacta have the limit property. He also constructed the following example (ibidem, 5.62) of a sequence of homeomorphisms of a continuum onto a continuum the uniform limit of which is not confluent. We prove that the uniform limit of this sequence is not an h-pseudo confluent mapping.

Example 7.3. Let C denote the Cantor ternary set lying in the unit interval $I = [0, 1]$. There is a sequence $\{f_n\}$ of homeomorphisms from I onto I such that $f_n(C) = C$ for $n = 1, 2, \ldots$ and such that it converges uniformly to the mapping f_0, where

$$f_0(t) = \begin{cases} 0 & \text{if } t \in [0, 2/3] \\ 3t - 2, & \text{if } t \in [2/3, 1]. \end{cases}$$

Put $N = (I \times \{0\}) \cup (C \times I)$ and $g_n(x, y) = (f_n(x), y)$ for each $(x, y) \in N$ and $n = 0, 1, 2, \ldots$. The mappings g_n are homeomorphisms for $n = 1, 2, \ldots$ and they converge uniformly to g_0. Define a mapping ψ of I onto itself as follows:

$$\psi(t) = \begin{cases} -t + 1/4, & \text{if } 0 \leqq t \leqq 1/4 \\ 3t - 3/4, & \text{if } 1/4 \leqq t \leqq 1/2 \\ -t + 5/4, & \text{if } 1/2 \leqq t \leqq 3/4 \\ 2t - 1 \quad , & \text{if } 3/4 \leqq t \leqq 1. \end{cases}$$

Consider the equivalence relation R in N defined as follows: $(x, y)R(x', y')$ if and only if either $(x, y) = (x', y')$ or $x = x' = 0$ and $\psi(y) = \psi(y')$. Let φ be the canonical mapping of N onto N/R. Put $M = N/R$ and $h_n(q) = \varphi(g_n(\varphi^{-1}(q)))$ for each $q \in M$ and $n = 0, 1, 2, \ldots$ The sequence $\{h_n\}$ is a sequence of homeomorphisms which converges uniformly to a mapping h_0. We shall prove that h_0 is not h-pseudo confluent. Note that $h_0 = \lim_{n\to\infty} h_n = \varphi g_0 \varphi^{-1}$, so $h_0^{-1} = \varphi g_0^{-1} \varphi^{-1}$. Consider now the following subset of N:

$$A = ((0, 1] \times \{0\}) \cup [(C\backslash\{0\}) \times [0, 1/2]] \cup \{(0, 1/8)\} \cup \{(0, y)\colon 5/8 \leqq y \leqq 1\}.$$

Then A is a connected subset of N, therefore $K = \varphi(A)$ is a connected subset of M. Consider the points $y = \varphi(0, 1/8)$ and $z = \varphi(0, 1)$ of K. Then there is no quasi-component of $h_0^{-1}(K)$ the image of which under h_0 contains both points y and z (see Diagram 2). Thus, h_0 is not h-pseudo confluent.

The following table summarizes the properties of mappings from Paragraphs 3, 4, 5, 6 and 7.

TABLE I.

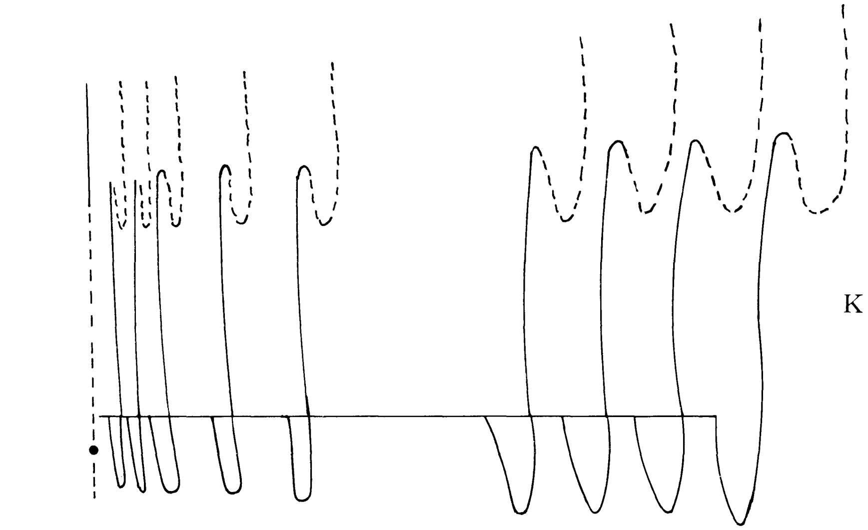
	composition property	composition factor property	product property	product factor property	weak limit property	general limit property
H-confluent	+ (4.1)	+ (4.2)	+ (5.3)	+ (5.8)	+ (7.1)	+ (7.3)
H-weakly confluent	+ (4.1)	+ (4.2)	+ (5.5)	+ (5.8)	+ (7.1)	− (7.3)
H-pseudo confluent	+ (4.1)	+ (4.2)	− (5.5)	+ (5.8)	+ (7.1)	− (7.3)
h-confluent	?	+ (4.3)	− (5.5)	+ (5.8)	+ (7.1)	− (7.3)
h-weakly confluent	?	+ (4.3)	− (5.5)	+ (5.8)	+ (7.1)	− (7.3)
h-pseudo confluent	?	+ (4.3)	− (5.5)	+ (5.8)	+ (7.1)	− (7.3)
confluent	+ (4.8)	+ (4.8)	− (5.1)	+ (5.7)	+ (7.1)	− (7.3)
weakly confluent	+ (4.8)	+ (4.8)	− (5.1)	+ (5.7)	+ (7.2)	+ (7.2)
pseudo confluent	+ (4.8)	+ (4.8)	− (5.1)	+ (5.7)	+ (7.2)	+ (7.2)

8. Mappings of rational continua. In [**12**, problem III] A. Lelek asked the question "Do confluent mappings preserve rational continua?". Recently, E. D. Tymchatyn (see [**18**]) constructed a confluent mapping of a rational curve onto a non-rational curve. It is not difficult to check that Tymchatyn's example is a confluent but not h-pseudo confluent, hence not h-confluent or h-weakly confluent, mapping.

The following theorem generalizes Theorems 3.8 and 3.9 of [**12**], where A. Lelek proves that open mappings as well as monotone mappings preserve rational continua.

THEOREM 8.1. *H-pseudo confluent mappings preserve rational continua.*

Proof. Let $f: X \to Y$ be an H-pseudo confluent mapping of a rational continuum X onto a continuum Y. In order that Y possesses a base of open sets with countable boundaries it is necessary and sufficient to show that any two points y and z in Y, with $y \neq z$, are separated in Y by a countable set (see [**19**, V 4.3]). To show this, let y and z be two distinct points of Y. Consider the compact sets $f^{-1}(y)$ and $f^{-1}(z)$ in X. Since X is a rational continuum, there exists a countable closed subset A of X disjoint from $f^{-1}(y)$ and $f^{-1}(z)$ such



The dashed lines represent the set $M \backslash K$.

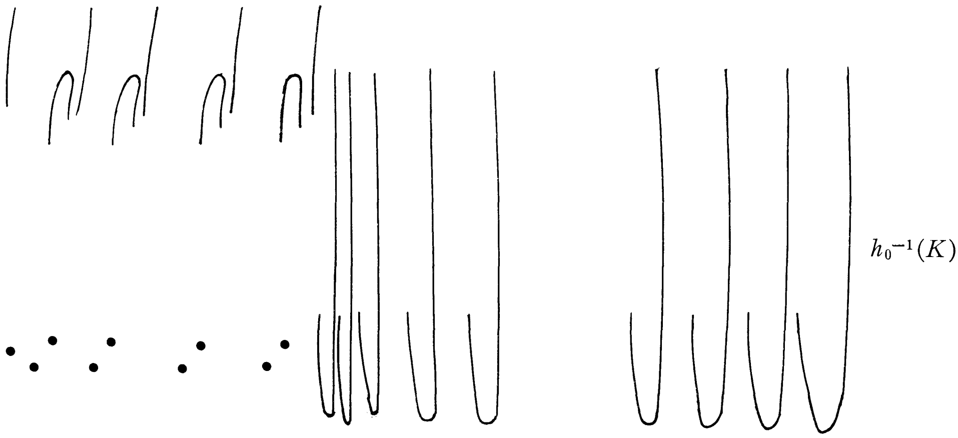


DIAGRAM 2.

that $X \backslash A = M \cup N$, $f^{-1}(y) \subset M$, $f^{-1}(z) \subset N$ and M and N are separated. Since f is H-pseudo confluent, by Proposition 3.13 (iii), we conclude that $f(A)$ separates y and z in Y. Finally, since $f(A)$ is a countable and closed subset of Y, we infer that Y is rational.

COROLLARY 8.2. *The classes of open, monotone, quasi-interior, H-confluent and H-weakly confluent mappings preserve rational continua.*

The following problem now can be raised:

Problem 1. Do h-confluent (h-weakly confluent or h-pseudo confluent) mappings preserve rational continua?

We also have the following:

PROPOSITION 8.3. *The h-pseudo confluent image of a rational continuum can be written as the union of a countable set and a hereditarily disconnected set.*

Proof. Let $f: X \to Y$ be an h-pseudo confluent mapping of a rational continuum X onto a continuum Y. Let $X = P \cup Q$ be a decomposition of X into a totally disconnected set P and a countable set Q (see [**8**, Theorem]). Consider the following decomposition for Y, $Y = (Y \backslash f(Q)) \cup f(Q)$. $f(Q)$ is countable. To show that $Y \backslash f(Q)$ is hereditarily disconnected, let K be a connected subset of $Y \backslash f(Q)$. Then $f^{-1}(K) \subset P$, which is totally disconnected. Therefore, all the quasi-components of $f^{-1}(K)$ are degenerate. Assume that K is non-degenerate and y and z are two distinct points of K. Since f is h-pseudo confluent, there exists a quasi-component Q of $f^{-1}(K)$ such that $y, z \in f(Q)$. But, since Q is degenerate, we conclude that y and z coincide, a contradiction. Thus, each connected subset of $Y \backslash f(Q)$ is degenerate, which means that $Y \backslash f(Q)$ is hereditarily disconnected.

COROLLARY 8.4. *If $f: X \to Y$ is an h-confluent or h-weakly confluent mapping from a rational continuum X onto a continuum Y, then $Y = P \cup Q$, where P is hereditarily disconnected and Q is countable.*

9. Weakly σ-connected and σ-connected spaces.

We say that a connected space is *σ-connected* (respectively, *weakly σ-connected*) provided that it cannot be decomposed into countably many mutually separated (respectively, mutually separated connected) non-empty subsets. Clearly, each σ-connected space is weakly σ-connected and each weakly σ-connected is connected (see [**2**] and [**6**]). We now define a space to be *hereditarily σ-connected* (respectively, *hereditarily weakly σ-connected*) provided that it is connected and each connected subset of it is σ-connected (respectively, weakly σ-connected.)

The following notions have already been introduced: We say that a topological space is *hereditarily* ($q = c$) provided each subset of it has connected quasi-components (see [**15**]). A continuum is said to be *finitely Suslinian* provided that for each number $\epsilon > 0$, every collection of mutually disjoint subcontinua of it with diameters greater than ϵ is finite (see [**2**]).

The following theorems are known:

THEOREM 9.1. *A continuum is hereditarily ($q = c$) if and only if it is h.l.c.*

THEOREM 9.2. *A continuum is finitely Suslinian $\Leftrightarrow$ it is hereditarily σ-connected $\Leftrightarrow$ it is hereditarily weakly σ-connected, provided it is h.l.c.*

For their proofs, see [**19**] and [**2**], respectively.

Added in proof: In a recent paper, J. Grispolakis and E. D. Tymchatyn constructed an h-confluent mapping from an arc-like rational continuum onto a non-rational continuum.

10. Mappings of hereditarily σ-connected spaces.

PROPOSITION 10.1. *Strongly confluent mappings preserve hereditarily σ-connected spaces.*

Proof. Let f be a strongly confluent mapping of the hereditarily σ-connected space X onto the topological space Y, and suppose that K is a connected subset of Y such that $K = \bigcup_{i=1}^{\infty} F_i$, where F_i and F_j are non-empty mutually separated sets for $i \neq j$ and $i, j = 1, 2, \ldots$ Then $f^{-1}(F_i)$ and $f^{-1}(F_j)$ are non-empty mutually separated for $i \neq j$ and $i, j = 1, 2, \ldots$ Let C be a component of $f^{-1}(K)$. Since $f(C) = K$ we infer that $C \cap f^{-1}(F_i) \neq \emptyset$ for $i = 1, 2, \ldots$ Therefore, $C = \bigcup_{i=1}^{\infty} (f^{-1}(F_i) \cap C)$. But this contradicts the fact that C is σ-connected set. Thus, K is σ-connected.

COROLLARY 10.2. *Monotone mappings preserve hereditarily σ-connected spaces.*

PROPOSITION 10.3. *Monotone mappings preserve hereditarily weakly σ-connected spaces.*

Proof. Let f be a monotone mapping of a hereditarily weakly σ-connected space X onto a topological space Y. Let K be a connected subset of Y such that $K = \bigcup_{i=1}^{\infty} C_i$, where C_i, C_j are non-empty mutually separated connected sets for $i \neq j$ and $i, j = 1, 2, \ldots$ Since f is monotone $f^{-1}(K)$ and $f^{-1}(C_i)$ are connected for $i = 1, 2, \ldots$ We also have $f^{-1}(K) = \bigcup_{i=1}^{\infty} f^{-1}(C_i)$, which contradicts the fact that $f^{-1}(K)$ is weakly σ-connected. Thus, Y is hereditarily weakly σ-connected.

PROPOSITION 10.4. *Let f be an H-weakly confluent mapping of a hereditarily $(q = c)$ and hereditarily σ-connected space onto a topological space Y. Then Y is hereditarily $(q = c)$ and hereditarily σ-connected.*

Proof. Let K be a connected subset of Y and suppose that $K = \bigcup_{i=1}^{\infty} F_i$, where F_i and F_j are non-empty mutually separated sets for $i \neq j$ and $i, j = 1, 2, \ldots,$ Since f is H-weakly confluent there exists a quasi-component Q of $f^{-1}(K)$ such that $f(Q) = K$. Since X is hereditarily $(q = c)$ we infer that Q is connected. Therefore, we have that $Q \cap f^{-1}(F_i) \neq \emptyset$ for $i = 1, 2, \ldots,$ so that $Q = \bigcup_{i=1}^{\infty} (f^{-1}(F_i) \cap Q)$ is not σ-connected, contradicting the fact that X is hereditarily σ-connected. Thus, K is σ-connected. It is trivial also to check that Y is hereditarily $(q = c)$, so the proof of the theorem is complete.

Next, we prove an analogous theorem for more general mappings. First, we prove a lemma.

LEMMA 10.5. *Let f be a mapping of a topological space X onto a topological space Y. If $C = \bigcup_{i=1}^{\infty} F_i$ is a connected subset and F_i are mutually separated, then each* *component of* $f^{-1}(C)$ *intersects either one or infinitely many of the sets* $f^{-1}(F_i)$ $i = 1, 2, \ldots$

Proof. On the contrary, suppose that a component K of $f^{-1}(C)$ is such that $K \cap f^{-1}(F_{k_i}) \neq \emptyset$ for $i = 1, 2, \ldots, n$, $n > 1$ and $K \cap f^{-1}(F_j) = \emptyset$, if $j \neq k_i$, $i = 1, \ldots, n$. Then we have that $K = \bigcup_{i=1}^{n} (K \cap f^{-1}(F_{k_i}))$, and since $n > 1$ and $K \cap f^{-1}(F_{k_i})$, $f^{-1}(F_{k_j}) \cap K$ are non-empty and mutually separated for $i \neq j$; $i, j = 1, \ldots, n$, we have that K is not connected, contradicting the fact that K is a component of $f^{-1}(C)$. This completes the proof of the lemma.

THEOREM 10.6. *Let f be an h-pseudo confluent mapping of an h.l.c. and hereditarily σ-connected space X onto a topological space Y. Then Y is hereditarily σ-connected.*

Proof. Let K be a connected subset of Y and suppose that K is not σ-connected. Then $K = \bigcup_{i=1}^{\infty} F_i$, where F_i, F_j are mutually separated non-empty subsets of K for $i \neq j$; $i, j = 1, 2, \ldots$ Let i and j be two indices such that $i \neq j$ and let $x \in F_i$, $y \in F_j$. Since f is h-pseudo confluent there exists a quasi-component Q of $f^{-1}(K)$ such that $x, y \in f(Q)$. But X is h.l.c., therefore, Q is connected. Since $x, y \in f(Q)$ we have that $Q \cap f^{-1}(F_i) \neq \emptyset$ and $Q \cap f^{-1}(F_j) \neq \emptyset$ and so, by Lemma 12.5, there are infinitely many indices $n_1, \ldots, n_k, \ldots$ such that $Q \cap f^{-1}(F_{nk}) \neq \emptyset$, for $k = 1, 2, \ldots$. Therefore, $Q = \bigcup_{k=1}^{\infty} (Q \cap f^{-1}(F_{nk}))$, where $Q \cap f^{-1}(F_{n_i})$ and $Q \cap f^{-1}(F_{n_j})$ are non-empty mutually separated sets for $i \neq j$ and $i, j = 1, 2, \ldots$ Thus, Q is not σ-connected contrary to the hypothesis. Hence, Y is hereditarily σ-connected.

COROLLARY 10.7. *Pseudo confluent mappings preserve finitely Suslinian continua.*

Proof. By Theorem 4.7 in [**11**], if f is pseudo confluent and X is a hereditarily locally connected continuum, then $Y = f(X)$ is a hereditarily locally connected continuum. By Theorem 2.2 in [**2**], X is hereditarily σ-connected and by Theorem 3.3 of [**11**], the mapping f is h-pseudo confluent. Therefore, by Theorem 10.6 we infer that Y is hereditarily σ-connected, which in accordance with Theorem 2.2 of [**2**] implies that Y is finitely Suslinian continuum.

Remark 10.8. Corollary 10.7 was obtained by A. Lelek and E. D. Tymchatyn (see [**11**, Theorem 4.6]). Therefore, Theorem 10.6 generalizes Theorem 4.6 of [**11**].

PROPOSITION 10.9. *The h-weakly confluent image of an h.l.c. separable metric space onto a metric space is h.l.c.*

Proof. Let f be an h-weakly confluent mapping of an h.l.c. separable metric space X onto a metric space Y. Let K be a connected subset of Y. Since f is h-weakly confluent there exists a quasi-component Q of $f^{-1}(K)$ such that $f(Q) = K$. Since X is h.l.c. we have that X is hereditarily $(q = c)$ (see [**15**, Theorem 1.5]), so that Q is connected subset of X, hence Q is locally connected. Since the mapping $f|f^{-1}(K)$ onto K is closed, and since Q is a closed subset of $f^{-1}(K)$, we infer that the mapping $f|Q$ onto K is closed. Thus, $K = f(Q)$ is locally connected.

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